Journal of Statistics Applications & Probability Letters
*An International Journal*

http://dx.doi.org/10.18576/jsapl/070304

# Inference for Stress-Strength Models Based on the Bivariate General Farlie-Gumbel-Morgenstern Distributions

*Dina Ahmed*[1,*], *Sohair Khames*[2] *and Nahed Mokhli*[3]

[1]Department of Basic Science, Faculty of Computer and Information Sciences, Ain Shams University, Cairo, Egypt
[2]Department of Mathematics, University College for Women, Ain Shams University, Cairo, Egypt
[3]Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt.

Received: 1 Sep. 2019, Revised: 21 Feb. 2020, Accepted: 18 Apr. 2020
Published online: 1 Sep. 2020

**Abstract:** In this paper two forms of bivariate distributions are proposed, namely, the Bivariate General Exponential Farlie-Gumbel-Morgenstern (BGE-FGM) form and the Bivariate General Inverse Exponential Farlie-Gumbel-Morgenstern (BGI-FGM) form. Explicit general expressions for the stress-strength reliability $R = P(X_2 < X_1)$ are obtained, when both the random strength $X_1$ and the random stress $X_2$ are dependent with either BGE-FGM form or BGI-FG form. Also a characterization of the marginal of the bivariate forms is presented associated with $R$. Three estimators as well as a Bayesian estimator of $R$ are derived. The bivariate Weibull and the bivariate Burr type III FGM distributions are studied as special cases of the bivariate BGE-FGM and the BGI-FGM forms, respectively. A simulation study is carried out to detect the performance of the estimators obtained. Also a real life data example is presented as a practical example of the proposed models.

**Keywords:** Farlie- Gumbel- Morgenstern bivariate distribution; general exponential form; general inverse exponential form; Stress—Strength reliability; maximum likelihood estimator; MCMC methods.

## 1 Introduction

In the reliability literature, the stress-strength term mainly refers to a component which has a random strength $X_1$ and is exposed to a random stress $X_2$. The component works only whenever $X_2 < X_1$. Thus, $R = P(X_2 < X_1)$ is a measure of component reliability, which is commonly referred to as the "stress-strength reliability parameter". However, $X_1$ and $X_2$ do not necessarily always represent the strength of a component and the stress imposed on it, they may represent other variables of interest, for example the effectiveness of two different treatments on a certain disease. In this case R measures which treatment is more effective. The stress-strength models have been widely used in physics and engineering, and are spread across different disciplines, such as quality control, genetics, psychology and economics.

Many authors have estimated $R$ when $X_1$ and $X_2$ are independent variables belonging to the same univariate family of distributions. Kotz et al. [1] presented a review of almost all methods and results on the stress-strength model, up to year 2003. Mokhlis [2] discussed the stress-strength reliability, $R$, when $X_1$ and $X_2$ are independent, not identically, Burr type III distributed. Rezaei et al. [3] considered the estimation of $R$ when $X_1$ and $X_2$ are independent with the generalized Pareto distribution with different parameters. Wong [4] studied the interval estimation of $R$ when $X_1$ and $X_2$ have independent generalized Pareto distributions with a common scale parameter. Akgül et al. [5] considered the estimation of R, based on ranked set sampling data in the case of the Lindley distribution. Mokhlis et al. [6] obtained a general form for $R$, when the distributions of the stress and the strength are non-identical and independent with either the general exponential form

$$F(x;\alpha,\beta) = 1 - e^{-\alpha g_1(x,\beta)}; \quad a < x < b\ ,\ \alpha > 0, \beta > 0, \tag{1}$$

where $g_1(x,\beta)$ is any continuous, monotone increasing and differential function of $x$ and $\beta$, not depending on $\alpha$, such that $\lim_{x\to a^+} g_1(x,\beta) = 0$, and $\lim_{x\to b^-} g_1(x,\beta) = \infty$,

* Corresponding author e-mail: DinaAhmed@cis.asu.edu.eg

or the general inverse exponential form

$$F(x;\alpha,\beta) = e^{-\alpha g_2(x,\beta)}; \quad a < x < b\ ,\ \alpha > 0, \beta > 0, \tag{2}$$

where $g_2(x,\beta)$ is any continuous, monotone decreasing and differentiable function of $x$ and $\beta$, not depending on $\alpha$, such that $\lim_{x\to a^+} g_2(x,\beta) = \infty$, and $\lim_{x\to b^-} g_2(x,\beta) = 0$. It is to be noticed that Gurvich et al. [7] considered the form in (1) as a class of extended Weibull distribution.

Note that the definitions (1) and (2) hold for $a = 0,\ b = \infty$, and also for any real value of the parameter $\beta$ and $\beta$ could be a vector parameter. The importance of models (1) and (2) lies in the fact that many statistical distributions are obtained by selecting the appropriate form of $g_i(x,\beta),\ i = 1,2$ and the appropriate parameters. Some examples of statistical distributions that follow model (1) are: Burr XII, with $g_1(x,\beta) = \ln\left(1+x^\beta\right)$, Pareto with $g_1(x,\beta) = \ln\left(\frac{x}{\beta}\right)$, and Weibull with $g_1(x,\beta) = x^\beta$, see Santos et al. [8]. Some examples of the statistical distributions that follow model (2) are: Burr III, with $g_2(x,\beta) = \ln\left(1+x^{-\beta}\right)$, inverse Weibull with $g_2(x,\beta) = x^{-\beta}$, and generalized exponential with $g_2(x,\beta) = -\ln\left(1-e^{-\beta x}\right)$, for more details see [6].

Nadrajah [9] considered the estimation of when $X_1$ and $X_2$ have a bivariate gamma distribution, Mokhlis [10] considered the estimation of $R$ for a system consisting of two parallel components under four different stress models, and the strengths of the components have a Friday and Patial bivariate exponential distribution. Nadarajah and Kotz [11] studied the estimation of $R$ when $X_1$ and $X_2$ have a bivariate exponential distribution. Domma and Giordano [12] considered the problem of evaluating $R$ when $X_1$ and $X_2$ are dependent considering the association between $X_1$ and $X_2$ via a copula-based approach. Also, Abdel-Hamid [13] considered the problem of estimation of $R$ when the vector $(X_1,\ X_2)$ follows some general bivariate distribution.

In the present paper we propose two forms of the bivariate Farlie-Gumbel-Morgenstern (FGM) distributions, which we call: the Bivariate General Exponential FGM (BGE- FGM) form, and the Bivariate General inverse Exponential FGM (BGI-FGM) form. We obtain explicit general expressions for R, when the vector $(X_1,\ X_2)$ has either BGE- FGM form or BGI-FGM form. A characterization of the marginal distribution associated with $R$, is presented for both forms. Four different estimators are obtained for $R$.

In Section 2, the proposed models are introduced .In Section 3, explicit expressions of $R$ are obtained for the proposed BGE-FGM or BGI- FGM forms. Estimation of $R$ is introduced in Section 4. As an illustration of the results of Sections 3 and 4, the results of a simulation study using the bivariate Weibull FGM and the bivariate burr III FGM distributions are given in Section 5. A real life data example is also introduced in Section 5. Finally, the paper is concluded in Section 6.

## 2 The Proposed models

Let $X_1$ and $X_2$ be continuous, non-negative and dependent random variables, with continuous marginal distribution functions $F_{X_1}(.)$ and $F_{X_2}(.)$ respectively. Using the one parameter Farlie-Gumbel-Morgenstern copula function, with association parameter $\theta,\ -1 \le \theta \le 1$, the joint distribution function and the joint density function of $X_1$ and $X_2$ are given respectively by

$$F_{X_1,X_2}(x_1,x_2) = F_{X_1}(x_1)F_{X_2}(x_2)\{1+\theta(1-F_{X_1}(x_1))(1-F_{X_2}(x_2))\};\ x_1,x_2 > 0 \tag{3}$$

and

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2)\{1+\theta(1-2F_{X_1}(x_1))(1-2F_{X_2}(x_2))\};x_1,x_2 > 0 \tag{4}$$

where $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$ are the marginal density functions of $X_1$ and $X_2$ respectively.
Since the Kendall's tau coefficient is

$$\tau = \frac{2\theta}{9} \tag{5}$$

and $-1 \le \theta \le 1$, so $-0.222 \le \tau \le 0.222$. This means that FGM copula describes a weak dependence between $X_1$ and $X_2$. In practical life weak dependence may exist between variables such as in hydrology problems, the rainfall intensity $X_1$ and the rainfall depth $X_2$ are weakly associated, see Long and Krzysztofowicz [14]. Also in the medical field the recurrence times of some diseases have weak dependence.
Suppose that the vector $(X_1,\ X_2)$ has a bivariate FGM distribution in (3), with $F_{X_1}(.)$ and $F_{X_2}(.)$ having either the general exponential form given in (1) or the general inverse exponential form given in (2) with parameters $(\alpha_1,\beta)$ and $(\alpha_2,\beta)$ respectively. Then $(X_1,\ X_2)$ is said to have BGE-FGM form with distribution function

$$F_{X_1,X_2}(x_1,x_2) = \left(1-e^{-\alpha_1 g_1(x_1,\beta)}\right)\left(1-e^{-\alpha_2 g_1(x_2,\beta)}\right)\left\{1+\theta e^{-\alpha_1 g_1(x_1,\beta)-\alpha_2 g_1(x_2,\beta)}\right\} \tag{6}$$

On the other hand suppose that the marginal distributions have the general inverse exponential form given by (2) with parameters $(\alpha_1, \beta)$ and $(\alpha_2, \beta)$ respectively, then $(X_1,\ X_2)$ is said to have BGI-FGM form with distribution function given by

$$F_{X_1,X_2}(x_1,x_2) = e^{-\alpha_1 g_2(x_1,\beta)-\alpha_2 g_2(x_2,\beta)}\left\{1+\theta\left(1-e^{-\alpha_1 g_2(x_1,\beta)}\right)\left(1-e^{-\alpha_2 g_2(x_2,\beta)}\right)\right\}. \tag{7}$$

The joint density function for either forms is

$$f_{X_1,X_2}(x_1,x_2) = \alpha_1\alpha_2 e^{-\alpha_1 g_r(x_1,\beta)-\alpha_2 g_r(x_2,\beta)} g_r'(x_1,\beta)\, g_r'(x_2,\beta)\left\{1+\theta\left(1-2e^{-\alpha_1 g_r(x_1,\beta)}\right)\left(1-2e^{-\alpha_2 g_r(x_2,\beta)}\right)\right\}; \tag{8}$$

where $g_r'(.,\beta)$ is the first derivative of $g_r(.,\beta)$ with respect to $x$, and $r = 1,2$ for BGE-FGM and BGI-FGM respectively.

## 3 Stress-Strength Reliability

Suppose that $(X_1, X_2)$ has FGM distribution in (3), then the stress-strength reliability parameter is given by

$$R = P(X_2 < X_1) = \int_0^\infty \int_0^x f_{X_1,X_2}(x,y)\,dydx. \tag{9}$$

Using (4), we get

$$R = R_1 + \theta(R_1 + 2I),$$

where

$$I = \int_0^\infty \int_0^x \left(F_{X_1}(x)(2F_{X_2}(y)-1) - F_{X_2}(y)\right) f_{X_1}(x) f_{X_2}(y)\,dydx$$

and

$$R_1 = \int_0^\infty \int_0^x f_{X_1}(x) f_{X_2}(y)\,dydx.$$

Clearly $R_1$ is the stress-strength reliability when $X_1$ and $X_2$ are independent. (i.e., when $\theta = 0$).
Explicit expressions for $R$ are obtained by Theorems 1 and 2, when $(X_1, X_2)$ has bivariate distribution with BGE-FGM and BGI-FGM respectively. Theorems 1 and 2 present characterizations of the marginal distributions of the BGE-FGM and BGI-FGM associated with $R$.

**Theorem 1** Let $(X_1, X_2)$ be a bivariate vector having FGM distribution in (3) with marginal distribution functions $F_{X_1}(.)$, and $F_{X_2}(.)$ respectively. Then the stress-strength reliability, $R$, is given by

$$R = P(X_2 < X_1) = \frac{\alpha_2}{\alpha_1+\alpha_2}\left\{1+\theta\frac{\alpha_1(\alpha_2-\alpha_1)}{(\alpha_1+2\alpha_2)(2\alpha_1+\alpha_2)}\right\}, \tag{10}$$

if and only if the marginal distribution functions are

$$F_{X_i}(x_i) = 1 - e^{-\alpha_i g_1(x_i,\beta)};\ i = 1,2, \tag{11}$$

where $\alpha_i, \beta > 0$ ; $i = 1,2$; $g_1(x_i,\beta)$ is continuous, monotone increasing and differentiable function, such that $\lim_{x_i\to 0} g_1(x_i,\beta) = 0$, and $\lim_{x_i\to\infty} g_1(x_i,\beta) = \infty$, $i = 1,\ 2$.

**Proof:** If $(X_1, X_2)$ have marginal distribution functions given by (11), then $(X_1, X_2)$ will have a BGE-FGM form as in (6) and joint probability density function given by (8). Using (8) in (9) we get (10). Putting $\theta = 0$ in (10), we get $R = R_1 = \frac{\alpha_2}{\alpha_1+\alpha_2}$ the case of independence.
Conversely, suppose that $(X_1, X_2)$ has FGM distribution given by (3) and that (10) holds. Setting $\theta = 0$ in (10) we get the independent case, $R = R_1 = \frac{\alpha_2}{\alpha_1+\alpha_2}$. Hence, according to Theorem (1) in [6], if $R = \frac{\alpha_2}{\alpha_1+\alpha_2}$ then the two independent random variables must have distribution functions with the forms $F_{X_i}(x_i) = 1 - e^{-\alpha_i g_1(x_i,\beta)}$, where $g_1(x_i,\beta)$ is continuous, monotone increasing and differential function, such that $\lim_{x_i\to 0} g_1(x_i,\beta) = 0$, and $\lim_{x_i\to\infty} g_1(x_i,\beta) = \infty$, $i = 1,\ 2$. Hence the proof is complete.

**Theorem 2**
Let $(X_1, X_2)$ be a bivariate vector having FGM distribution in (3) with marginal distribution functions $F_{X_1}(.)$, $F_{X_2}(.)$ respectively, then the stress-strength reliability, $R$, is

$$R = P(X_2 < X_1) = \frac{\alpha_1}{\alpha_1+\alpha_2}\{1+\theta\frac{\alpha_2(\alpha_1-\alpha_2)}{(\alpha_1+2\alpha_2)(2\alpha_1+\alpha_2)}\} \tag{12}$$

if and only if the marginal distribution functions are

$$F_{X_i}(x_i) = e^{-\alpha_i g_2(x_i,\beta)};\ i = 1,2. \tag{13}$$

where $\alpha_i, \beta > 0$ ; $i = 1,2$; $g_2(x_i,\beta)$ is continuous, monotone decreasing and differentiable function, such that $\lim_{x_i\to 0} g_2(x_i,\beta) = \infty$ and $\lim_{x_i\to\infty} g_2(x_i,\beta) = 0$, $i = 1,\ 2$.

**Proof:**

If $(X_1,X_2)$ have marginal distribution functions given by (13), then $(X_1,X_2)$ will have a BGI-FGM form as in (7) and joint probability density function given by (8). Using (8) in (9) we get (12). If putting $\theta = 0$, we get $R = R_1$, the independence case.

Conversely, suppose that $(X_1,X_2)$ has FGM distribution given by (3) and that (12) holds, putting $\theta = 0$ in (12) we get the case of independent and, $R = R_1 = \frac{\alpha_1}{\alpha_1+\alpha_2}$. Hence, according to Theorem (2) in [6], if $R = \frac{\alpha_1}{\alpha_1+\alpha_2}$ then the two independent random variables must have distribution functions with the forms $F_{X_i}(x_i) = e^{-\alpha_i g_2(x_i,\beta)}$, where $g_2(x_i,\beta)$ is continuous, monotone decreasing and differential function, such that $\lim_{x_i\to 0} g_2(x_i,\beta) = \infty$ and $\lim_{x_i\to\infty} g_2(x_i,\beta) = 0$, $i = 1,\ 2$. The proof is complete.

From Theorems (1) and (2) we notice:

1. The parameter $\beta$ does not appear neither in (10) nor in (12). Thus, if the parameters $\alpha_1,\alpha_2$ and $\theta$ are known $R$ could be calculated exactly, even without knowing $\beta$. However if $\alpha_1,\alpha_2$ and $\theta$ are unknown their estimators may depend on $\beta$ and consequently the estimator of $R$.
2. Clearly if the stress and strength variables have identical distributions ($i.e.\ \alpha_1 = \alpha_2$) then $R = \frac{1}{2}$ regardless whether the variables are dependent or not.
3. $R > R_1$ (The reliability in the dependence case is greater than that in the independence case) if, and only if $\theta(\alpha_2-\alpha_1) > 0$, $(\theta(\alpha_2-\alpha_1) < 0)$ for BGE-FGM (BGI-FGM).
4. $R < R_1$ (the reliability in the dependence case is less than that in the independence case) if, and only if $\theta(\alpha_2-\alpha_1) < 0$, $(\theta(\alpha_2-\alpha_1) > 0)$ for BGE-FGM (BGI-FGM).
5. $R = R_1$ if, and only if

$$\theta(\alpha_2-\alpha_1) = 0.$$

## 4 Point estimation of R

According to Theorems (1) and (2), the value of $R$ can be calculated if the values of the parameters $\alpha_1,\alpha_2$ and $\theta$ are known. So if $\alpha_1,\alpha_2$ and $\theta$ are unknown, in order to estimate $R$ we may first estimate the parameters $\alpha_1,\alpha_2$ and $\theta$, then calculate the estimate of $R$. As mentioned before the estimation of $\alpha_1$ *and* $\alpha_2$ may depend on the estimation of $\beta$. The parameter $\theta$, association parameter, is the copula parameter so, it is appropriate to estimate $\theta$ by using a measure of association as Kendall's tau coefficient or the Spearman's rho. Here we shall use the Kendall's tau coefficient. Using (5), an unbiased estimator of $\theta$ is given by $\widehat{\theta} = \frac{9T}{2}$, where $T$ is the unbiased estimator of the Kendall's tau $\tau$ and is called Kendall's sample tau coefficient, see Domma and Giordano [15]. For estimating $\alpha_1,\alpha_2$ and $\beta$ we apply two methods as follows.

### 4.1 Method I

Method I, is applied by getting the maximum likelihood estimators (MLEs) $\widehat{\alpha}_{11},\widehat{\alpha}_{12}$ *and* $\widehat{\beta}_1$ of $\alpha_1,\alpha_2$ *and* $\beta$ using the joint density function (8) after replacing $\theta$ by $\widehat{\theta}$. Then calculating $\hat{R}_1$ by replacing the parameters $\alpha_1,\alpha_2$ *and* $\theta$ in the Equations (10) and (12) by their estimators $\widehat{\alpha}_{11},\widehat{\alpha}_{12}$ *and* $\widehat{\theta}$, respectively.

Let $(X_{1i},X_{2i})$ ; $i = 1,2,..,n$ be a random sample from BGE-FGM or BGI-FGM density given by (8). the log-likelihood function after replacing $\theta$ by $\widehat{\theta}$ is given by

$$\begin{aligned} ln(L) = nln(\alpha_1) + nln(\alpha_2) + \textstyle\sum_{i=1}^{n} \ln g_r'(x_{1i},\beta) + \sum_{i=1}^{n} \ln g_r'(x_{2i},\beta) - \alpha_1 \sum_{i=1}^{n} g_r(x_{1i},\beta) \\ -\alpha_2 \textstyle\sum_{i=1}^{n} g_r(x_{2i},\beta) + \sum_{i=1}^{n} \ln\left\{1+\widehat{\theta}\left(1-2e^{-\alpha_1 g_r(x_{1i},\beta)}\right)\left(1-2e^{-\alpha_2 g_r(x_{2i},\beta)}\right)\right\}, \end{aligned} \tag{14}$$

where $r = 1$ for BGE-FGM form and $r = 2$ for BGI-FGM form. Thus the likelihood equations are given by

$$\frac{n}{\alpha_1} - \sum_{i=1}^{n} g_r(x_{1i},\beta) + 2\widehat{\theta}\sum_{i=1}^{n} \frac{\left(e^{-\alpha_1 g_r(x_{1i},\beta)} g_r(x_{1i},\beta)\right)\left(1-2e^{-\alpha_2 g_r(x_{2i},\beta)}\right)}{1+\widehat{\theta}\left(1-2e^{-\alpha_1 g_r(x_{1i},\beta)}\right)\left(1-2e^{-\alpha_2 g_r(x_{2i},\beta)}\right)} = 0,$$

$$\frac{n}{\alpha_2}-\sum_{i=1}^{n} g_r(x_{2i},\beta)+2\widehat{\theta}\sum_{i=1}^{n}\frac{\left(1-2e^{-\alpha_1 g_r(x_{1i},\beta)}\right)\left(e^{-\alpha_2 g_r(x_{2i},\beta)}g_r(x_{2i}\beta)\right)}{1+\widehat{\theta}\left(1-2e^{-\alpha_1 g_r(x_{1i},\beta)}\right)\left(1-2e^{-\alpha_2 g_r(x_{2i},\beta)}\right)}=0\,,$$

$$\sum_{j=1}^{2}\sum_{i=1}^{n}\frac{\frac{\partial\left(g_r'(x_{ji},\beta)\right)}{\partial\beta}}{g_r'(x_{ji},\beta)}-\sum_{j=1}^{2}\sum_{i=1}^{n}\alpha_j\frac{\partial g_r(x_{ji},\beta)}{\partial\beta}+2\widehat{\theta}\sum_{j=1}^{2}\sum_{i=1}^{n}\frac{\left(1-2e^{-\alpha_j g_r(x_{ji},\beta)}\right)\left(\alpha_{3-j}e^{-\alpha_{3-j}g_r\left(x_{(3-j)i},\beta\right)}\frac{\partial\left(g_r\left(x_{(3-j)i},\beta\right)\right)}{\partial\beta}\right)}{1+\widehat{\theta}\left(1-2e^{-\alpha_1 g_r(x_{1i},\beta)}\right)\left(1-2e^{-\alpha_2 g_r(x_{2i},\beta)}\right)}=0$$

Solving the above equations with respect to $\alpha_1,\alpha_2$ *and* $\beta$, we obtain the estimators $\widehat{\alpha}_{11},\widehat{\alpha}_{12}$ *and* $\widehat{\beta}_1$ respectively. The parameters estimates are obtained numerically using a statistical software, like maple or R.

## *4.2 Method II*

This method depends on the marginal (MLEs) of the parameters, $\alpha_1,\alpha_2$ *and* $\beta$ using marginal distributions in (11) or (13).

Suppose that $(X_{1i},X_{2i})\ ;i=1,2,\ldots n$ is a random sample from either BGE-FGM or BGI-FGM. This means that $X_{ji}\ ;j=1,2,\ i=1,..n$ are independent and identically distributed with marginal distribution functions given by either (11) or (13). Thus the marginal log-likelihood functions of $X_{ji}'s;j=1,2$ are given by,

$$\ln(L_j)\ =n\,ln(\alpha_j)+\sum_{i=1}^{n}\ln\left(g_1'(x_{ji},\beta)\right)\ -\alpha_j\sum_{i=1}^{n}g_1(x_{ji},\beta);j=1,2 \tag{15}$$

and

$$\ln(L_j)\ =n\,ln(\alpha_j)+\sum_{i=1}^{n}\ln\left(-g_2'(x_{ji},\beta)\right)\ -\alpha_j\sum_{i=1}^{n}g_2(x_{ji},\beta);j=1,2, \tag{16}$$

where $L_j$ is the marginal likelihood function of $X_{ji}'s$. Then, the marginal likelihood equations are given by

$$\alpha_j=\frac{n}{\sum_{i=1}^{n}g_r(x_{ji},\beta)}\quad ;j=1,2 \tag{17}$$

$$\sum_{i=1}^{n}\frac{(\partial g_r'(x_{1i},\beta)/\partial\beta)}{g_r'(x_{1i},\beta)}-\frac{n}{\sum_{i=1}^{n}g_r(x_{1i},\beta)}\sum_{i=1}^{n}\frac{\partial g_r(x_{1i},\beta)}{\partial\beta}=0 \tag{18}$$

$$\sum_{i=1}^{n}\frac{(\partial g_r'(x_{2i},\beta)/\partial\beta)}{g_r'(x_{2i},\beta)}-\frac{n}{\sum_{i=1}^{n}g_r(x_{2i},\beta)}\sum_{i=1}^{n}\frac{\partial g_r(x_{2i},\beta)}{\partial\beta}=0, \tag{19}$$

With $r=1$ for BGE-FGM and $r=2$ for BGI-FGM.

For obtaining the estimator of $\beta$ we apply one of the following two methods. The first method is to solve Equations (18) and (19) independently with respect to $\beta$, obtaining $\widehat{\beta_{21}}$ and $\widehat{\beta_{22}}$ respectively and then taking their average, to obtain the estimator $\widehat{\beta}_2$ of $\beta$. Replacing $\beta$ with $\widehat{\beta}_2$ in (17) we obtain the estimators $\widehat{\alpha}_{21}$ *and* $\widehat{\alpha}_{22}$ of $\alpha_1$*and* $\alpha_2$ respectively. In the second method we add Equations (18) and (19), to obtain

$$\sum_{i=1}^{n}\frac{(\partial g_r'(x_{1i},\beta)/\partial\beta)}{g_r'(x_{1i},\beta)}-\frac{n}{\sum_{i=1}^{n}g_r(x_{1i},\beta)}\sum_{i=1}^{n}\frac{\partial g_r(x_{1i},\beta)}{\partial\beta}+\sum_{i=1}^{n}\frac{(\partial g_r'(x_{2i},\beta)/\partial\beta)}{g_r'(x_{2i},\beta)}-\frac{n}{\sum_{i=1}^{n}g_r(x_{2i},\beta)}\sum_{i=1}^{n}\frac{\partial g_r(x_{2i},\beta)}{\partial\beta}=0 \tag{20}$$

Solving (20) with respect to $\beta$ we obtain the estimator $\widehat{\beta}_3$ of $\beta$ and substituting with $\widehat{\beta}_3$ in (17) we obtain $\widehat{\alpha}_{3j},\ j=1,2$. Replacing the parameters in Equations (10) or (12) by their estimators we obtain the estimator of $R$, given by

$$\hat{R}(\boldsymbol{i})=\begin{cases}\frac{\widehat{\alpha}_{i2}}{\widehat{\alpha}_{i1}+\widehat{\alpha}_{i2}}\left(1+\widehat{\theta}\frac{\widehat{\alpha}_{i1}(\widehat{\alpha}_{i2}-\widehat{\alpha}_{i1})}{(\widehat{\alpha}_{i1}+2\widehat{\alpha}_{i2})(2\widehat{\alpha}_{i1}+\widehat{\alpha}_{i2})}\right),\ \text{for BGE}-\text{FGM}\\ \frac{\widehat{\alpha}_{i1}}{\widehat{\alpha}_{i1}+\widehat{\alpha}_{i2}}\left(1+\widehat{\theta}\frac{\widehat{\alpha}_{i2}(\widehat{\alpha}_{i1}-\widehat{\alpha}_{i2})}{(\widehat{\alpha}_{i1}+2\widehat{\alpha}_{i2})(2\widehat{\alpha}_{i1}+\widehat{\alpha}_{i2})}\right),\ \text{for BGI}-\text{FGM}\end{cases}\qquad i=1,2,3. \tag{21}$$

### 4.3 Bayesian estimator of R

For the Bayesian estimator of $R$, we assume that $\alpha_1, \alpha_2$ and $\beta$ are independent random variables having the following gamma priors,

$$\pi_i(\alpha_i) = \frac{{b_i}^{a_i}}{\Gamma a_i} \alpha_i^{a_i-1} e^{-b_i\alpha_i},\ i = 1,\ 2,$$

and

$$\pi_3(\beta) = \frac{{b_3}^{a_3}}{\Gamma a_3} \beta^{a_3-1} e^{-b_3\beta}.$$

The hyper parameters $a_i,\ b_i\ > 0\ ;\ i = 1,\ 2,\ 3$, can be estimated by using the method of moments of the gamma distribution.

Since $-1 \le \theta \le 1$ we assume that the prior distribution of $\theta$ is uniform over $(-1,\ 1)$, i.e.,

$$\pi_4(\theta) = \frac{1}{2}, -1 \le \theta \le 1.$$

We assume that $\theta$ is independent of $\alpha_1,\ \alpha_2$ *and* $\beta$. Therefore the joint posterior distribution of $\alpha_1,\ \alpha_2,\ \beta,\ \theta$ given the data is given by

$$\pi^*(\alpha_1,\ \alpha_2,\ \beta,\ \theta|\,data) \propto L\pi_1(\alpha_1)\,\pi_2(\alpha_2)\,\pi_3(\beta)\,\pi_4(\theta),$$

where the likelihood function $L$ is given by,

$$L = {\alpha_1}^n {\alpha_2}^n e^{-\alpha_1\sum_{i=1}^n g_r(x_{1i},\beta)-\alpha_2\sum_{i=1}^n g_r(x_{2i},\beta)} \prod_{i=1}^n g'_r(x_{1i},\beta)\,g'_r(x_{2i},\beta)(1+\widehat{\theta}\left(1-2e^{-\alpha_1 g_r(x_{1i},\beta)}\right)\left(1-2e^{-\alpha_2 g_r(x_{2i},\beta)}\right)).$$

The Bayesian estimator of R with respect to the squared-error loss function is given by

$$\widetilde{R_B} = \int_{-1}^{1}\int_0^\infty\int_0^\infty\int_0^\infty R\,\pi^*(\alpha_1,\ \alpha_2,\ \beta,\ \theta|\,(X_1,X_2))\ \ d\alpha_1 d\alpha_2 d\beta d\theta$$

The above integration is difficult to obtain in an explicit form. So we can use MCMC methods to obtain $\widetilde{R_B}$. The MCMC methods depends on generating observations from the conditional posterior distributions of the parameters, for more details see Robert et al. [16].

The conditional posterior distributions of $\alpha_1, \alpha_2,\ \beta$ and $\theta$ are respectively

$$\pi^*_{\alpha_1}(\alpha_1|\,\alpha_2,\ \beta,\ \theta, data) \propto {\alpha_1}^{n+a_1-1} e^{-b_1\alpha_1-\alpha_1\sum_{i=1}^n g_r(x_{1i},\beta)} \prod_{i=1}^n \left(1+\theta\left(1-2e^{-\alpha_1 g_r(x_{1i},\beta)}\right)\left(1-2e^{-\alpha_2 g_r(x_{2i},\beta)}\right)\right), \tag{22}$$

$$\pi^*_{\alpha_2}(\alpha_2|\,\alpha_1,\ \beta,\ \theta, data) \propto {\alpha_2}^{n+a_2-1} e^{-b_2\alpha_2-\alpha_2\sum_{i=1}^n g_r(x_{2i},\beta)} \prod_{i=1}^n \left(1+\theta\left(1-2e^{-\alpha_1 g_r(x_{1i},\beta)}\right)\left(1-2e^{-\alpha_2 g_r(x_{2i},\beta)}\right)\right), \tag{23}$$

$$\begin{aligned}\pi^*_\beta(\beta|\,\alpha_1,\alpha_2\ ,\ \theta, data)\ \propto\ & \beta^{a_3-1} e^{-b_3\beta-\alpha_1\sum_{i=1}^n g_r(x_{1i},\beta)-\alpha_2\sum_{i=1}^n g_r(x_{2i},\beta)} \prod_{i=1}^n g'_r(x_{1i},\beta)\,g'_r(x_{2i},\beta) \\ \left(1+\ \ \theta\left(1-2e^{-\alpha_1 g_r(x_{1i},\beta)}\right)\left(1-2e^{-\alpha_2 g_r(x_{2i},\beta)}\right)\right)\end{aligned} \tag{24}$$

and

$$\pi^*_\theta(\ \theta|\,\alpha_1,\alpha_2,\ \beta, data) \propto \prod_{i=1}^n \left(1+\theta\left(1-2e^{-\alpha_1 g_r(x_{1i},\beta)}\right)\left(1-2e^{-\alpha_2 g_r(x_{2i},\beta)}\right)\right). \tag{25}$$

Using the Metropolis-Hasting algorithm which is one of the MCMC methods, we get $\widetilde{R_B}$ by the following steps.

1. Initialize the vector parameter $a = (a_1, a_2, a_3, a_4)$ with any value say $a_0 = (a_{10}, a_{20}, a_{30}, a_{40})$ from the support of the target distributions, (where $a_1 = \alpha_1,\ a_2 = \alpha_2\ , a_3 = \beta$ *and* $a_4 = \theta$).
2. Set $t = 1$.
3. Generate a proposal point $(a_{iprop})\,; i = 1,2,3,4$ from the proposal density $q_i\left(a_{i(t-1)}, a_{it}\right) = p(a_{i(t-1)} \to a_{it})$, i.e., the probability of returning a value $a_{it}$ from a previous value of $a_{i(t-1)}$.

4.Calculate the acceptance probability

$$\rho_{ai} = \min\left(1, \left(\frac{\pi^*_{a_i}\left(a_{iprop}\right), q\left(a_{iprop}, a_{i(t-1)}\right)}{\pi^*_{a_i}\left(a_{i(t-1)}\right), q\left(a_{i(t-1)}, a_{iprop}\right)}\right)\right)$$ , where $\pi^*_{a_i}$ is the conditional posterior distribution of $a_i; i = 1,2,3,4$.

Notice that when the proposal density $q_i\left(a_{iprop}, a_{it}\right)$ is symmetric $\rho_{ai}$ became as following:

$$\rho_{ai} = \min\left(1, \left(\frac{\pi^*_{a_i}\left(a_{iprop}\right)}{\pi^*_{a_i}\left(a_{i(t-1)}\right)}\right)\right).$$

5.Generate $U$ from uniform on $(0,1)$.
6.If $U < \rho_{ai}$, accept the move, i.e. $a_{iprop}$ for $i = 1,2,3,4$ and set $a_{it} = a_{iprop}$. Otherwise, reject it and set $a_{it} = a_{i(t-1)}$.
7.calculate $\widetilde{R_{Bt}}$ by substituting the parameters in the Equations (10) or (12) in each move.
8.Set $t = t+1$.
9.Repeat steps 3-8 N times.
10.Calculate $\widetilde{R_B}$ as: $\widetilde{R_B} = E\left(R \mid data\right) = \frac{\sum_{t=M+1}^{N} \widetilde{R_{Bt}}}{N-M}$, where $M$ is burn-in.

## 5 Numerical Illustration

Analyses of simulated and real life data sets are provided in this section.

### *5.1 Simulated Data*

A simulation study is performed to detect the performance of the different estimators, obtained in Section 4. The simulation study is implemented for BGE-FGM and BGI-FGM forms in (6) and (7). As illustration of BGE-FGM, we take $g_1\left(x_i, \beta\right) = x_i^{\beta}\ ; i = 1,2$ obtaining a bivariate Weibull FGM, while as an illustration of BGI-FGM, we take $g_2\left(x_i, \beta\right) = \ln\left(1 + x_i^{-\beta}\right); i = 1,2$, obtaining Burr III FGM distribution.
1000 samples of different sizes (n=10, 25, 50) were generated from the bivariate Weibull FGM, and the bivariate Burr III FGM distribution. For the Weibull case we take $\beta = 4$ and $\theta = 0.4$, while for Burr III, we take $\beta = 0.5$ and $\theta = 0.4$. Different values of the parameters $\alpha_1$ *and* $\alpha_2$ are chosen so that $R$ takes moderate and high values, *(R*= 0.6, 0.7, 0.9*)* approximately. The values $\widehat{R_1}, \widehat{R_2}, \widehat{R_3}$ *and* $\widetilde{R_B}$ shown in the accompanying tables are the mean of the 1000 replicates of the corresponding estimates. To evaluate the performance of the estimators we calculated estimates of the mean squared error (MSE) and the bias (Bias) of the different estimators, where $MSE(\hat{R}) = \frac{1}{1000}\sum_{i=1}^{1000}\left(\widehat{R}_i - R\right)^2$ and $Bias(\hat{R}) = \frac{1}{1000}\sum_{i=1}^{1000}\left(\widehat{R}_i - R\right)$. Notice that, for calculating the Bayesian estimate of $R$, in each replicate we generated *N=10,000* realizations of the Markov chains using Metropolis–Hastings algorithms. According to the graphs of the conditional posterior distributions of the parameters $\alpha_1,\ \alpha_2\ , \beta$ we found it reasonable to use the normal distribution $N\left(a_{i(t-1)}, \sigma^2\right); i = 1,2,3$ (we take $\sigma^2$ is the variance of MLE) as the proposal distribution $q_i$ for $\alpha_1,\ \alpha_2$ *and* $\beta$. While for the parameter $\theta$, we considered the truncated normal distribution on the interval (-1,1), $N\left(a_{4(t-1)}, 0.5\right)$ as the proposed distribution $q_4$. It was observed that after *M=5000* burn-in periods, all the Markov chains reached their stationary condition.
The results of the simulation study are summarized in Tables 1–2. For all the numerical computations, the programs are developed in R-software.
From Tables 1 and 2 we notice that differences in the bias and the MSE among the four estimates are small. Also, based on the bias and the MSE all estimators perform well.

Table 1. The estimates of R using simulated data from bivariate Weibull FGM distribution, with different sample sizes, and different parameters.

| $R$ | 0.6085 | | | 0.7190 | | | 0.9372 | | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | 10 | 25 | 50 | 10 | 25 | 50 | 10 | 25 | 50 |
| $\widehat{R_1}$ | 0.5898 | 0.6108 | 0.6073 | 0.7459 | 0.7373 | 0.7423 | 0.8911 | 0.8790 | 0.8755 |
| Bias($\widehat{R_1}$) | -0.0187 | 0.0023 | -0.0011 | 0.0269 | 0.0183 | 0.0233 | -0.0461 | -0.0582 | -0.0617 |
| MSE($\widehat{R_1}$) | 0.0155 | 0.0034 | 0.00225 | 0.0114 | 0.0037 | 0.0023 | 0.0062 | 0.0054 | 0.0044 |
| $\widehat{R_2}$ | 0.5860 | 0.6052 | 0.6016 | 0.7477 | 0.7343 | 0.7390 | 0.8850 | 0.8725 | 0.8705 |
| Bias($\widehat{R_2}$) | -0.0225 | -0.0033 | -0.0069 | 0.0287 | 0.0153 | 0.0200 | -0.0522 | -0.0647 | -0.0667 |
| MSE($\widehat{R_2}$) | 0.0178 | 0.0039 | 0.00229 | 0.010 | 0.0040 | 0.0020 | 0.0075 | 0.0066 | 0.0050 |
| $\widehat{R_3}$ | 0.5840 | 0.6036 | 0.6010 | 0.7408 | 0.7313 | 0.7375 | 0.8738 | 0.8613 | 0.8585 |
| Bias($\widehat{R_3}$) | -0.0245 | -0.0049 | -0.0075 | 0.0218 | 0.0123 | 0.0185 | -0.0634 | -0.0759 | -0.0787 |
| MSE($\widehat{R_3}$) | 0.0160 | 0.0037 | 0.00226 | 0.0097 | 0.0038 | 0.0019 | 0.0092 | 0.0083 | 0.0068 |
| $\widetilde{R_B}$ | 0.5852 | 0.6005 | 0.6007 | 0.7185 | 0.7159 | 0.7272 | 0.8596 | 0.8546 | 0.8587 |
| Bias($\widetilde{R_B}$) | -0.0233 | -0.0080 | -0.0078 | -0.0005 | -0.0031 | 0.0082 | -0.0777 | -0.0826 | -0.0785 |
| MSE($\widetilde{R_B}$) | 0.0125 | 0.0028 | 0.0020 | 0.0085 | 0.0030 | 0.0016 | 0.0109 | 0.0087 | 0.0066 |

Table 2. The estimates of R using simulated data from bivariate burr III FGM distribution, with different sample sizes, and different parameters.

| $R$ | 0.6085 | | | 0.7499 | | | 0.9072 | | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | 10 | 25 | 50 | 10 | 25 | 50 | 10 | 25 | 50 |
| $\widehat{R_1}$ | 0.6273 | 0.6020 | 0.6133 | 0.7442 | 0.7581 | 0.7505 | 0.8638 | 0.9053 | 0.9116 |
| Bias($\widehat{R_1}$) | 0.0188 | -0.0065 | 0.0048 | -0.0057 | 0.0082 | 0.0006 | -0.0434 | -0.0019 | 0.0044 |
| MSE($\widehat{R_1}$) | 0.0394 | 0.00896 | 0.0028 | 0.0231 | 0.0051 | 0.0022 | 0.1031 | 0.0017 | 0.0007 |
| $\widehat{R_2}$ | 0.6033 | 0.5997 | 0.6051 | 0.6812 | 0.7045 | 0.7104 | 0.723 | 0.7837 | 0.7960 |
| Bias($\widehat{R_2}$) | -0.0052 | -0.0088 | -0.0034 | -0.0687 | -0.0454 | - 0.0395 | -0.1842 | -0.1235 | -0.1112 |
| MSE($\widehat{R_2}$) | 0.0129 | 0.0054 | 0.0028 | 0.0134 | 0.0054 | 0.0033 | 0.0505 | 0.0222 | 0.0250 |
| $\widehat{R_3}$ | 0.5968 | 0.5974 | 0.6032 | 0.6910 | 0.7162 | 0.7222 | 0.8143 | 0.8396 | 0.8641 |
| Bias($\widehat{R_3}$) | -0.0117 | -0.0111 | -0.0053 | -0.0589 | -0.0337 | -0.0277 | -0.0929 | -0.0676 | -0.0431 |
| MSE($\widehat{R_3}$) | 0.0110 | 0.0051 | 0.0025 | 0.0113 | 0.0043 | 0.0025 | 0.0129 | 0.0062 | 0.0025 |
| $\widetilde{R_B}$ | 0.6405 | 0.5904 | 0.6070 | 0.7248 | 0.7460 | 0.7431 | 0.8292 | 0.8891 | 0.9061 |
| Bias($\widetilde{R_B}$) | 0.0320 | -0.0181 | -0.0015 | -0.0251 | -0.0039 | -0.0068 | -0.078 | -0.0181 | -0.0011 |
| MSE($\widetilde{R_B}$) | 0.1518 | 0.0082 | 0.0027 | 0.0223 | 0.0041 | 0.0020 | 0.1190 | 0.0095 | 0.0006 |

## *5.2 Real life Data*

The following data, quoted from McGilchrist et al. [17], and used by Elaal MKA and Jarwan RS [18], represent the recurrence times to infection at point of insertion of the catheter for kidney patients using portable dialysis equipment. $X_1$ refers to the first recurrence time and $X_2$ to second recurrence time. The data for 30 patients are reported in Table 3.

Table 3. Recurrence times of infection for kidney patients.

| **patient** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** | **13** | **14** | **15** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X_1$ | 8 | 23 | 22 | 447 | 30 | 24 | 7 | 511 | 53 | 15 | 7 | 141 | 96 | 149 | 536 |
| $X_2$ | 16 | 13 | 28 | 318 | 12 | 245 | 9 | 30 | 196 | 154 | 333 | 8 | 38 | 70 | 25 |
| **Patient** | **16** | **17** | **18** | **19** | **20** | **21** | **22** | **23** | **24** | **25** | **26** | **27** | **28** | **29** | **30** |
| $X_1$ | 152 | 402 | 13 | 39 | 12 | 113 | 132 | 34 | 2 | 130 | 17 | 185 | 292 | 22 | 15 |
| $X_2$ | 362 | 24 | 66 | 46 | 40 | 201 | 156 | 30 | 25 | 26 | 4 | 117 | 114 | 159 | 108 |

The value of Kendall's sample tau coefficient $T$, from the data is 0.1110. It is clear that there is a weak positive association between $X_1$ and $X_2$. So FGM copula is suitable to describe the joint distribution of $X_1$ and $X_2$. Here we shall suggest three bivariate distributions to fit the data: the bivariate Weibull (BGE-FMG form), Burr III (BGI-FMG form), and the generalized exponential FGM distributions (BGI-FMG form). The bivariate Weibull has the joint CDF in (6), with $g_1(x_i,\beta) = x_i^{\beta}$ ; $i = 1.2$. The bivariate Burr III FGM has the joint CDF in (7), with $g_2(x_i,\beta) = \ln\left(1+x_i^{-\beta}\right)$ ; $i = 1.2$. Also, the bivariate generalized exponential FGM has the joint CDF in (7), with $g_2(x_i,\beta) = -\ln\left(1-e^{-\beta x_i}\right)$ ; $i = 1.2$. for testing the goodness of fit of the marginal, we first estimate the parameters $\alpha_1,\alpha_2$, $\beta$ and $\theta$ using Method I. Table 4 shows ML estimates of $\alpha_1,\alpha_2$ *and* $\beta$, the Kolmogorov-Smirnov distance between the fitted and the empirical distribution functions $(i.e\ D1_{30}, D2_{30})$ for $X_1$ and $X_2$, respectively.

Table 4. ML estimates of $\alpha_1,\alpha_2$ and $\beta$, and $D1_{30}, D2_{30}$ for $X_1$ and $X_2$.

| | **ML estimates** | | | | |
|---|---|---|---|---|---|
| | $\alpha_1$ | $\alpha_2$ | $\beta$ | $D1_{30}$ | $D2_{30}$ |
| **Weibull** | 0.0214 | 0.0239 | 0.825 | 0.1770 | 0.1052 |
| **Burr III** | 13.70 | 16.82 | 0.823 | 0.1294 | 0.1225 |
| **general exponential** | 0.733 | 0.799 | 0.0075 | 0.1730 | 0.1458 |

Since $D_{30,\ 0.05} = 0.24$, the marginals of the three distributions fit. To determine which of the three bivariate distributions fits better the data, we apply the Akaike information criterion (AIC) and the Bayesian information criterion (BIC). The AIC and BIC are defined respectively as

$$AIC = -2loglikelihood + 2k,$$

$$BIC = -2loglikelihood + k\ln(n)\ ,$$

where k equals the number of parameters in the model. The following table shows the values of $AIC$ and $BIC$ .

Table 5. The values of $AIC$ and $BIC$ for Weibull, Burr III and generalized exponential models.

| **model** | **AIC** | **BIC** |
|---|---|---|
| **Weibull** | 543.945 | 549.550 |
| **Burr III** | 689.210 | 694.81 |
| **Generalized exponential** | 688.473 | 649.077 |

So the bivariate FGM Weibull distribution fits better the data. Using Equation (21) we find that $\widehat{R_1} = 0.5030$. This indicates that there is no significance difference between the first and the second recurrence times.

## 6 Conclusion

In this paper, two bivariate FGM forms are proposed, namely BGE-FGM form and the BGI-FGM form. In the literature many bivariate FGM distributions possess these forms. An explicit expression for $R = P(X_2 < X_1)$ is obtained when the random variables $X_1$ and $X_2$ are dependent having a bivariate distribution with either the BGE- FGM form in (6) or the BGI- FGM form in (7). We have shown that $R$ does not depend on the form of the function $g_i(.,\ \beta)$ ; $i = 1,2$. The estimation of $R$ is studied for both forms. Three estimators in addition to a Bayesian estimator are presented. A simulation study is performed, to compare numerically the four estimators obtained. The comparison is based on the bias and the MSE. Also, characterizations of the marginal of the BGE-FGM and BGI-FGM forms, associated with $R$, is presented. Finally, the applicability of the proposed forms, is presented by fits a real life data set.

## Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article

## References

[1] Kotz, S., & Pensky, M., & Lumelskii, Y. (2003). *The stress-strength model and its generalizations: theory and applications*. World Scientific.

[2] Mokhlis, N. A. (2005). Reliability of a stress-strength model with Burr type III distributions. *Communications in Statistics-Theory and Methods,* 34(7), 1643–1657.
[3] Rezaei, S., Tahmasbi, R., & Mahmoodi, M. (2010). Estimation of P [Y< X] for generalized Pareto distribution. *Journal of Statistical Planning and Inference*, *140*(2), 480-494.
[4] Wong, A. (2012). Interval estimation of P (Y< X) for generalized Pareto distribution. *Journal of Statistical Planning and Inference*, *142*(2), 601-607.
[5] Akgül, F. G., Acıtaş, Ş., & Şenoğlu, B. (2018). Inferences on stress–strength reliability based on ranked set sampling data in case of Lindley distribution. *Journal of Statistical Computation and Simulation*, *88*(15), 3018-3032.
[6] Mokhlis, N. A., Ibrahim, E. J., & Gharieb, D. M. (2017). Stress− strength reliability with general form distributions. *Communications in Statistics-Theory and Methods*, *46*(3), 1230-1246.
[7] Gurvich, M. R., Dibenedetto, A. T., & Ranade, S. V. (1997). A new statistical distribution for characterizing the random strength of brittle materials. *Journal of Materials Science*, *32*(10), 2559-2564.
[8] Santos-Neto, M., Bourguignon, M., Zea, L. M., Nascimento, A. D., & Cordeiro, G. M. (2014). The Marshall-Olkin extended Weibull family of distributions. *Journal of Statistical Distributions and Applications*, *1*(1), 9.
[9] Nadarajah, S. (2005). Reliability for some bivariate gamma distributions. *Mathematical Problems in Engineering*, *2005*(2), 151-163.
[10] Mokhlis, N. A. (2006). Reliability of stress-strength model with a bivariate exponential distribution, *Journal of the Egyptian Mathematical Society*, 14(1), 69-78.
[11] Nadarajah, S., & Kotz, S. (2006). Reliability models based on bivariate exponential distributions. *Probabilistic engineering mechanics*, *21*(4), 338-351.
[12] Domma, F., & Giordano, S. (2013). A copula-based approach to account for dependence in stress-strength models. *Statistical Papers*, *54*(3), 807-826.
[13] Abdel-Hamid, A. H. (2016). Stress-strength reliability for general bivariate distributions. *Journal of the Egyptian Mathematical Society*, *24*(4), 617-621.
[13] Long, D., & Krzysztofowicz, R. (1992). Farlie-Gumbel-Morgenstern bivariate densities: Are they applicable in hydrology? *Stochastic Hydrology and Hydraulics*, *6*(1), 47-54.
[14] Domma, F., & Giordano, S. (2016). Concomitants of m-generalized order statistics from generalized Farlie–Gumbel–Morgenstern distribution family. Journal of Computational and Applied Mathematics, 294, 413-435.
[15] Robert, C. P., Casella, G., & Casella, G. (2010). Introducing monte carlo methods with R (Vol. 18). New York: Springer.
[16] McGilchrist, C. A., & Aisbett, C. W. (1991).Regression with frailty in survival analysis. *Biometrics*, 461 - 466.
[17] Elaal, M. K. A., & Jarwan, R. S. (2017). Inference of Bivariate Generalized Exponential Distribution Based on Copula Functions. *Applied Mathematical Sciences*, *11*(24), 1155-1186.

---

**Dina Ahmed** is a demonstrator at the Basic Science Department at the Faculty of Computer and Information Science, Ain Shams University. She has received her BSc degree in 2015, from Department of Mathematics, Faculty of Science, Ain Shams University. She is an MSc student at Ain Shams University. She has attended several international conferences in statistics. Her research interest is the inference of reliability based on different types of samples.

**Sohair Khames** received the BSc. degree (2005), MSc. degree (2010) in Mathematical Statistics and PhD degree (2015) in Mathematical statistics at faculty of Women for Arts , science & educaton, Ain Shams University, Egypt. Her research interests are in the area of mathematical statistics and its applications such as reliability theory and concomitants of ordered variables. She has published research articles in reputed statistical journals. She has participated in many international conferences of mathematical and statistical sciences.

**Nahed Mokhlis** is a professor of Mathematical Statistics at the Department of Mathematics, Faculty of Science, Ain Shams University, Egypt. She received her BSc. degree (1973), MSc. degree (1978) in Mathematical Statistics from Ain Shams University, and the PhD degree (1983) in Mathematical Statistics from Zagazig University, Egypt. She has published over 43 research articles in reputed international journals of statistical sciences. She is a reviewer in reputed international statistical journals. She has participated in many international conferences of mathematical and statistical sciences. Her research interests are in reliability theory, statistical inference, and order statistics.