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On Convex Structures in Pseudo Metric Spaces and Fixed Point Theorems

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Abstract: In this paper, we introduce convex pseudo metric space and illustrate this concept with an example, we discuss properties of convex structures in pseudo metric spaces. We apply these properties to obtain fixed point theorems in complete pseudo metric spaces. We present an example of convex structure.

Keywords: Fixed point,complete pseudo metric space,convex structure.

1 Introduction and preliminaries

Takahashi [1] introduced the notion of convexity in metric spaces and studied some fixed point theorems for nonexpansive mappings. Subsequently, Beg [2], Beg and Abbas [3], Chang, Kim and Jin [4], Ciric [5], Shimizu and Takahashi [6], Ding [7], and many others presented fixed point theorems in convex metric spaces.
Oussaeif, Abdelkrim [8], presented following result in partial metric spaces endowed with convex structure.

Theorem 1.[8] Let C be a nonempty closed convex subset of a convex complete partial metric space (X,p,W) and f be a self-mapping of C. If there exist $k \in (0,\frac{1}{4})$ such that

$$p(x,f(y)) + p(f(x),f(y)) \leq kp(y,f(x)) \text{ for all } x,y \in C,$$

then f has at least one fixed point.

E. Karapinar [9], obtained the following fixed point theorem.

Theorem 2.[9] Let C be a closed and convex subset of a cone Banach space X with the norm $\|x\|_p = d(x,0)$ and $T : C \to C$ be a mapping satisfying following condition for $2 \leq q < 4$ condition

$$d(x,Tx) + d(y,Ty) \leq qd(x,y) \; \forall \, x,y \in C.$$

Then T has at least one fixed point.

In this paper, we shall prove Theorem 1 and Theorem 2 in the framework of pseudo metric spaces.
Let us recall mathematical basics needed in the sequel.
A mapping $T : X \to X$ is said to have fixed point $x \in X$ if $T(x) = x$ and $F(T)$ denotes the set of all fixed points of T. $F(T,f)$ represents set of common fixed points of T and f.

Definition 1.[10] Let X be a non empty set. A function $d_s : X \times X \to \mathbb{R}^+$ is called pseudo metric if for any $x,y,z \in X$, the following conditions hold:

$(p_1) d_s(x,x) = 0.$
$(p_2) d_s(x,y) = d_s(y,x).$
$(p_3) d_s(x,z) \leq d_s(x,y) + d_s(y,z).$

Note that $d_s(x,y) = 0$ may not imply that $x = y$.

*Example 1.*Let $X = \mathbb{R}$ and $d_s : X \times X \to \mathbb{R}_0^+$ be defined by $d_s(x,y) = |sin(x) - sin(y)|$, then (X,d_s) is a pseudo metric space.

*Example 2.*Let F represents space of all real valued functions which are not injective and $d_s(x,y)$ is defined by $d_s(x,y) = |f(x) - f(y)|$, then (F,d_s) is a pseudo metric space.

*Example 3.*If X is a vector space and p is a semi-norm on X then $d_s(x,y) = p(x-y)$ is a pseudo metric on X.

*Example 4.*Consider the set $C[0,1] = \{f \in \mathbb{R}^{[0,1]} : \text{ f is continuous}\}$ and define a

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distance function d_s on $C[0,1]$ as

$$d_s(f,g) = \int_0^1 |f(x) - g(x)|\,dx,$$

then d_s is a pseudo metric on C.

Following [10], a pseudo metric space generates a topology $\tau(d_s)$ on X, whose base is the family of open balls $\{\mathbb{B}(x_0,\varepsilon)\ x_0 \in X, \varepsilon > 0\}$, where

$$\mathbb{B}(x_0,\varepsilon) = \{y \in X : d_s(x,y) < \varepsilon\}.$$

Definition 2.[10] *For a pseudo metric space (X,d_s), we have following concepts*

(1)*The sequence $\{x_n\}$ in X is said to be convergent to a point x, if $\lim_{n\to\infty} d_s(x_n,x) = 0$.*
(2)*The sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for any $\varepsilon > 0$ there exists a positive integer N such that $d_s(x_n,x_k) < \varepsilon$ for all $n,k > N$.*
(3)*The pseudo metric space is said to be complete if every Cauchy sequence converges to a point in it.*
(4)*Let $T : X \to X$ be a mapping. The mapping T is said to be sequentially continuous if*

$$x_n \to x \Rightarrow T(x_n) \to T(x) \text{ as } n \to \infty.$$

Definition 3.[11] The ordered pair (T,f) of two self-maps of a pseudo metric space (X,d_s) is called a Banach operator pair if $F(f)$ is T-invariant, namely $T(F(f)) \subseteq F(f)$.

Definition 4. Let (X,d_s) be a pseudo metric space and $I = [0,1]$. A mapping $W : X \times X \times I \to X$ is said to be a pseudo convex structure on X if for each $(x,y,\lambda) \in X \times X \times I$ and $u \in X$

$$d_s(u,W(x,y,\lambda)) \le \lambda d_s(u,x) + (1-\lambda)d_s(u,y).$$

A pseudo metric space (X,d_s) endowed with a convex structure W is called a convex pseudo metric space, denoted by (X,d_s,W).

Example 5.Let $X = \mathbb{R}^2$ and consider the function $d_s : X \times X \to [0,\infty)$ defined by

$$d_s(u,v) = |x_1 - y_1| \quad \forall \quad u = (x_1,x_2), v = (y_1,y_2) \in X.$$

Define $W(x,y,\lambda) = \lambda x + (1-\lambda)y$, then (X,d_s,W) is convex pseudo metric space.
Indeed, for $u = (z_1,z_2)$, $x = (x_1,x_2)$, $y = (y_1,y_2)$ in X.

$$\begin{aligned}
&d_s(x,W(y,u,\lambda))d_s(x,\lambda y + (1-\lambda)u),\\
&= d_s((x_1,x_2),(\lambda y_1 + (1-\lambda)z_1, \lambda y_1 + (1-\lambda)z_1)),\\
&= |x_1 - (\lambda y_1 + (1-\lambda)z_1)|,\\
&= |[\lambda + (1-\lambda)]x_1 - (\lambda y_1 + (1-\lambda)z_1)|,\\
&= |(\lambda x_1 - \lambda y_1) + [(1-\lambda)x_1 - (1-\lambda)z_1]|,\\
&\le \lambda|x_1 - y_1| + (1-\lambda)|x_1 - z_1|,\\
&\le \lambda d_s(x,y) + (1-\lambda)d_s(x,u).
\end{aligned}$$

Definition 5. Let (X,d_s,W) be a convex pseudo metric space. A non-empty subset C of X is said to be convex if $W(x,y,\lambda) \in C$ whenever $(x,y,\lambda) \in C \times C \times I$

Proposition 1. Let (X,d_s,W) be convex pseudo metric space then for all $x,y \in X$, $\lambda \in [0,1]$, following properties hold:

(a)$d_s(x,y) = d_s(x,W(x,y,\lambda)) + d_s(W(x,y,\lambda),y)$.
(b)$d_s(x,W(x,y,\lambda)) = (1-\lambda)d_s(x,y)$.
(c)$d_s(y,W(x,y,\lambda)) = \lambda d_s(x,y)$.

*Proof.*Due to (p_3), we have

$$d_s(x,y) \le d_s(x,W(x,y,\lambda)) + d_s(W(x,y,\lambda),y), \quad (1)$$

and definition of convex structure gives

$$d_s(x,W(x,y,\lambda)) \le \lambda d_s(x,x) + (1-\lambda)d_s(x,y)$$

and

$$d_s(y,W(x,y,\lambda)) \le \lambda d_s(y,x) + (1-\lambda)d_s(y,y).$$

Since $d_s(x,x) = d_s(y,y) = 0$, therefore, above two inequalities implies that

$$d_s(x,W(x,y,\lambda)) + d_s(y,W(x,y,\lambda)) \le d_s(x,y). \quad (2)$$

Combining equations (1) and (2), we get required result. Proofs of (b) and (c) can be obtained by using definition of convex structure similarly.

2 The results

In this section, we present fixed point theorems in pseudo metric spaces endowed with pseudo convex structure. We begin with following lemma.

Lemma 1.*Let (X,d_s) be a pseudo metric space and $C \subseteq X$ be a closed subset of X. Let $T : C \to C$ be a sequentially continuous mapping then $d_s(x,T(C)) = 0$ implies $x \in T(C)$.*

*Proof.*Since T is a self mapping and C is closed subset of X, so, $T(C)$ is a closed subset of X. Due to definition of distance of a point of (X,d_s) from a subset of (X,d_s), we have

$$d_s(x,T(C)) = \inf\{d_s(x,y) | y \in T(c)\}$$

and since it is given that $d_s(x,T(C)) = 0$, therefore,

$$d_s(x,T(C)) = \inf\{d_s(x,y) | y \in T(c)\} = 0.$$

Sequentially continuity of T implies that there exists at least one distance between x,y such that $d_s(x,y) = 0$, which in turn implies that x is a limit point of $T(C)$. Since $T(C)$ is a closed subset of X, therefore, $x \in T(C)$.

Our first main result:

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Theorem 3. *Let C be a nonempty closed convex subset of a complete convex pseudo metric space (X, d_s, W) and T be a sequentially continuous self-mapping defined on C. If there exists $k \in \left[0, \frac{1}{2}\right)$ such that*

$$d_s(x, T(y)) + d_s(T(x), T(y)) \leq k d_s(y, T(x)), \quad (3)$$

for all $x, y \in C$. Then T has at least one fixed point.

Proof. From the definition of pseudo convex structure we obtain

$$d_s(x, W(x,y,\lambda)) \leq \lambda d_s(x,x) + (1-\lambda) d_s(x,y), \quad (4)$$

and

$$d_s(y, W(x,y,\lambda)) \leq \lambda d_s(y,x) + (1-\lambda) d_s(y,y). \quad (5)$$

Since $d_s(x,x) = d_s(y,y) = 0$, therefore, using (4) and (5) we get,

$$d_s(x, W(x,y,\lambda)) + d_s(y, W(x,y,\lambda)) \leq d_s(x,y). \quad (6)$$

Now let $x_0 \in C$ and define a sequence $\{x_n\}$ as follows,

$$x_n = W(x_{n-1}, Tx_{n-1}, \lambda);\ n = 1,2,3,\ldots. \quad (7)$$

Since C is convex so $x_n \in C$ and using equations (4), (6) and (7), we obtain

$$d_s(x_n, x_{n+1}) \leq \lambda d_s(x_n, x_n) + (1-\lambda) d_s(x_n, Tx_n), \quad (8)$$

and

$$d_s(Tx_n, x_{n+1}) \leq \lambda d_s(x_n, Tx_n) + (1-\lambda) d_s(Tx_n, Tx_n). \quad (9)$$

Adding (8) and (9), we get

$$d_s(x_n, x_{n+1}) + d_s(Tx_n, x_{n+1}) \leq d_s(x_n, Tx_n). \quad (10)$$

Now using p_3, we obtain

$$d_s(x_n, Tx_n) \leq d_s(x_n, Tx_{n-1}) + d_s(Tx_{n-1}, Tx_n). \quad (11)$$

From equations (10) and (11), we get

$$\begin{aligned} & d_s(x_n, x_{n+1}) + d_s(Tx_n, x_{n+1}) \\ \leq\ & d_s(x_n, Tx_{n-1}) + d_s(Tx_{n-1}, Tx_n). \end{aligned}$$

This implies

$$\begin{aligned} & d_s(x_n, x_{n+1}) + d_s(Tx_n, x_{n+1}) - d_s(x_n, Tx_{n-1}) \\ \leq\ & d_s(Tx_{n-1}, Tx_n). \end{aligned} \quad (12)$$

By replacing x with x_n and y with x_{n-1} in (3), we have

$$d_s(x_n, Tx_{n-1}) + d_s(Tx_n, Tx_{n-1}) \leq k d_s(x_{n-1}, Tx_n).$$

This implies that

$$d_s(Tx_n, Tx_{n-1}) \leq k d_s(x_{n-1}, Tx_n) - d_s(x_n, Tx_{n-1}). \quad (13)$$

Simplifying equations (12) and (13), we obtain

$$d_s(x_n, x_{n+1}) + d_s(x_{n+1}, Tx_n) \leq k d_s(x_{n-1}, Tx_n). \quad (14)$$

Again using p_3, we get

$$d_s(x_{n-1}, Tx_n) \leq d_s(x_{n-1}, x_n) + d_s(x_n, Tx_n). \quad (15)$$

Combining (14) and (15), we have

$$\begin{aligned} d_s(x_n, x_{n+1}) + d_s(x_{n+1}, Tx_n) &\leq k[d_s(x_{n-1}, x_n) + d_s(x_n, Tx_n)], \\ \Rightarrow\ d_s(x_n, Tx_n) &\leq k[d_s(x_{n-1}, x_n)] + k[d_s(x_n, Tx_n)], \\ (1-k) d_s(x_n, Tx_n) &\leq k d_s(x_{n-1}, x_n). \end{aligned}$$

Due to pseudo convex structure, we have

$$d_s(x_n, x_{n+1}) \leq (1-\lambda) d_s(x_n, Tx_n) \leq d_s(Tx_n, x_n).$$

That is

$$d_s(x_n, x_{n+1}) \leq d_s(x_n, Tx_n).$$

Therefore, we obtain

$$\begin{aligned} (1-k) d_s(x_n, x_{n+1}) &\leq k d_s(x_{n-1}, x_n), \\ \Rightarrow\ d_s(x_n, x_{n+1}) &\leq \frac{k}{1-k} d_s(x_{n-1}, x_n), \\ \Rightarrow\ d_s(x_n, x_{n+1}) &\leq \theta d_s(x_{n-1}, x_n);\ \theta = \frac{k}{1-k} \\ \Rightarrow d_s(x_n, x_{n+1}) &\leq \theta^n d_s(x_0, x_1). \end{aligned}$$

Similarly,

$$d_s(x_{n+1}, x_{n+2}) \leq \theta^{n+1} d_s(x_0, x_1).$$

Continuing in the similar way, we obtain

$$d_s(x_{n+k-1}, x_{n+k}) \leq \theta^{n+k-1} d_s(x_0, x_1).$$

By p_3 we have

$$\begin{aligned} d_s(x_n, x_{n+k}) &\leq d_s(x_n, x_{n+1}) + d_s(x_{n+1}, x_{n+2}) + \ldots \\ &\quad + d_s(x_{n+k-1}, x_{n+k}), \\ &\leq (\theta^n + \theta^{n+1} + \ldots + \theta^{n+k-1}) d_s(x_0, x_1), \\ &\leq \frac{\theta^n}{1-\theta} d_s(x_0, x_1). \end{aligned}$$

Since $0 \leq \theta < 1$, therefore, we get

$$\lim_{n\to\infty} d_s(x_n, x_{n+k}) = 0.$$

This implies that $\{x_n\}$ is a Cauchy sequence in C. Since C is closed subset of a complete space (X, d_s), so C is also complete subspace. Consequently, $\{x_n\}_{n=1}^{\infty}$ converges to a point v in C and

$$\lim_{n\to\infty} x_n = v.$$

Now we are left to prove that v is fixed point of T. Since T is sequentially continuous therefore,

$$x_n \to v\ \Rightarrow Tx_n \to Tv \text{ as } n \to \infty.$$

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Due to (p_1),

$$d_s(x_n,Tx_n) \le \lambda d_s(Tx_{n-1},Tx_{n-1})+(1-\lambda)d_s(x_{n-1},Tx_{n-1}),$$

implies

$$d_s(x_n,Tx_n) \le (1-\lambda)d_s(x_{n-1},Tx_{n-1}).$$

Letting $n\to\infty$ we have

$$d_s(v,Tv) \le (1-\lambda)d_s(v,Tv) \le d_s(v,Tv).$$

Hence $d_s(v,Tv)=0$ and by Lemma 1, we deduce that $v=T(v)$, which completes the proof.

Theorem 4.*Let C be a nonempty closed convex subset of a complete convex pseudo metric space (X,d_s,W) and T and f be a sequentially continuous self mapping of C. Suppose that*

(1)there exists $k\in\left[0,\frac{1}{2}\right)$ such that for all $x,y\in C$

$$d_s(f(x),T(y))+d_s(T(x),T(y)) \le kd_s(f(y),T(x)). \tag{16}$$

(2)(T,f) is a Banach operator pair and f has the property $f(W(x,y,\lambda))=W(f(x),f(y),\lambda)$.

(3)$F(f)$ is a nonempty closed subset of C.

Then $F(T,f)$ is nonempty.

*Proof.*Since, (T,f) is a Banach operator pair, therefore from (16), we obtain

$$d_s(x,T(y))+d_s(T(x),T(y)) \le kd_s(y,T(x)) \ \forall\ x,y\in F(f).$$

$F(f)$ is convex because f has the property

$$f(W(x,y,\lambda))=W(f(x),f(y),\lambda).$$

It follows from Theorem 3 that $F(T,f)$ is nonempty.

Theorem 5.*Let C be a nonempty closed convex subset of a complete convex pseudo metric space (X,d_s,W) and T be a sequentially continuous self-mapping of C. If there exist $k\in[2,4)$ such that,*

$$d_s(x,Tx)+d_s(y,Ty) \le kd_s(x,y) \ \forall\ x,y\in C. \tag{17}$$

Then T has at least one fixed point.

*Proof.*let $x_0\in C$ and define a sequence $\{x_n\}$ by

$$x_n=W(x_{n-1},Tx_{n-1},\lambda);\ n=1,2,3,....$$

As C is convex, so $x_n\in C$ and from the proposition 1, we get following equations for $\lambda=\frac{1}{2}$

$$d_s(x_n,x_{n+1})=\frac{1}{2}d_s(x_n,Tx_n), \tag{18}$$

and

$$d_s(x_n,x_{n-1})=\frac{1}{2}d_s(x_{n-1},Tx_{n-1}). \tag{19}$$

Replacing x with x_n and y with x_{n-1} in equation (17), we obtain

$$d_s(x_n,Tx_n)+d_s(x_{n-1},Tx_{n-1}) \le kd_s(x_n,x_{n-1}) \ \forall\ n\in N. \tag{20}$$

Using equations (18), (19) and (20), we have

$$\begin{aligned}
2d_s(x_n,x_{n+1})+2d_s(x_n,x_{n-1}) &\le kd_s(x_n,x_{n-1}),\\
2d_s(x_n,x_{n+1}) &\le (k-2)d_s(x_n,x_{n-1}),\\
d_s(x_n,x_{n+1}) &\le \frac{k-2}{2}d_s(x_{n-1},x_n)\ \forall\ n\in\mathbb{N}.\\
d_s(x_n,x_{n+1}) &\le \lambda d_s(x_n,x_{n-1}),\ \text{where } \lambda=\frac{k-2}{2}.\\
\Rightarrow d_s(x_n,x_{n+1}) &\le \lambda^n d_s(x_0,x_1).
\end{aligned}$$

Similarly,

$$d_s(x_{n+1},x_{n+2}) \le \lambda^{n+1}d_s(x_0,x_1).$$

Continuing in a similar way we obtain

$$d_s(x_{n+k-1},x_{n+k}) \le \lambda^{n+k-1}d_s(x_0,x_1).$$

By p_3, we have

$$\begin{aligned}
d_s(x_n,x_{n+k}) &\le d_s(x_n,x_{n+1})+d_s(x_{n+1},x_{n+2})+\cdots+d_s(x_{n+k-1},x_{n+k}),\\
&\le (\lambda^n+\lambda^{n+1}+\cdots+\lambda^{n+k-1})d_s(x_0,x_1),\\
&\le \frac{\lambda^n}{1-\lambda}d_s(x_0,x_1).
\end{aligned}$$

Since $0\le\lambda<1$, therefore, we get

$$\lim_{n\to\infty} d_s(x_n,x_{n+k})=0.$$

This implies that $\{x_n\}$ is a Cauchy sequence in C. Since C is closed subset of a complete space (X,d_s), so C is also complete subspace. Consequently, $\{x_n\}_{n=1}^{\infty}$ converges to a point v in C and

$$\lim_{n\to\infty} x_n=v.$$

Now we are left to prove that v is a fixed point of T. Since T is sequentially continuous, thus,

$$x_n\to v\ \Rightarrow T(x_n)\to T(v)\ as\ n\to\infty.$$

Also, due to p_1,

$$d_s(x_n,Tx_n) \le \lambda d_s(Tx_{n-1},Tx_{n-1})+(1-\lambda)d_s(x_{n-1},Tx_{n-1}).$$

implies

$$d_s(x_n,Tx_n) \le (1-\lambda)d_s(x_{n-1},Tx_{n-1}).$$

Letting $n\to\infty$ we have

$$d_s(v,T(v)) \le (1-\lambda)d_s(v,T(v)) \le d_s(v,T(v)).$$

Hence $d_s(v,Tv)=0$ and by Lemma 1, we deduce that $v=T(v)$, which completes the proof.

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Theorem 6. *Let C be a nonempty closed convex subset of a complete convex pseudo metric space (X, d_s, W) and T and f be a sequentially continuous self mapping of C. Suppose that*

(1)there exists $k \in \left[0, \frac{1}{2}\right)$ such that for all $x, y \in C$

$$d_s(f(x), T(x)) + d_s(f(y), T(y)) \leq k d_s(f(x), f(y)). \tag{21}$$

(2)(T, f) is a Banach operator pair and f has the property $f(W(x, y, \lambda)) = W(f(x), f(y), \lambda)$.
(3)$F(f)$ is a nonempty closed subset of C.

Then $F(T, f)$ is nonempty.

Proof. Since (T, f) is a Banach operator pair, therefore from (21), we obtain

$$d_s(x, Tx) + d_s(y, Ty) \leq k d_s(x, y) \ \forall \ x, y \in F(f).$$

$F(f)$ is convex because f has the property

$$f(W(x, y, \lambda)) = W(f(x), f(y), \lambda).$$

It follows from Theorem 5 that $F(T, f)$ is nonempty.

3 Conclusion

In this article, we extend the study of pseudo metric spaces to fixed point theory endowed with convex structures. We discuss properties of convex structures in pseudo metric spaces and apply these properties to obtain fixed point theorems in complete pseudo metric spaces. Theorems established in this article involve implicit type contraction mappings and using Banach Operator we derive some common fixed point theorems.

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References

[1] Takahashi, T: *A convexity in metric spaces and nonexpansive mapping* I. Kodai Math. Sem. Rep. 22, 142-149(1970).
[2] Beg, I, Abbas, M: *Common fixed points and best approximation in convex metric spaces*. Soochow Journal of Mathematics. 33:4, 729-738(2007).
[3] Beg, I, Abbas, M: *Fixed-point theorem for weakly inward multivalued maps on a convex metric space.* Demonstratio Mathematica. 39:1,149-160(2006).
[4] Chang, SS, Kim, JK, Jin, DS: *Iterative sequences with errors for asymptotically quasi nonexpansive mappings in convex metric spaces.* Arch. Inequal. Appl. 2, 365-374(2004).
[5] Ciric, L: *On some discontinuous fixed point theorems in convex metric spaces.* Czech. Math. J. 43:188, 319-326(1993).
[6] Shimizu, T, Takahashi, W: *Fixed point theorems in certain convex metric spaces.* Math. Japon. 37, 855-859(1992).
[7] Ding, XP: *Iteration processes for nonlinear mappings in convex metric spaces.* J. Math. Anal. Appl. 132, 114-122(1998).
[8] Oussaeif, Abdelkrim:*Fixed Point Theorems in Convex Partial Metric Space*, Konuralp Journal of Mathematics Volume 2 No. 2 pp. 96-101 (2014).
[9] Karapinar, E: *Fixed point theorems in cone Banach spaces.* Fixed Point Theory Appl. 2009, Article ID 609281 (2009). doi:10.1155/2009/609281.
[10] Ronald C. Freiwald *Introduction to Set Theory and Topology.* Edition 2013-2014.
[11] M. Moosaei, *Fixed Point Theorems in Convex Metric Spaces*, Fixed Point Theory and Applications 2012, 2012:164 doi:10.1186/1687-1812-2012-164

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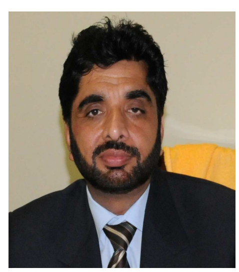

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