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Adomian Decomposition Method (ADM) for Solving the Nonlinear Generalized Regularized Long Wave Equation

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Abstract: In this paper, the Adomian decomposition method is implemented to find the solution of the nonlinear generalized regularized long wave (GRLW) equation with appropriate initial conditions. The methodology of the method is investigated. The accuracy of the algorithm is illustrated by solving three test examples of the problem. We study the convergence analysis of the method. The results that will be obtained are compared with the analytical solutions, where we observe that only some terms of the series expansion are required to obtain approximation solutions with good accuracy. And also, this studying is extending to discuss the interaction of two and three solitons.

Keywords: Adomian decomposition method, Nonlinear generalized regularized long wave equation, Convergence analysis.

1 Introduction

In many frontier problems, we have strong nonlinear terms, so that it becomes important to find an improved general method which is provides high accurate solutions without the usual modeling. This technique so-called the ADM was introduced by George Adomian [1, 2]. It is used for finding the exact and approximate solutions of large class of equations. These include differential equations containing polynomial, exponential and trigonometric terms, negative or irrational powers, and product nonlinearities. These include also the partial differential equations, delay differential equations, algebraic equations and matrix equations describing physical systems such as in [3-12].

The various methods for obtaining explicit solutions to nonlinear equations have a common feature which uses the transformations to reduce the equation into a more simple equation and then solve it. Unlike classical techniques, the nonlinear equations are solved easily and elegantly without transformation using ADM. The technique has many advantages over the classical techniques, mainly; it avoids linearization and perturbation in order to find solutions of given nonlinear equations. It is providing an efficient explicit solution with high accuracy, minimal calculation and avoidance of physically unrealistic. In general, the solution is usually not in closed form; we get a series with a convergence region in which numerical solutions stabilize quickly to a solution with the range of accuracy needed. The approximation method based on ADM produced a good accurate approximation; we don't make any discretization, in fast convergent series.

ADM avoids linearization and we don't need higher computer storage. Also little number of terms is needed to get a good accurate approximation. The GRLW equation is discussed by many authors in last years, see [13-17].

In this paper, the ADM to obtain the solution of the nonlinear generalized regularized long wave equation is presented. Also, the convergence of the method according to [12,18] and [19] is investigated. The accuracy of the algorithm is illustrated through solving some of test problems. Finally, we point out that, the series solution is in the form

$$g(x,t)=\sum_{n=0}^{\infty} g_n(x,t) \ , \qquad (1)$$

where $g(x,t)$ are the components which are determined recursively. In fact, the decomposition series (1) is a fast converging series, so we can write the partial sum as:

$$\varphi_n=\sum_{k=0}^{n-1} g_k(x,t) \ , n>0 \qquad (2)$$

The approximation for the solution $g(x,t)$ is equation (2), and then to make numerical comparison, we write the solution $g(x,t)$ as

$$g(x,t)=\lim_{n\to\infty}\varphi_n$$

Where $$\varphi_n=\sum_{k=0}^{n-1} g_k(x,t) \ , n>0.$$

On the other hand, we can increase the accuracy of the solution by taken more terms which is quite easy using one

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of the symbolic programming packages Mathematica, Mable, etc. Here, in our study, we use the Mathematica.

2 Governing Equation and Analysis of the Method

Consider the GRLW equation

$$g_t + g_x + \varepsilon g^p g_x - \mu g_{xxt} = 0, \qquad (3)$$

where $\varepsilon = p(p+1)$ and subscripts x and t denote differentiation. We considered the boundary conditions as $g \to 0$ as $x \to \pm\infty$ and the initial conditions at $t=0$. In this paper, we take the periodic boundary conditions on the closed region $a \le x \le b$ are assumed in the form:

$$g(a,t) = g(b,t) = 0, \qquad t \ge 0, \qquad (4)$$

the initial condition

$$g(x,0) = \sqrt[p]{\frac{(p+2)\upsilon}{2p}\sec h^2\left(\frac{p}{2}\sqrt{\frac{\upsilon}{\mu(\upsilon+1)}}(x-x_0)\right)} = f(x). \qquad (5)$$

The analytical solution of equation (3) write in this form.[14]

$$g(x,t) = \sqrt[p]{\frac{(p+2)\upsilon}{2p}\sec h^2\left(\frac{p}{2}\sqrt{\frac{\upsilon}{\mu(\upsilon+1)}}(x-(\upsilon+1)t-x_0)\right)}, \qquad (6)$$

where x_0 and υ are any choice constants. Actually in most times we can't get analytic solutions for the nonlinear partial differential equations, so we try to obtain the approximate or numerical methods to solve these problems. To the purposes of illustration of the ADM, in this paper, we write equation (3) in the operator form as:

$$Lg(x,t) = -(g_t + g_x + \varepsilon g^p g_x - \mu g_{xxt}), \qquad (7)$$

where $L = \frac{\partial}{\partial t}$. Suppose that the inverse operator L^{-1} and we can defined it as integral with respect to t from t_0 to t, i.e.

$$L^{-1} = \int_{t_0}^{t} (.)\,dt. \qquad (8)$$

$$g(x,t) = -L^{-1}(g_t + g_x + \varepsilon g^p g_x - \mu g_{xxt}), \qquad (9)$$

where $g(x,0) = f(x)$ and $N(g) = g^p g_x$ represents the nonlinear term. The ADM assumes that the unknown function $g(x,t)$ can be expressed by a sum of components defined by the decomposition series of the form

$$g(x,t) = \sum_{n=0}^{\infty} g_n(x,t). \qquad (10)$$

By using (9) and (10) we get

$$\sum_{n=0}^{\infty} g_n(x,t) = f(x) - L^{-1}\left(\varepsilon \sum_{n=0}^{\infty} A_n + \left(\sum_{n=0}^{\infty} g_n\right)_x - \mu\left(\sum_{n=0}^{\infty} g_n\right)_{xxt}\right), \qquad (11)$$

where $\sum_{n=0}^{\infty} A_n = N\left(\sum_{n=0}^{\infty} g_n\right) = \left(\sum_{n=0}^{\infty} g_n^{\ p}\right)\left(\sum_{n=0}^{\infty} (g_n)_x\right)$.

Identifying the zero[th] component $g_0(x,t)$ as $f(x)$, the remaining components $g_n(x,t), n \ge 1$ can be determined by using the recurrence relation

$$g_0(x,t) = f(x),$$

$$g_{n+1}(x,t) = f(x) - L^{-1}(\varepsilon A_n + (g_n)_x - \mu(g_n)_{xxt}), \; n \ge 0, \qquad (12)$$

where A_n are Adomian polynomials that represent the nonlinear term $g^p g_x$. For the problem (7), these polynomials can be calculated as:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty} \lambda^i g_i\right)\right]_{\lambda=0} = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}(f\,m)\right]_{\lambda=0}, \qquad (13)$$

where

$$f = \sum_{i=0}^{\infty} \lambda^i g_i^{\ p} = (g_0 + \lambda g_1 + \lambda^2 g_2 + \lambda^3 g_3 + \cdots)^p,$$

and

$$m = \sum_{i=0}^{\infty} \lambda^i g_{ix} = g_{0x} + \lambda g_{1x} + \lambda^2 g_{2x} + \lambda^3 g_{3x} + \cdots.$$

By some simple calculations we get

$$A_n == \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}(fm)\right]_{\lambda=0} = \frac{1}{n!}\left[(D^n f)m + C_1^n (D^{n-1} f)(Dm) + C_2^n (D^{n-2} f)(D^2 m) + \cdots + f(D^n m)\right]_{\lambda=0},$$

Where $C_r^n = \frac{n!}{r!(n-r)!}$ and $D^n = \frac{d^n}{d\lambda^n}$.

Thus $A_0 = g u_0^p g_{0x}$, $A_1 = g_0^p g_{1x} + p g_0^{p-1} g_1 g_{0x}$, $A_2 = g_0^p g_{2x} + p g_0^{p-1} g_2 g_{0x} + p g_0^{p-1} g_1 g_{1x} + \frac{p(p-1)}{2} g_0^{p-2} g_1^2 g_{0x}$,

and so on. The rest of the polynomials can be obtained by the similar way. The system (12) can be determine the components $g_n(x,t),\ n \geq 0$ and the first little components of $g_n(x,t)$ follows immediately of the form

$$\begin{aligned}
&g_o(x,t) = f(x),\\
&g_1(x,t) = -L^{-1}(\varepsilon A_0 + (g_0)_x - \mu(g_0)_{xxt}),\\
&g_2(x,t) = -L^{-1}(\varepsilon A_1 + (g_1)_x - \mu(g_1)_{xxt}),\\
&g_3(x,t) = -L^{-1}(\varepsilon A_2 + (g_2)_x - \mu(g_2)_{xxt}),\\
&g_4(x,t) = -L^{-1}(\varepsilon A_3 + (g_3)_x - \mu(g_3)_{xxt}),\\
&\vdots
\end{aligned} \tag{15}$$

It is in principle, possible to calculate more components in the decomposition series to enhance the approximation. Consequently, on can recursively determine each individual term of the series $\sum_{n=0}^{\infty} g_n(x,t)$, and hence the solution $g(x,t)$ is readily obtained in a series form.

The accuracy of the proposed method, based on the ADM, is investigated by considering three test examples for the GRLW equation at $p = 1,2,3$. The obtained numerical approximate solution $g_{appr.}(x,t)$ for each example, is compared with the exact solution where the four terms approximation is

$$g_{appr.}(x,t) = g_0(x,t) + g_1(x,t) + g_2(x,t) + g_3(x,t)\ . \tag{16}$$

3 The Convergence Analysis

The solutions obtained by decomposition series is converge very fast in real physics models. The convergence of the series has been studied by many researchers see [12, 18] and [19]. In [12], the new condition for obtaining convergence of the series to the ADM is introduced by Ngarhasta et al. Here, we introduce the convergence analysis of the series by the same way as in [12], applied this methodology to the RLW, the MRLW and GRLW equations. Now, we consider the Hilbert space H which may be defined in this form $H = L^2((a,b)\times[0,T])$ the set of applications

$$(g,v)_H = \int_{(a,b)\times[0,T]} g(x,s)v(x,s)\,ds\,d\tau,$$

And the associate norm

$$\|g\|_H^2 = \int_{(a,b)\times[0,T]} g^2(x,s)\,ds\,d\tau\ .$$

We take some shorts for some components such as

$$L(g) = \frac{\partial}{\partial t}g,\quad M(g) = \frac{\partial}{\partial x}g,\quad N(g) = g^p g_x,\quad R(g) = \frac{\partial^2}{\partial x^2}g_t.$$

Thus, equation (3) can be written in the form

$$L(u) = -M(g) - \varepsilon N(g) + \mu R(g) = -\frac{\partial}{\partial t}g - \varepsilon g^p g_x + \mu\frac{\partial^2}{\partial x^2}g_t,$$

Where $p = 1,2,3$.

As the same in [19], if the next two hypotheses are satisfied therefor, the ADM is converges

(a) $(L(g) - L(v), g - v) \geq \|g - v\|^2,\ k > 0\ \forall g,v \in H$.

(b) $\forall M > 0$, there exists a constant $C(M) > 0$, such that for $g,v \in H$ with $\|g\| \leq M, \|v\| \leq M$ we have:

$$(L(g) - L(v), w) \leq C(M)\|g - v\|\|w\|\ , w \in H\ .$$

Theorem 1. (Appropriate condition to convergence of the RLW, the MRLW and the GRLW equations). The ADM applied to the RLW, the MRLW and the GRLW equations in the next form:

$$L(g) = \frac{\partial}{\partial t}g = -\frac{\partial}{\partial x}g - \varepsilon g^p g_x + \mu\frac{\partial^2}{\partial x^2}g_t,\quad \text{where}$$

$$p = 1,2,3\,.$$

Without initial and boundary conditions, converges towards a particular solution in each case; ($p = 1$, $p = 2$ and $p = 3$).

Proof: Now, we try to prove the conditions (a) and (b) for the convergence.

$\forall g,v \in H\ \exists k > 0$, we can write:

$$\begin{aligned}
L(g) - L(v) &= (-\frac{\partial}{\partial x}g - \varepsilon g^p\frac{\partial}{\partial x}g + \mu\frac{\partial^2}{\partial x^2}g_t) - (-\frac{\partial}{\partial x}v - \varepsilon v^p\frac{\partial}{\partial x}v + \mu\frac{\partial^2}{\partial x^2}v_t)\\
&= -\frac{\partial}{\partial x}(g - v) - \varepsilon(g^p\frac{\partial}{\partial x}g - v^p\frac{\partial}{\partial x}v) + \mu(\frac{\partial^2}{\partial x^2}g_t - \frac{\partial^2}{\partial x^2}v_t)\\
&= -\frac{\partial}{\partial x}(g - v) - \frac{\varepsilon}{p+1}\frac{\partial}{\partial x}(g^{p+1} - v^{p+1}) + \mu\frac{\partial^2}{\partial x^2}(g_t - v_t), \text{ where } p = 1,2,3\ .
\end{aligned}$$

The scalar product

$$(L(g)-L(v),g-v)=(-\frac{\partial}{\partial x}(g-v),g-v)+\frac{\varepsilon}{p+1}(-\frac{\partial}{\partial x}(g^{p+1}-v^{p+1}),g-v)-$$

$$\mu(-\frac{\partial^2}{\partial x^2}(g_t-v_t),g-v), \quad (17)$$

where the differential operators in H are $\frac{\partial}{\partial x},\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^3}{\partial t\partial x^2}$, therefore we get the real positive constants β_1,β_2 and β_3 such that:

$$(-\frac{\partial}{\partial x}(g-v),g-v)\geq\beta_1\|g-v\|^2,$$

$$\mu(-\frac{\partial^2}{\partial x^2}(g_t-v_t),g-v)\geq\mu\beta_2\|g-v\|^2, \quad (18)$$

and according to Schwartz inequality, we get

$$(\frac{\partial}{\partial x}(g^{p+1}-v^{p+1}),g-v)-\leq\beta_3\|g^{p+1}-v^{p+1}\|\,\|g-v\|,\text{ w}$$

By stratifying the mean value theorem, we get

$$\exists\, g<\eta<v,\|g^{p+1}-v^{p+1}\|=(p+1)\eta^p\|g-v\|,\text{where}$$

Then, we get

$$\frac{\varepsilon}{p+1}(\frac{\partial}{\partial x}(g^{p+1}-v^{p+1}),g-v)\leq\varepsilon\beta_3\eta^p\|g-v\|^2=\varepsilon\beta_3M^p\|g-v\|^2,$$

where $p=1,2,3$ and $\|g\|\leq M,\|v\|\leq M$, so we have

$$\frac{\varepsilon}{p+1}(\frac{\partial}{\partial x}(g^{p+1}-v^{p+1}),g-v)\leq\varepsilon\beta_3M^p\|g-v\|^2$$

$$\Leftrightarrow\frac{\varepsilon}{p+1}(-\frac{\partial}{\partial x}(g^{p+1}-v^{p+1}),g-v)\leq\varepsilon\beta_3M^p\|g-v\|^2. \quad (19)$$

Substituting (18) and (19) into (17) we get

$$(L(g)-L(v),g-v)\geq(\beta_1-\mu\beta_2+\varepsilon\beta_3M^p)\|g-v\|^2=k\|g-v\|^2,$$

where $k=\beta_1-\mu\beta_2+\varepsilon\beta_3M^P$ with $\beta_1+\varepsilon\beta_3M^P>\mu\beta_2$. Thus, the first condition (a) is achieved. Now, for the second condition (b), for every $M>0$, there exists a constant $C(M)>0$ such that for $g,v\in H$ with $\|g\|\leq M,\|v\|\leq M$ we get

$$(L(g)-L(v),g-v)\geq C(M)\|g-v\|\|w\|,\forall w\in H.$$

So, we obtain

$$(L(g)-L(v),w)=(-\frac{\partial}{\partial x}(g-v),w)+\frac{\varepsilon}{p+1}(-\frac{\partial}{\partial x}(g^{p+1}-v^{p+1}),w)-\mu(-\frac{\partial^2}{\partial x^2}(g_t-v_t),w)$$

$$\begin{aligned}&\leq\|g-v\|\|w\|+\varepsilon M^P\|g-v\|\|w\|-\mu\|g-v\|\|w\|\\&=(1+\varepsilon M^P-\mu)\|g-v\|\|w\|\\&=C(M)\|g-v\|\|w\|,\text{ where }p=1,2,3,\end{aligned}$$

where, with $C(M)=(1+\varepsilon M^P-\mu)$ therefore the second condition (b) is $(1+\varepsilon M^P>\mu)$ achieved. Hence, we prove the theorem 1 for the convergence to ours method.

4 Application and Numerical Results

We know that the GRLW equation has an analytical solution in the form (6). In this paper, we take $\mu=1$ and introduced some numerical problems to show the numerical solution of single solitary wave, in addition we can discussed the solution obtaining from the interaction between two and three soliton at different time levels.
In this article, we have applied the ADM to obtaining the solution for the GRLW equation, with the following initial condition.

$$g(x,0)=\sqrt[p]{\frac{(p+2)\upsilon}{2p}\sec h^2(\frac{p}{2}\sqrt{\frac{\upsilon}{\mu(\upsilon+1)}}(x-x_0))}.$$

The L_2- norm and L_∞- norm are used to compare the numerical results with the analytical values and the quantities I_1,I_2 and I_3 can be computing to discuss the conservation for the schemes.
Now, we can show the numerical results for MRLWE (i.e. at $p=2$) and show the numerical results for GRLWE (i.e. at $p=3$).

4.1 Numerical Results for the MRLW Equation

(I) Single Solitary Waves

Consider the MRLW equation in the form (3) with $p=2,\varepsilon=6$

$$g_t+g_x+6g^2g_x-\mu g_{xxt}=0, \quad (20)$$

it has the exact solution in the form (6) with $p=\varepsilon=1$:

$$g(x,t)=\sqrt{\upsilon}\,\sec h(\sqrt{\frac{\upsilon}{\mu(\upsilon+1)}}(x-(\upsilon+1)t-x_0)),$$

(21)

with initial condition in the form (5) with $p = 2, \varepsilon = 6$:

$$g(x,0) = \sqrt{\upsilon}\,\operatorname{sec}h\left(\sqrt{\frac{\upsilon}{\mu(\upsilon+1)}}(x - x_0)\right). \qquad (22)$$

We characterize the zero[th] ingredient $g_0 = g(x,0)$ by terms emerge from the initial conditions, to get the next components g_0, g_1, g_2, g_3 etc. By using (12) and (13), we get g_0, g_1, g_2, g_3 and so on.

Here we take the first 4-terms from the series expand (1). We can write the series of approximate solution is this form

$$\varphi_4 = \sum_{k=0}^{3} g_k(x,t) \ , \qquad (23)$$

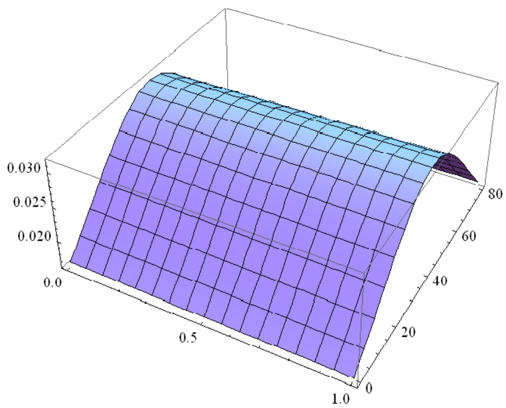
and the accuracy of our results can be compute from the absolute error $\left|g(x,t) - \varphi_4(x,t)\right|$. The three quantities I_1, I_2 and I_3 can be calculate to ensure the conservation laws in the Adomian scheme. The results are given in Tables 1 and 3 show the difference between the analytical solution (21) and our numerical solution (23). Tables 2 and 4 show the values of the norms L_2, L_∞ and the quantities I_1, I_2 and I_3. From these results we show that we obtain a good degree of accuracy and efficiency for the scheme. Also, Figs.1 and 2 illustrate that the results that we obtained it from the series solution is close to the results from the analytical solutions.

Table 1: Absolute error between the numerical and analytical results for MRLW equation at $\upsilon = 0.001,\ x_0 = 40,\ \mu = 1, n = 4$, [0, 80].

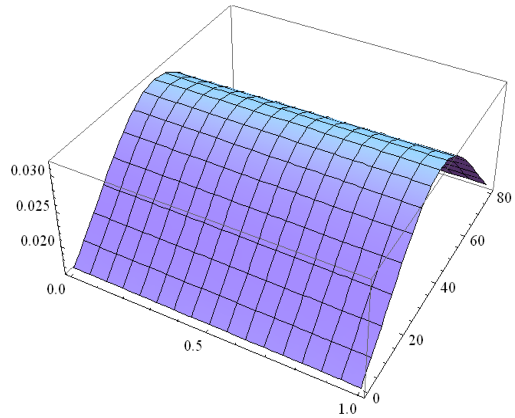
t / x	0	0.2	0.4	0.6	0.8	1
0	0	2.3043E-12	3.12075E-11	5.05216E-11	4.50791E-11	4.2914E-10
10	0	1.4843E-11	4.75276E-11	4.3597E-11	5.06655E-10	1.69592E-9
20	0	3.08454E-11	3.37058E-11	4.68804E-10	1.61374E-9	3.87421E-9
30	0	1.96029E-11	3.21599E-10	1.04201E-9	2.20036E-9	3.70394E-9
40	0	1.05561E-10	2.95508E-10	1.89703E-10	8.45214E-10	3.69552E-9
50	0	6.02677E-12	3.44057E-10	1.56118E-9	4.28965E-9	9.31133E-9
60	0	8.71128-11	4.46151E-10	1.17417E-9	2.31384E-9	3.85609E-9
70	0	3.78862E-11	5.63691E-11	1.23929E-10	7.89819E-10	2.3357E-9
80	0	9.57312E-12	1.31258E-10	5.41795-10	1.49074E-9	3.30138E-9

Table 2: the quantities I_1, I_2 and I_3 to MRLW equation at $\upsilon = 0.001,\ x_0 = 40,\ \mu = 1, n = 4$, [0, 80].

t	I_1	I_2	I_3	L_2**-norm**	L_∞**-norm**
0	2.05799	0.0542118	0.0000277044	0	0
0.2	2.05797	0.0542112	0.0000277044	4.66469E-10	1.05561E-10
0.4	2.05792	0.0542095	0.0000277042	2.28785E-9	4.46151E-10
0.6	2.05783	0.0542066	0.0000277041	7.30521E-9	1.56118E-9
0.8	2.05771	0.0542025	0.0000277038	1.84617E-8	4.28965E-9
1.0	2.05755	0.0541973	0.0000277034	3.98654E-8	9.31133E-9



(a)



(b)

Fig.1: (a) Adomian solution for MRLWE (b) The analytical solution to MRLWE at $\upsilon = 0.001,\ t \in [0,1]$

Table 3: Absolute error between the numerical and analytical results for MRLW equation at $\upsilon = 0.002,\ x_0 = 40,\ \mu = 1, n = 4$, [0, 80].

T / x	0	0.2	0.4	0.6	0.8	1
0	0	1.61874E-11	2.38488E-11	2.42083E-10	6.86923E-10	1.33621E-9
10	0	2.74642E-11	1.93556E-10	6.96867E-10	1.18996E-9	1.06102E-9
20	0	1.67146E-10	8.10387E-10	9.73227E-10	9.12008E-10	7.02369E-9
30	0	4.72659E-10	6.5168E-10	5.53556E-9	1.61914E-8	3.4513E-8
40	0	1.25822E-9	4.31461E-9	7.01495E-9	5.77142E-9	4.43428E-9
50	0	2.74696E-10	3.85516E-9	1.35464E-8	3.24329E-8	6.39128E-8
60	0	6.54108E-10	1.86354E-9	2.75286E-9	1.83873E-9	2.9698E-9
70	0	9.11954E-11	1.00889E-9	3.80703E-9	9.89782E-9	2.10507E-8
80	0	1.26203E-10	6.73359E-10	2.00148E-9	4.5629E-9	8.88776E-9

Table 4: the quantities I_1, I_2 and I_3 to MRLW equation at $\upsilon = 0.002,\ x_0 = 40,\ \mu = 1, n = 4$, [0, 80].

t	I_1	I_2	I_3	L_2 **-norm**	L_∞ **-norm**
0	2.49489	0.0688217	0.0000681802	0	0
0.2	2.49487	0.0688221	0.000068181	4.7443E-9	1.25822E-9
0.4	2.49479	0.0688226	0.0000681834	1.89162E-8	4.31461E-9
0.6	2.49467	0.068824	0.0000681875	5.25705E-8	1.35464E-8
0.8	2.49451	0.0688241	0.0000681933	1.20589E-7	3.24329E-8
1.0	2.49428	0.068826	0.0000682006	2.43437E-7	6.39128E-8

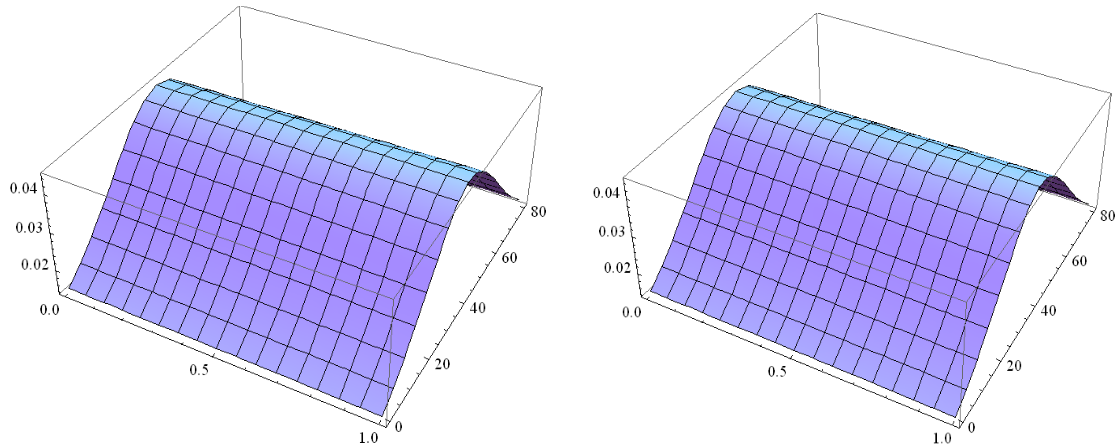


Fig.2: (a) Adomian solution for MRLWE (b) The analytical solution to MRLWE at $\upsilon = 0.002,\ t \in [0,1]$.

(II) The interaction between two solitons

The interaction between two solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider the MRLW equation with initial conditions given by the linear sum of two well separated solitary waves of various amplitudes

$$g(x,0) = \sqrt{\upsilon_i}\ \sec h(\sqrt{\frac{\upsilon_i}{\mu(\upsilon_i+1)}}(x-(\upsilon_i+1)t-x_i)) \tag{24}$$

where, $i = 1,2,\ \ x_i$ and υ_i are any choice constants. By the same way as single solitons, we take the approximation solution series as form (1) and we characterize the zero^th ingredient g_0 in the next form:

$$g_0 = g(x,0), \tag{25}$$

and the next components g_0, g_1, g_2, g_3 etc. are get in sequence using (12). Now, we applied the ADM for discuss the interaction between two solitary waves using the first 4-terms from the series (1), the results can be calculated using the from

$$\varphi_4 = \sum^{3} g_k(x,t) \tag{26}$$

To compute our results in this paper, we choose $\upsilon_1 = 0.02, \upsilon_2 = 0.01, x_1 = 20, x_2 = 60, n = 4$ at interval [0, 80], the three quantities I_1, I_2 and I_3 in Table 5. Fig.3 shows graph of the interaction of solitary waves at: (a) $t \in [0,1]$ and (b) $t \in [1,2]$.

Table 5: Invariants for interaction two solitary waves (MRLW) equation with $\upsilon_1 = 0.02, \upsilon_2 = 0.01, x_1 = 20, x_2 = 60, n = 4$.

t	I_1	I_2	I_3
0	5.95073	0.527446	0.00369009
0.2	5.94881	0.527364	0.00369026
0.4	5.94669	0.527275	0.00369054
0.6	5.94438	0.527178	0.00369091
0.8	5.94188	0.527072	0.00369134
1.0	5.93919	0.526955	0.00369178

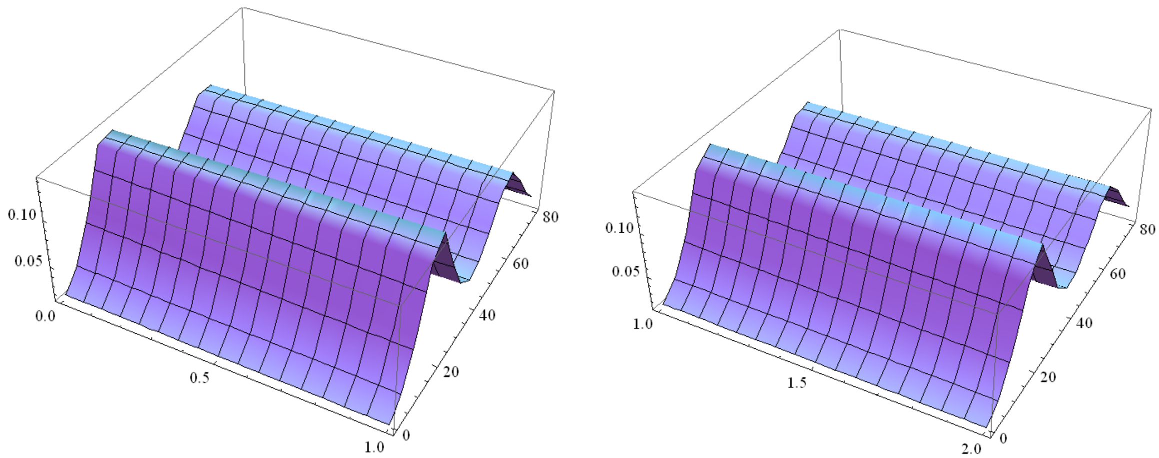


Fig.3: the graph of the interaction of two solitons to MRLW equation at times:

(a) $t \in [0,1]$ and (b) $t \in [1,2]$

(III) The interaction between three solitons

The interaction between three solitary waves for MRLW with different amplitudes in the same direction is discussed. We take the MRLWE with initial conditions given by the linear sum of three well separated solitary waves of different amplitudes

$$g(x,0) = \sqrt{\upsilon_i}\ \sec h\left(\sqrt{\frac{\upsilon_i}{\mu(\upsilon_i+1)}}(x-(\upsilon_i+1)t-x_i)\right) \tag{27}$$

where, $i = 1,2,3,$ x_i and υ_i are any choice constants. By the same way as single solitons and interaction between two solitons, we take the approximation solution series as form (1) and we characterize the zeroth ingredient g_0 in the next form: $g_0 = g(x,0)$, in sequence using (12). Now, we applied the ADM for discuss the interaction between two solitary waves using the first 4-terms from the series (1), the results can be calculated using the from

$$\varphi_4 = \sum_{k=0}^{3} g_k(x,t) \tag{28}$$

To make our computational, choose

$$\upsilon_1 = 0.03, \upsilon_2 = 0.02, \upsilon_3 = 0.01, x_1 = 10, x_2 = 40, x_3 = 75,$$

$n = 4$ in interval [0, 80], the three quantities I_1, I_2 and I_3 in Table 6. Fig.4 shows the graph of this interaction between three solitary waves at: (a) $t \in [0,1]$ and (b) $t \in [1,2]$.

Table 6: The three quantities I_1, I_2 and I_3 to interaction between three solitons at $\upsilon_1 = 0.03, \upsilon_2 = 0.02, \upsilon_3 = 0.01, x_1 = 10, x_2 = 40, x_3 = 75, n = 4$.

t	I_1	I_2	I_3
0	8.11469	0.909253	0.0106676
0.2	8.10847	0.908349	0.0106445
0.4	8.10169	0.907376	0.0106216
0.6	8.09436	0.906334	0.0105988
0.8	8.0865	0.90522	0.0105758
1.0	8.07813	0.904032	0.0105523

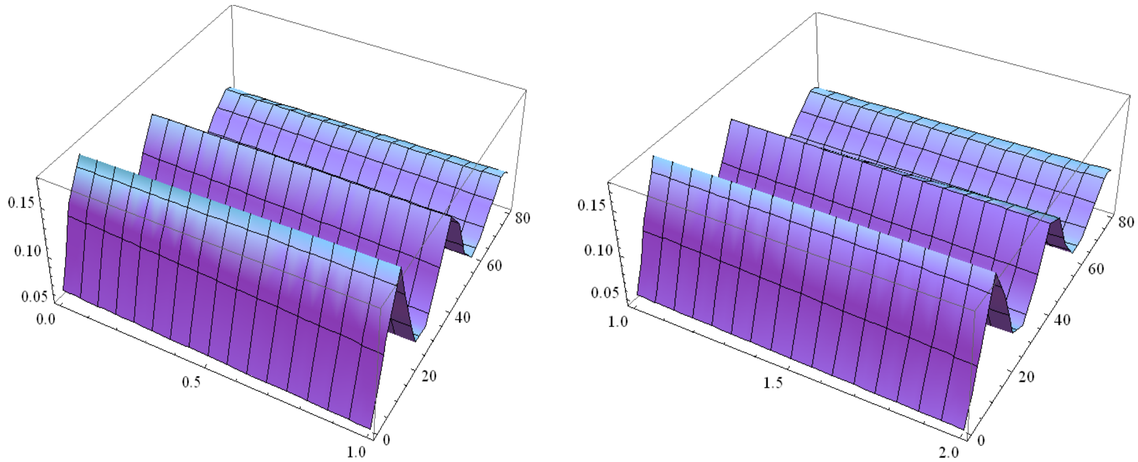


Fig. 4: The graph of interaction between three solitons to MRLW equation at:

(a) $t \in [0,1]$ and (b) $t \in [1,2]$

4.2 Numerical Results for the GRLW Equation

(I) The Single Soliton

We take the GRLW equation at the form (3) with

$$p = 3, \varepsilon = 12$$

$$g_t + g_x + 12g^3 g_x - \mu g_{xxt} = 0. \qquad (29)$$

It has the exact solution in the form (6) with $p = 3, \varepsilon = 12$:

$$g(x,t) = \sqrt[3]{\frac{5\upsilon}{6}\sec h^2\left(\frac{3}{2}\sqrt{\frac{\upsilon}{\mu(\upsilon+1)}}(x-(\upsilon+1)t-x_0)\right)} \qquad (30)$$

with initial condition in the form Eq. (5.5) with $p = 3, \varepsilon = 12$:

$$g(x,0) = \sqrt[3]{\frac{5\upsilon}{6}\sec h^2\left(\frac{3}{2}\sqrt{\frac{\upsilon}{\mu(\upsilon+1)}}(x-x_0)\right)}. \qquad (31)$$

We characterize the zeroth ingredient $g_0 = g(x,0)$ by terms emerge from the initial conditions, to get the next components g_0, g_1, g_2, g_3 etc. By using (12) and (13), we get g_0, g_1, g_2, g_3 and so on.

Here we take the first 4-terms from the series expand (1). We can write the series of approximate solution is this form

$$\varphi_4 = \sum_{k=0}^{3} g_k(x,t). \qquad (32)$$

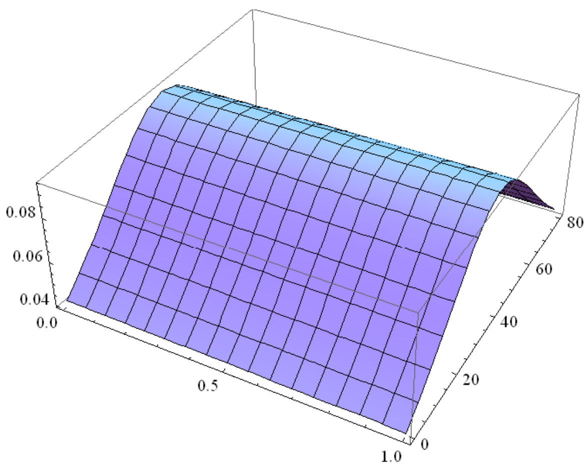
We can find the accuracy of our results from compute the absolute error $|g(x,t) - \varphi_4(x,t)|$.

Table 7: Absolute error between the numerical and analytical results for GRLW equation with $\upsilon = 0.001,\ x_0 = 40,\ \mu = 1, n = 4$, [0, 80].

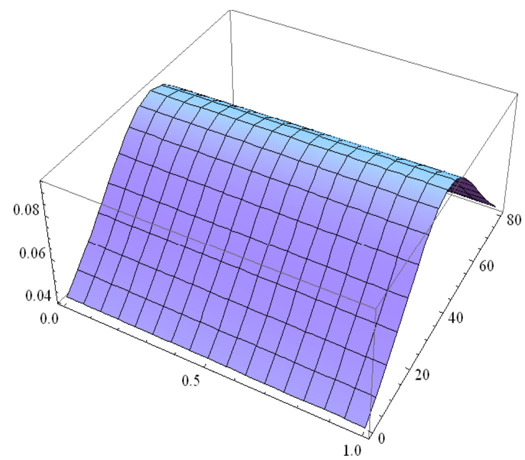
t / x	0	0.2	0.4	0.6	0.8	1
0	0	1.74601E-11	3.9998E-11	2.78292E-10	6.61587E-10	1.01643E-9
10	0	2.83276E-11	2.40501E-11	7.5693E-10	1.00162E-9	5.78338E-12
20	0	2.08776E-10	9.8352E-10	9.93087E-10	1.90268E-9	1.06532E-8
30	0	6.50497E-10	8.48984E-10	7.40792E-9	2.16823E-8	4.61179E-8
40	0	1.75708E-9	6.01018E-9	9.70561E-9	7.75715E-9	6.94807E-9
50	0	4.06067E-10	5.41711E-9	1.89679E-8	4.54179E-8	8.96221E-8
60	0	8.80107E-10	2.56682E-9	3.97014E-9	3.19243E-9	2.44205E-9
70	0	1.04026E-10	1.2595E-9	4.87765E-9	1.28659E-8	2.76808E-8
80	0	1.59975E-10	8.99464E-10	2.77557E-9	6.51405E-9	1.29715E-8

Table 8: Invariants for GRLW equation with $\upsilon = 0.001,\ x_0 = 40,\ \mu = 1, n = 4$, [0, 80].

t	I_1	I_2	I_3	L_2-norm	L_∞-norm
0	5.73098	0.429444	0.00258283	0	0
0.2	5.73093	0.429436	0.00258282	6.54462E-9	1.75708E-9
0.4	5.73078	0.429424	0.00258278	2.6049E-8	6.01018E-9
0.6	5.73053	0.429404	0.00258273	7.24859E-8	1.89679E-8
0.8	5.73018	0.429375	0.00258266	1.66637E-7	4.54179E-8
1.0	5.72973	0.429338	0.00258256	3.37157E-7	8.96221E-8



(a)



(b)

Fig.5: (a) Adomian solution for GRLW and (b) The analytical solution for GRLW at $\upsilon = 0.001,\ t \in [0,1]$

Table 9: Absolute error between the numerical and analytical results for GRLW equation with $\upsilon = 0.002,\ x_0 = 40,\ \mu = 1, n = 4$, [0, 80].

T / x	0	0.2	0.4	0.6	0.8	1
0	0	1.89252E-11	5.40381E-11	2.3883E-10	1.25684E-9	3.4534E-9
10	0	1.62957E-10	2.22616E-10	8.65315E-10	3.66649E-9	8.29024E-9
20	0	9.06732E-10	1.51937E-9	8.01222E-9	1.65794E-8	2.25669E-8
30	0	7.55065E-9	1.84556E-8	1.47812E-8	2.37873E-8	1.20305E-7
40	0	1.9649E-8	7.20024E-8	1.43247E-7	2.11209E-7	2.35985E-7
50	0	3.16153E-8	8.32617E-8	7.25355E-8	3.69832E-7	5.15458E-7
60	0	3.48704E-8	8.71465E-8	5.40681E-8	4.28513E-7	1.0142E-6
70	0	5.42828E-8	1.99611E-7	2.56645E-7	6.25324E-7	6.90094E-7
80	0	2.36188E-8	8.69421E-8	1.71118E-8	5.35342E-7	4.932E-7

Table 10: The three quantities I_1, I_2 and I_3 for GRLWE at $\upsilon = 0.002,\ x_0 = 40,\ \mu = 1, n = 4$, [0, 80].

t	I_1	I_2	I_3	L_2 **-norm**	L_∞**-norm**
0	6.06369	0.521896	0.00452927	0	0
0.2	6.06364	0.521893	0.00452927	2.25474E-7	5.42828E-8
0.4	6.06348	0.521883	0.00452928	7.64984E-7	1.99611E-7
0.6	6.06321	0.521867	0.00452928	1.40022E-6	2.56645E-7
0.8	6.06383	0.521844	0.00452929	3.25026E-6	6.25324E-7
1.0	6.06335	0.521815	0.00452931	5.11942E-6	1.0142E-6

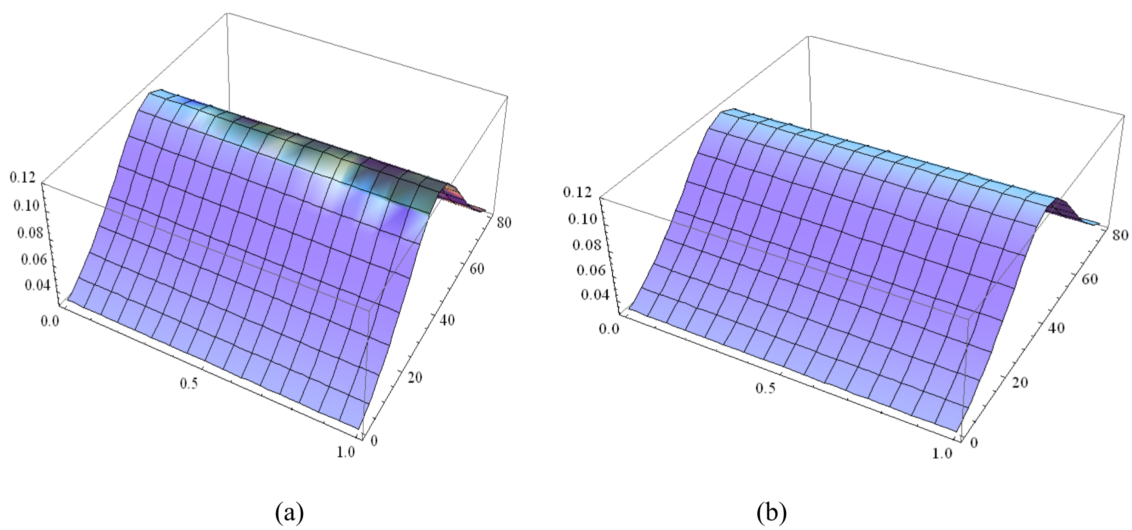


Fig.6: (a) Adomian solution for GRLW. (b)The analytical solution to GRLW $\upsilon = 0.002,\ t \in [0,1]$

In Tables 7 and 9 show the difference between the analytical solution (30) and our numerical solution (32). Tables 8 and 10 show the values of the norms L_2, L_∞ and the quantities I_1, I_2 and I_3 . The results that we obtained it show a good degree of fineness of the suggested method. Also, Figs.5 and 6 illustrate that the results that we obtained it from the series solution is close to the results from the analytical solutions.

(II) The interaction between two solitons

The interaction between two solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider the GRLW equation with initial conditions given by the linear sum of two well separated solitary waves of various amplitudes

$$g(x,0) = \sqrt[3]{\frac{5\upsilon_i}{6}\sec h^2\left(\sqrt{\frac{\upsilon_i}{\mu(\upsilon_i+1)}}(x-(\upsilon_i+1)t-x_i)\right)}, \quad (33)$$

Where, $i = 1,2,\ \ x_i$ and υ_i are any choice constants. By the same way as single solitons, we take the approximation solution series as form (1) and we characterize the zeroth ingredient g_0 in the next form:

$$g_0 = g(x,0),$$

and the next components g_0, g_1, g_2, g_3 etc. are get in sequence using (12). Now, we applied the ADM for discuss the interaction between two solitary waves using the first 4-terms from the series (1), the results can be calculated using the from

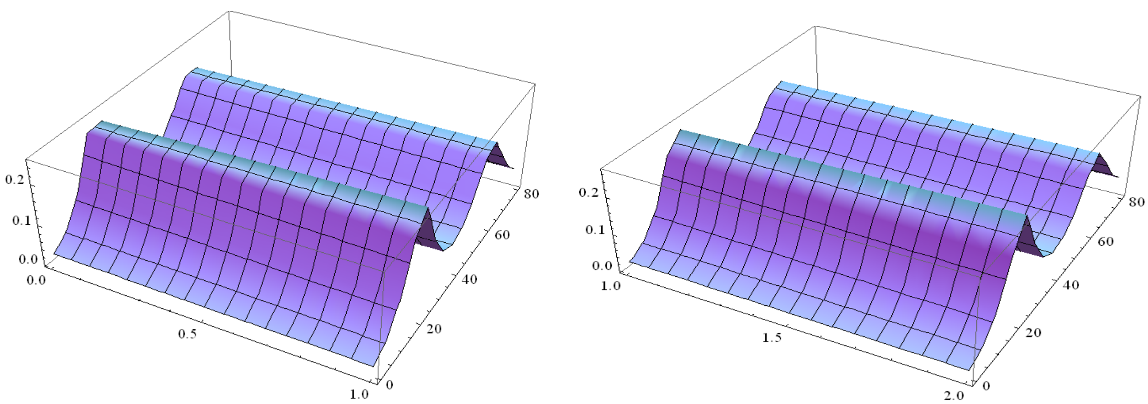
$$\varphi_4 = \sum_{k=0}^{3} g_k(x,t) \quad . \qquad (34)$$

To compute our results in this paper, we choose $\upsilon_1 = 0.02, \upsilon_2 = 0.01, x_1 = 20, x_2 = 60, n = 4$ at interval [0, 80], the three quantities I_1, I_2 and I_3 in Table 11. Fig.7 shows graph of the interaction of solitary waves at: (a) $t \in [0,1]$ and (b) $t \in [1,2]$.

Table 11: Invariants for interaction two solitary waves (GRLW) equation with

$\upsilon_1 = 0.02, \upsilon_2 = 0.01, x_1 = 20, x_2 = 60, n = 4$.

t	I_1	I_2	I_3
0	10.2369	1.62386	0.0499454
0.2	10.2333	1.62408	0.0500072
0.4	10.2296	1.62521	0.0501859
0.6	10.226	1.62723	0.0504784
0.8	10.2223	1.63014	0.0508803
1.0	10.2187	1.63391	0.0513855



(a) (b)

Fig.7: interaction of two solitary waves of GRLW equation at times:

(a) $t \in [0,1]$ and (b) $t \in [1,2]$.

(III) The interaction between three solitons

The interaction between three solitary waves for GRLW with different amplitudes in the same direction is discussed. We take the GRLWE with initial conditions given by the linear sum of three well separated solitary waves of different amplitudes

$$g(x,0) = \sqrt{\upsilon_i}\ \operatorname{sec}h\left(\sqrt{\frac{\upsilon_i}{\mu(\upsilon_i+1)}}(x-(\upsilon_i+1)t-x_i)\right) \tag{35}$$

where, $i = 1,2,3,$ x_i and υ_i are any choice constants. By the same way as single solitons and interaction between two solitons, we take the approximation solution series as form (1) and we characterize the zeroth ingredient g_0 in the next form:

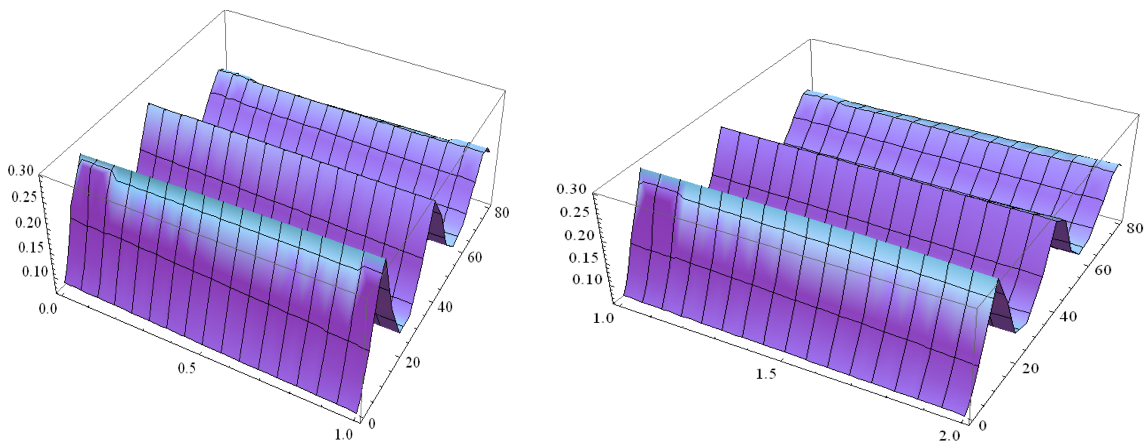
And the next components g_0, g_1, g_2, g_3 etc. are get in Sequence using (12). Now, we applied the ADM for discuss the interaction between two solitary waves using the first 4-terms from the series (1), the results can be calculated using the from

$$\varphi_4 = \sum_{k=0}^{3} g_k(x,t) \quad . \tag{36}$$

To make our computational, choose $\upsilon_1 = 0.03, \upsilon_2 = 0.02, \upsilon_3 = 0.01, x_1 = 10, x_2 = 40, x_3 = 75,$ $n = 4$ in interval [0, 80], the three quantities I_1, I_2 and I_3 in Table 12. Fig.8 shows the graph of this interaction between three solitary waves at: (a) $t \in [0,1]$ and (b) $t \in [1,2]$.

Table 12: Invariants for interaction between three solitons (GRLW) equation at $\upsilon_1 = 0.03, \upsilon_2 = 0.02, \upsilon_3 = 0.01, x_1 = 10, x_2 = 40, x_3 = 75, n = 4$.

t	I_1	I_2	I_3
0	13.3451	2.52088	0.102143
0.2	13.327	2.51644	0.101982
0.4	13.308	2.51182	0.101822
0.6	13.2879	2.50699	0.101656
0.8	13.2669	2.50194	0.101472
1.0	13.2651	2.49662	0.101252



(a) (b)

Fig.8: interaction of three solitary waves of GRLW equation at times:
(a) $0 \le t \le 1$ (b) $1 \le t \le 2$

Now we make comparison between finite difference method (FDM), a collocation method using cubic B-spline (CUBIC), the Non-polynomial spline method (NON) Adomian decomposition method (ADM). Adomian obtained by other method but this happens only through the lowest levels. We show that in the next tables.

Table 13: L_2-norm and L_∞-norm to the single soliton with $p = 3, \upsilon = 0.001, h = 0.1, k = 0.1$ and $x_0 = 40, t = 1, 0 \le x \le 80$.

Method	L_2-norm	L_∞-norm
FDM	3.37086E-5	2.58831E-5
CUBIC	4.06386E-6	3.85976E-6
NON	3.44797E-5	3.68169E-5
ADM	3.37157E-7	8.96221E-8

Table 14: Absolute errors to one solitary wave at $p=3, \upsilon=0.001, k=0.1$ and $x_0=40,\ 0\le x\le 80\ \ h=0.1,$

x	$t=0.2$		$t=0.8$	
	Absolute error ADM	Absolute error FDM	Absolute error ADM	Absolute error FDM
10	2.8276E-11	2.87239E-10	1.00162E-9	4.66695E-9
20	2.08776E-10	1.56472E-13	1.90268E-9	9.83699E-12
30	6.50497E-10	5.32808E-11	2.16823E-8	2.03929E-10
60	8.80107E-10	6.26915E-13	3.19243E-9	1.77853E-11
70	1.04026E-10	2.89972E-10	1.28659E-8	9.73824E-10

Table 15: Absolute errors to single soliton $p=3, \upsilon=0.001, k=0.1$ and. $x_0=40,\ 0\le x\le 80\ \ h=0.1,$

x	$t=0.2$		$t=0.8$	
	Absolute error ADM	Absolute error CUBIC	Absolute error ADM	Absolute error CUBIC
10	2.8276E-11	2.36659E-9	1.00162E-9	8.66633E-9
20	2.08776E-10	4.32149E-9	1.90268E-9	1.70174E-8
30	6.50497E-10	2.70892E-9	2.16823E-8	1.18256E-8
60	8.80107E-10	6.38803E-9	3.19243E-9	2.61289E-8
70	1.04026E-10	3.12317E-9	1.28659E-8	1.31417E-8

Table 16: Absolute errors to one solitary wave $p=3, \upsilon=0.001, k=0.1$ and $h=0.1,\ x_0=40,\ 0\le x\le 80.$

x	t=0.2		t=0.8	
	Absolute error ADM	Absolute error NON	Absolute error ADM	Absolute error NON
10	2.8276E-11	7.21727E-10	1.00162E-9	6.56367E-9
20	2.08776E-10	9.81521E-10	1.90268E-9	3.72341E-9
30	6.50497E-10	1.45786E-9	2.16823E-8	5.88889E-9
60	8.80107E-10	1.01528E-9	3.19243E-9	4.26216E-9
70	1.04026E-10	4.39855E-10	1.28659E-8	1.18029E-9

5 Conclusions

In this article, we used Adomians decomposition method to discuss solitary waves. We chick our scheme through study a single solitary wave that the analytic solution of it is known, and the studding extend to discuss the interaction between solitons (two and three solitons) where in this case the analytic solution of the interactions unknown but we can tested the accuracy from compute the three quantities I_1, I_2 and I_3. The convergence of the method was presented. The comparison between finite difference method (FDM), a collocation method using cubic B-spline (CUBIC), the Non-polynomial spline method (NON) and Adomian decomposition method (ADM) was introduced. From our results we show that the Adomian decomposition method (ADM) has very accurate approximate solutions than the approximate solutions obtained by other method but this happens only through the lowest levels.

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