Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090631

A New Index for Fuzzy Distance Measure

Mostafa Ali Beigi[1,*], *Tayebeh Hajjari*[2] *and Erfan Ghasem Khani*[3]

[1] Young Researchers and Elite Club, Firoozkooh Branch, Islamic Azad University, Firoozkooh, Iran
[2] Department of Mathematics, Firoozkooh Branch, Islamic Azad University, Firoozkooh, Iran
[3] Department of Industrial Engineering, Qazvin Branch, Islamic Azad University, Qazvin, Iran

Received: 28 Feb. 2015, Revised: 28 Apr. 2015, Accepted: 29 Apr. 2015
Published online: 1 Nov. 2015

Abstract: The hesitant fuzzy linguistic term sets (HELTSs), which can be used to represent an expert's hesitant preferences when assessing a linguistic variable, increase the flexibility of eliciting and representing linguistic information. The HELTSs have attracted a lot of attention recently due to their distinguished power and efficiency in representing uncertainty and vagueness within the process of decision making. To enhance and extend the applicability of HELTSs, in this paper we first review on some fuzzy distance methods, then we present a new fuzzy distance measure based on end points and core, which is far simple, and easier than previous methods. Finally, some numerical examples illustrate the presented method as well as comparing it with other various ones.

Keywords: Fuzzy distance measure, Trapezoidal fuzzy numbers, Triangular Fuzzy numbers (TFN), Image processing

1 Introduction

Fuzzy distance can be widely usage in attribute importance. Many fuzzy distance indices have been proposed since 1965. Some of the methods used crisp number to calculate the distance between two trapezoidal fuzzy numbers [1,2,3,4,5,6,7,8,9]. Human intuition says that the distance between two uncertain numbers should as a collection of points with different degrees of belongingness, then the distance between two fuzzy numbers is noting but the collection of pairwise distance between the elements of the respective fuzzy numbers [10,11,12]. Therefore, we pay to other methods for fuzzy distance, which used fuzzy distance to calculate the distance between two fuzzy numbers and introduce a fuzzy distance for normal fuzzy numbers.

One of the first fuzzy distance method was introduced by voxman [12] through using r-cut. In 2006, Chakraborty and Chakraborty [10] introduced a fuzzy value distance measured by using r-cut of two generalized fuzzy numbers. Then Guha and Chakraborty [13] showed that their previous method could be developed to a new similarity measure with the help of the fuzzy distance measure. It is obvious that, the distance between the fuzzy number A and zero is more suitable to be A. However, by Chakraborty method the distance between the fuzzy number A and zero is not A.

Hajjari [16] presented a fuzzy Euclidean distance for fuzzy data. Hajjari first described positive fuzzy number, negative fuzzy number and fuzzy zero to represent some new definitions, and then the author discussed fuzzy absolute, equality and inequality of fuzzy numbers based on these concepts and some useful properties too. The aforementioned concepts used to produce the distance between two fuzzy numbers as a trapezoidal fuzzy number.

Abbasbandy and Hajighasemi [15] introduced a symmetric triangular fuzzy number (*TFN*) as a fuzzy distance based on r-cut concept. All the above-mentioned distance methods used the r-cut concept to calculate the fuzzy distance. There are some other distance methods, which have used other fuzzy concept. Many distance methods for fuzzy numbers have been discussed [16,17,18]. Most of these approaches applied r-cut concept and can consider general fuzzy numbers in one dimension space. Nowadays one of the most applicable types of fuzzy numbers is triangular fuzzy numbers. In this paper, we apply this kind of fuzzy numbers for a new distance measure.

The rest of the paper is organized as follows: Section 2 contains the basic definitions and notations that will be used in the remaining parts of the paper. In Section 3, we review some different fuzzy distance methods. Section 4 includes a new Approach to determine fuzzy distance

* Corresponding author e-mail: mostafaalibeigi@gmail.com

measure and some properties. Some numerical examples demonstrate the advantages of the reviewed methods and compared results in section 5. The paper is concluded in Section 6.

2 Preliminaries

In general, a generalized fuzzy number A is described as any fuzzy subset of real line R, whose membership $\mu_A(x)$ can be defined as [20]:

$$\mu_A(x) = \begin{cases} L_A(x), & a \leq x \leq b, \\ \omega, & b \leq x \leq c, \\ R_A(x), & c \leq x, \\ 0, & otherwise, \end{cases} \tag{1}$$

where $0 \leq \omega \leq 1$ is a constant, $L_A(x) : [a,b] \rightarrow [0,\omega]$ and $R_A(x) : [c,d] \rightarrow [0,\omega]$ are two strictly monotonically and continuous mapping. If $\omega = 1$, then A is a normal fuzzy number. If $L_A(x) = \omega(x-a)/(b-a)$, and $R_A(x) = \omega(d-x)/(d-c)$ then it is a trapezoidal fuzzy number and is usually denoted by $A = (a,b,c,d;\omega)$ or $A = (a,b,c,d)$ if $\omega = 1$. In particular, when $b = c$, the trapezoidal fuzzy number is reduced to a triangular fuzzy number denoted by $A = (a,b,c,d;\omega)$ or $A = (a,b,c,d)$ if $\omega = 1$. Therefore, triangular fuzzy numbers are special cases of trapezoidal fuzzy numbers.
Since $L_A(x)$ and $R_A(x)$ are both strictly monotonically and continuous functions, their inverse functions exist and should also be continuous and strictly monotonically. Let $\mu_L^{-1} : [0,\omega] \rightarrow [a,b]$ and $\mu_R^{-1} : [0,\omega] \rightarrow [c,d]$ be the inverse functions of $L_A(x)$ and $R_A(x)$, respectively. Then $L_A^{-1}(r)$ and $R_A^{-1}(r)$ should be integrable on the close interval $[0,\omega]$. In other words, both $\int_0^{\omega} L_A^{-1}(r)dr$ and $\int_0^{\omega} R_A^{-1}(r)dr$ should exist. In the case of trapezoidal fuzzy number, the inverse functions $L_A^{-1}(r)$ and $R_A^{-1}(r)$ can be analytically expressed as

$$L_A^{-1}(r) = a + (b-c)r/\omega, \qquad 0 \leq \omega \leq 1, \tag{2}$$

$$R_A^{-1}(r) = d - (d-c)y/\omega, \qquad 0 \leq \omega \leq 1. \tag{3}$$

The set of all elements that have a nonzero degree of membership in a is called the *support* of A, i.e.

$$supp(A) = \{x \in X \mid \mu_A(x) > 0\}. \tag{4}$$

The set of elements having the largest degree of membership in $\tilde{A}$ is called the *core* of A, i.e.

$$core(A) = \{x \in X \mid \mu_A(x) = \sup_{x \in X} A(x)\}. \tag{5}$$

In the following, we will always assume that A is continuous and bounded support $supp(A) = (a,d)$. The strong support of A should be $\overline{supp}(A) = [a,d]$.

Definition 1. *A function $s : [0,1] \longrightarrow [0,1]$ is a reducing function if s is increasing and $s(0) = 0$ and $s(1) = 1$. We say that s is a regular function if $\int_0^1 s(r)\mathrm{d}r = \frac{1}{2}$.*

Definition 2. *If A is a fuzzy number with r-cut representation, $\left[L_A^{-1}(r), R_A^{-1}(r)\right]$, and s is a reducing function then the value of A (with respect to s) is defined by*

$$Val(A) = \int_0^1 s(r)\left[L_A^{-1}(r) + R_A^{-1}(r)\right]\mathrm{d}r. \tag{6}$$

Definition 3. *If A is a fuzzy number with r-cut representation, $\left[L_A^{-1}(r), R_A^{-1}(r)\right]$, and s is a reducing function then the ambiguity of A (with respect to s) is defined by*

$$Amb(A) = \int_0^1 s(r)\left[R_A^{-1}(r) - L_A^{-1}(r)\right]\mathrm{d}r. \tag{7}$$

Definition 4. *Let A is a fuzzy number. The absolute value of the fuzzy number A is denoted by $|A|$ and defined as follows [21]:*

$$\mid A(x) \mid = \begin{cases} 0, & x < 0, \\ A(x) \vee A(-x), & x \geq 0, \end{cases} \tag{8}$$

And for all $r \in [0.1]$,

$$[|Ax|]_r = \begin{cases} [A]_r, & L_A^{-1}(r) \geq 0, \\ [0, |L_A^{-1}(r)| \vee R_A^{-1}(r)], & L_A^{-1}(r) \leq 0 \leq R_A^{-1}(r), \\ [-R_A^{-1}(r), -L_A^{-1}(r)], & L_A^{-1}(r) \leq R_A^{-1}(r) \leq 0, \end{cases} \tag{9}$$

where $[A]_r = \lfloor L_A^{-1}(r), R_A^{-1}(r) \rfloor$ is the r-cut representation of A and $[|Ax|]_r$ is the r-cut representation of $|A|$, respectively.

3 Some existing fuzzy distance methods

In this section, we briefly review some existing fuzzy distance measure. Different authors have constructed different fuzzy distance measure between two fuzzy numbers. Some of them are discussed here.

3.1 Voxman's fuzzy distance measure [12]

Here, we briefly describe the fuzzy distance measure by Voxman [12]. The fuzzy distance function of F,

$$\begin{cases} \Delta : F \times F \longrightarrow F \\ \Delta(A,B)(z) = supmin_{|x-y|=z}\{\mu_A(x), \mu_B(y)\} \end{cases}$$

For each pair of fuzzy number A, B let $\Delta_{A,B}$ denoted the fuzzy number $\Delta(A,B)$.

If the r- cut representation of $\Delta_{AB} = (L(r), R(r))$ is given by

$$L(r) = \begin{cases} max\{\underline{\mu_B}(r) - \overline{\mu_A}(r), 0\}, \text{if} \\ \qquad (\underline{\mu_A}(1) + \overline{\mu_A}(1)) \le \underline{\mu_B}(1) + \overline{\mu_B}(1) \\ max\{\underline{\mu_A}(r) - \overline{\mu_B}(r), 0\}, \text{if} \\ \qquad (\underline{\mu_B}(1) + \overline{\mu_B}(1)) \le \underline{\mu_A}(1) + \overline{\mu_A}(1) \end{cases} \tag{10}$$

and

$$R(r) = max\{\overline{\mu_A}(r) - \underline{\mu_B}(r), \overline{\mu_B}(r) - \underline{\mu_A}(r)\} \tag{11}$$

3.2 Cheng's distance method [22]

In order to determine the centroid point (x_A, y_A) of a fuzzy number, Cheng [22] provided a formula. Wang et al. [23] found from the point of view of analytical geometry and showed the corrected centroid point as follows:

$$x_A = \frac{\int_a^b xL_A(x)dx + \int_b^c xdx + \int_c^d xR_A(x)dx}{\int_a^b L_A(x)dx + \int_b^c dx + \int_c^d R_A(x)dx}$$

$$y_A = \frac{[\int_0^\omega yR_A^-1(y)dy - \int_o^\omega yL_A^-1(y)dy}{\int_0^\omega R_A^-1(y)dy - \int_o^\omega L_A^-1(y)dy} \tag{12}$$

For non-normal trapezoidal fuzzy number $A = (a,b,c,d;\omega)$ formulas (14) lead to following results respectively.

$$x_A = 1/3[a+b+c+d - \frac{dc-ab}{(d+c)-(a+b)}]$$

$$y_A = \omega/3[1 + \frac{c-d}{(d+c)-(a+b)}]. \tag{13}$$

Since non-normal triangular fuzzy numbers are, special cases of normal trapezoidal fuzzy numbers with $b = c$, formulas (15) can be simplified as

$$x_A = 1/3[a+b+c+d], \qquad y_A = \omega/3. \tag{14}$$

Cheng [22] formulated his idea as follows:

$$R(A) = \sqrt{x_A^2 + y_A^2}, \tag{15}$$

where $R_(A)$ is the distance between the fuzzy number A and origin point.

3.3 Chakraborty et al.'s fuzzy distance measure [10]

Consider two generalized fuzzy numbers as $A = (a_1, a_2; \beta_1, \gamma_1)$ and $A_2 = (b_1, b_2; \beta_2, \gamma_2)$. Therefore, the r-cut of A_1 and A_2 represents two intervals, respectively $[A_1]_r = [A_1^L(r),\ A_1^R(r)]$ and $[A_2]_r = [A_2^L(r),\ A_2^R(r)]$, for all $r \in [0,1]$.

It is employed the interval-difference operation for the intervals $[A_1]_r = [A_1^L(r), A_1^R(r)]$ and $[A_2]_r = [A_2^L(r), A_2^R(r)]$ to formulate the fuzzy distance between A_1 and A_2. So, the distance between $[A_1]_\alpha$ and $[A_2]_\alpha$ for every $r \in [0,1]$ can be one of the following:
either (a)

$$[A_1]_r - [A_2]_r \quad if \quad \frac{A_1^L(1) + A_1^R(1)}{2} \ge \frac{A_2^L(1) + A_2^R(1)}{2},$$

or (b)

$$[A_2]_r - [A_1]_r \quad if \quad \frac{A_1^L(1) + A_1^R(1)}{2} < \frac{A_2^L(1) + A_2^R(1)}{2}.$$

To consider both notations together an indicator variable λ is used such that

$$\lambda([A_1]_r - [A_2]_\alpha) + (1-\lambda)([A_1]_r - [A_2]_r) = [d_r^L, d_r^R].$$

Where

$$\lambda = \begin{cases} 1, & if \quad \frac{A_1^L(1) + A_1^R(1)}{2} \ge \frac{A_2^L(1) + A_2^R(1)}{2}, \\ 0, & if \quad \frac{A_1^L(1) + A_1^R(1)}{2} < \frac{A_2^L(1) + A_2^R(1)}{2}, \end{cases} \tag{16}$$

Therefore, the fuzzy distance measure between A_1 and A_2 is defined by

$$d(A_1, A_2) = (d_{\alpha=1}^L, d_{\alpha=1}^R; \theta, \sigma), \tag{17}$$

Where

$$\theta = d_{\alpha=1}^L - max\left\{\int_0^1 d_\alpha^L\, d\alpha, 0\right\}, \sigma = \int_0^1 d_\alpha^R\, d\alpha - d_\alpha^R.$$

also in [10], the following metric properties are studied as following:

$(a1): d(A_1, A_2) = (d_{r=1}^L, d_{r=1}^R; \theta, \sigma)$is a positive fuzzy numbers,

$(a2): d(A_1, A_2) = d(A_2, A_1)$,

$(a3): d(A_1, A_3) \le d(A_1, A_2) + d(A_2, A_3)$

3.4 Fuzzy distance given by Shan-Huo Chen et al. [24]

Let A and B be two trapezoidal fuzzy numbers. Then the fuzzy distance of A and B is:

$$d(A,B) = |A - B| \tag{18}$$

3.5 Fuzzy distance given by Chen and Wang [25]

In 2007, Chen and Wang [25] introduced a fuzzy distance of two trapezoidal fuzzy numbers by using the GMIR and the spread of the fuzzy numbers. This new idea gives us two advantages, easy to calculate and easy to understand. Now, we describe the definition again as follows.

Let $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers, and their GMIR are $P(A)$, $P(B)$ respectively. Assume

$$\begin{aligned} P(A) &= \frac{a_1 + 2a_2 + 2a_3 + a_4}{6} \\ P(B) &= \frac{b_1 + 2b_2 + 2b_3 + b_4}{6} \\ s_i &= (a_i - P(A) + b_i - P(B))/2, \quad i = 1,2,3,4 \\ c_i &= |P(A) + P(B) + s_i|/i, \quad i = 1,2,3,4 \end{aligned}$$

then the fuzzy distance of A and B is

$$C = (c_1, c_2, c_3, c_4) \tag{19}$$

3.6 Fuzzy distance given by Hajjari [16]

Let A and B be two trapezoidal fuzzy numbers. The fuzzy distance between A, B denoted by $dist(A,B)$ and it was defined as:

$$dist(A,B) = absf(A - B), \tag{20}$$

where $Absf(A)$ is the fuzzy absolute of the fuzzy number A, i.e.,

$$Asbf(A) = \begin{cases} A, & A \succ 0 \\ 0, & A \approx 0 \\ -A, & A \prec 0 \end{cases} \tag{21}$$

and 0, $A \approx 0$ and $A \prec 0$, stand fuzzy zero number, positive fuzzy number and negative fuzzy number respectively as follows:

$$\begin{aligned} &\forall A \in E : A \succ 0 \Leftrightarrow \textstyle\int_a^\omega (L_A^{-1}(r) + R_A^{-1}(r))\, dr \succ 0, \\ &\forall A \in E : A \approx 0 \Leftrightarrow \textstyle\int_a^\omega (L_A^{-1}(r) + R_A^{-1}(r))\, dr = 0, \\ &\forall A \in E : A \prec 0 \Leftrightarrow \textstyle\int_a^\omega (L_A^{-1}(r) + R_A^{-1}(r))\, dr \prec 0, \end{aligned}$$

also ω be a constant such that $0 \le \omega \le 1$.

3.7 Fuzzy distance given by Guha and Chakraborty [13]

Let us consider two generalized trapezoidal fuzzy number as follows:

$A = (a_1, a_2; \beta_1, \gamma_1; \omega_1), \quad B = (b_1, b_2; \beta_2, \gamma_2; \omega_2)$
r- cut representation of A is denoted by:

$$[A]_r = \lfloor L_A^{-1}(r), R_A^{-1}(r) \rfloor, \quad \text{for} \quad 0 \prec \omega_1 \prec 1.$$

α- cut representation of B is denoted by:

$$[B]_r = \lfloor L_B^{-1}(r), R_B^{-1}(r) \rfloor, \quad \text{for} \quad 0 \prec \omega_2 \prec 1.$$

The r- cut representation of the distance measure between two fuzzy numbers A and B is denoted by $[d_r^L, a_r^R]$ for $r \in [0, \omega]$, $\omega = min(\omega_1, \omega_2)$.

Now the distance between A and B was defined by:

$$\begin{aligned} &d(A,B) = (d_{r=\omega}^L, d_{r=\omega}^R; \theta, \sigma) \\ &\omega = min(\omega_1, \omega_2) \end{aligned} \tag{22}$$

Where θ and σ is defined in the following way:

$$\begin{aligned} \theta &= d_{r=\omega}^L - max\{ \textstyle\int_0^\omega d_r^L\, dr, 0 \} \\ \sigma &= \left| \left[\textstyle\int_0^\omega d_r^R\, dr - d_{r=\omega}^R \right] \right| \end{aligned}$$

3.8 Improved centroid distance method [26]

To overcome the drawback of Cheng'S distance [22] method Abbasbandy and Hajjari [26] improved the centroid distance method. They first introduced a sign function as follows:

Definition 5. *Let E stands the set of non-normal fuzzy numbers, ω be a constant provided that $0 < \omega \le 1$ and $\gamma : E \to \{-\omega, 0, \omega\}$ be a function that is defined as:*

$$\forall A \in E : \gamma(A) = sign\Big[\int_0^\omega (L_A^{-1}(r) + R_A^{-1}(r))\, dx \Big],$$

i.e.

$$\gamma(A) = \begin{cases} 1 & \int_0^\omega (L_A^{-1}(r) + R_A^{-1}(r))\, dx \\ 0 & \int_0^\omega (L_A^{-1}(r) + R_A^{-1}(r))\, dx \\ -1 & \int_0^\omega (L_A^{-1}(r) + R_A^{-1}(r))\, dx \end{cases} \tag{23}$$

It is clear for normal fuzzy numbers $\omega = 1$.

Then they combined Chen's distance [22] method by corrected formulae with sign function and presented the improved centroid distance method as follows:

$$IR(A) = \gamma(A) R(A) \tag{24}$$

in other words

$$IR(A) = \gamma(A) \sqrt{x_A^2 + y_A^2} \tag{25}$$

3.9 Sadi-nezhad et al.'s fuzzy distance measure [27]

Sadi-Nezhad et al. [27] proposed a fuzzy distance measure for two triangular fuzzy numbers based on left and right points. They denoted fuzzy distance between $A = (x_1, x_2, x_3)$ and $B = (y_1, y_2, y_3)$ as $D_{AB} = (d_1, d_2, d_3)$ such that:

$$d_1 = \begin{cases} max\{x_1 - y_3, 0\} & x_2 \geq y_2 \\ max\{y_1 - x_3, 0\} & x_2 \leq y_2 \end{cases}$$

$$d_2 = |x_2 - y_2|$$
$$d_3 = \{max(y_3 - x_1, x_3 - y_1)\}.$$

Consider triangular fuzzy number $A = (0, 1, 2)$ by applying Sadi-Nezhad's method, the distance between A and A is $(0, 0, 2)$ which is unreasonable. In the following, we will present a new method to determine distance between two triangular fuzzy numbers.

4 New fuzzy distance measure

In this part, we present a new method for fuzzy distance measure.

Let $A = (x_1, x_2, x_3)$ and $B = (y_1, y_2, y_3)$ are two triangular fuzzy numbers and distance between A and B is denoted by $Dist_{AB}$ where $Dist_{AB} = (d_1, d_2, d_3)$. We calculate d_1, d_2 and d_3 as follows:

$$d_1 = \begin{cases} |x_1 - y_3| & x_2 \geq y_2 \\ |y_1 - x_3| & x_2 < y_2 \end{cases}$$

$$d_2 = max\{|x_2 - y_2|, d_1\}$$

$$d_3 = \begin{cases} 0 & x_3 = y_3 \\ \{max(y_3 - x_1, x_3 - y_1)\} & x_3 \neq y_3 \end{cases} \tag{26}$$

*Remark.*The proposed method is a metric.

1. $Dist_{AB} \geq 0$
2. $Dist_{AB} = Dist_{BA}$
3. $Dist_{AC} \leq Dist_{AB} + Dist_{BC}$

proof:

1. $Dist_{AB}$ is a positive fuzzy number. Based on definition, the parameters d_1, d_2 and d_3 are non-negative.
2. $Dist_{AB} = Dist_{BA}$, Based on new method it is clear.
3. The triangular inequality: $Dist_{AC} \leq Dist_{AB} + Dist_{BC}$: Let $A = (x_1, x_2, x_3)$, $B = (y_1, y_2, y_3)$ and $C = (z_1, z_2, z_3)$ are three triangular fuzzy numbers and the distance $Dist_{AC}$, $Dist_{AB}$ and $Dist_{BC}$ we should prove that:

$$Dist_{AC} \leq Dist_{AB} + Dist_{BC} \tag{27}$$

Let $Dist_{AC}$, $Dist_{AB}$ and $Dist_{BC}$ be three triangular fuzzy numbers and $Dist_{AB} + Dist_{BC}$ is a triangular fuzzy number too.

*Remark.*Let A is a triangular fuzzy number and O is original point $Dist_{AO} = Dist(A, O) = A$.

proof: Let $A = (x_1, x_2, x_3)$ and $O = (0, 0, 0)$, based on new fuzzy distance measure, proof is clear.

Now we must compare these triangular fuzzy numbers. According to A. kmbur et al. [28] if $A = (x_1, x_2, x_3)$ and $B = (y_1, y_2, y_3)$ be two triangular fuzzy numbers; we can use ranking function $R : F(R) \to R$ for ranking these two numbers that in this function $F(R)$ is a fuzzy set and R is a function that maps a fuzzy set to a real number so:

i. $A \succeq B$ if and only if $R(A) \geq R(B)$
ii.$A \preceq B$ if and only if $R(A) \leq R(B)$
iii.$A \approx B$ if and only if $R(A) = R(B)$

If B is a triangular fuzzy number ranking function can be defined as follows:

$$R(A) = \frac{1}{4}(y_1 + 2y_2 + y_3) \tag{28}$$

Suppose $x_2 \leq y_2 \leq z_2$ then distances will be defined According to our algorithm (we do not consider the equal cases i.e. $x_1 = y_1$, $x_2 = y_2$ and $x_3 = y_3$), The number of different cases is as follows:

$$d_2 = \binom{2}{1}, d_2' = \binom{2}{1}, d_2'' = \binom{2}{1}$$

$$d_3 = \binom{2}{1}, d_3' = \binom{2}{1}, d_3'' = \binom{2}{1}$$

If the combination of $4 * 4 * 4$ equal to 64, then the fuzzy number of $\binom{3}{2}$ could perform in 6 different outcomes. Hence, we could elaborate the $64 * 6 = 364$. Thus, we could prove the following similar hypotheses since they are having same concepts.

We will only prove two of them.

The first mode:

$$Dist_{AB} = (d_1, d_2, d_3) : d_1 = y_1 - x_3, d_2 = |x_2 - y_2|,$$
$$d_3 = y_3 - x_1.$$

$$Dist_{BC} = (d_1', d_2', d_3') : d_1' = z_1 - y_3, d_2' = |z_2 - y_2|,$$
$$d_3' = z_3 - y_1.$$

$$Dist_{AC} = (d_1'', d_2'', d_3'') : d_1'' = z_1 - x_3, d_2'' = |x_2 - z_2|,$$
$$d_3'' = z_3 - x_1.$$

Then we prove that:

$$Dist_{AC} \leq Dist_{AB} + Dist_{BC} \rightarrow$$

$$(z_1 - x_3, |x_2 - z_2|, z_3 - x_1)$$
$$\leq (y_1 - x_3, |x_2 - y_2|, y_3 - x_1)$$
$$+ (z_1 - y_3, |z_2 - y_2|, z_3 - y_1)$$

$Dist_{AC}$, $Dist_{AB}$ and $Dist_{BC}$ are defined as the new method. According to developed method in n-dimension, actually, we will prove the following relation:

$$\frac{1}{4}(z_1 - x_3 + 2|x_2 - z_2| + z_3 - x_1) \leq$$
$$\frac{1}{4}(y_1 - x_3 + 2|x_2 - y_2|$$
$$+ y_3 - x_1 + z_1 - y_3$$
$$2|z_2 - y_2| + z_3 - y_1)$$

From the above relation we have:

$$2|x_2 - z_2| \leq 2|x_2 - y_2| + 2|z_2 - y_2|$$

This unequal is satisfied based on the properties of absolute value.

The second mode:

$$Dist_{AB} = (d_1, d_2, d_3) : d_1 = y_1 - x_3, d_2 = |x_2 - y_2|,$$
$$d_3 = x_3 - y_1.$$

$$Dist_{BC} = (d'_1, d'_2, d'_3) : d'_1 = z_1 - y_3, d'_2 = |z_2 - y_2|,$$
$$d'_3 = y_3 - z_1.$$

$$Dist_{AC} = (d''_1, d''_2, d''_3) : d''_1 = z_1 - x_3, d''_2 = |x_2 - z_2|,$$
$$d''_3 = x_3 - z_1.$$

Now we prove that:

$$Dist_{AC} \leq Dist_{AB} + Dist_{BC} \rightarrow$$

$$(z_1 - x_3, |x_2 - z_2|, x_3 - z_1)$$
$$\leq (y_1 - x_3, |x_2 - y_2|, x_3 - y_1)$$
$$+ (z_1 - y_3, |z_2 - y_2|, y_3 - z_1)$$

from the first mode we have:

$$\frac{1}{4}(z_1 - x_3 + 2|x_2 - z_2| + x_3 - z_1) \leq$$
$$\frac{1}{4}(y_1 - x_3 + 2|x_2 - y_2|$$
$$x_3 - y_1 + z_1 - y_3$$
$$2|z_2 - y_2| + y_3 - z_1)$$

From the above relation we have:

$$2|x_2 - z_2| \leq 2|x_2 - y_2| + 2|z_2 - y_2|$$

This unequal is satisfied based on the properties of absolute value.

The proposed method can be developed in n-dimension. Let A and B are two points in n-Dimensional space with triangular fuzzy number values in each dimensions. The points A and B can be shown as:

$$A = \left(\left(x_1^n, x_2^n, x_3^n\right), n = 1, 2, ..., k\right)$$

$$B = \left(\left(y_1^n, y_2^n, y_3^n\right), n = 1, 2, ..., k\right)$$

$Dist_{AB} = (d_1^n, d_2^n, d_3^n)$ is the distance of the nth component of A from the nth component of B and d_1^n, d_2^n and d_3^n are related to from the left point, the centre and the right point of this distance respectively. The distance between each component can be calculated by the same method. Then we define the total fuzzy distance between and as following:

$$\begin{aligned} DistAB &= DistAB^1 + DistAB^2 + ... + DistAB^k \\ &= (d_1^1 + d_1^2 + ... + d_1^k, d_2^1 + d_2^2 + d_2^k, d_3^1 \\ &+ d_3^2 + ... + d_3^k). \end{aligned} \tag{29}$$

5 Numerical examples

The distance between zero and fuzzy number A by Chakraborty and Chakraborty [10] and Guha and Chakraborty [13] has a drawback, in case that is not A, i.e., $d(A,0) \neq A$, which is not a satisfactory result. But in our method can overcome the shortcoming of aforementioned methods:

$$Dist_{A0} = A$$

*Example 1.*Let $A = (0,0,0)$ and $B = (0,0,0.33)$ are two normalized fuzzy numbers, which are indicated in Fig. 1. The distance measure between A and B by Chakraborty and Chakraborty [10] is $d(A,B) = (0,0,0)$. Also by Guha and Chakraborty [13] obtained: $d(A,B) = (0,0,0.33/2)$. From Sadi-Nezhad's method [27] $D_{AB} = (0,0,0.33)$. By proposed method we get $Dist_{AB} = (0,0,0.33)$ which, the result is reasonable.

*Example 2.*Consider the data used in Sadi-Nezhad et al. [27] i.e. the triangular fuzzy number A_i, $i = 1,2,3,4$ as shown in Table 1.

Table 2 shows the distance results of the $Dist_{ij}$ by the proposed methods and some other approaches. According to Table 2, it is observed the method of Sadi-Nezhad et el. [27], voxman [12] and guha and chakraborty [13] sometimes is unreasonable and not consistent with human intuition. This example shows the proposed method is consistent with the distance obtained by other approaches

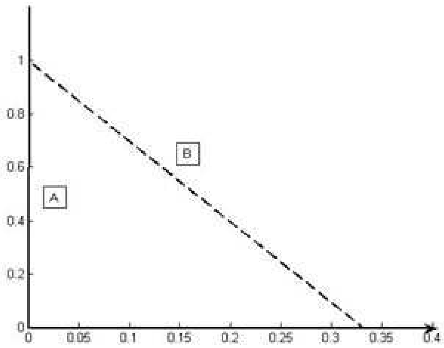

Fig. 1: Two Fuzzy numbers A and B in Example 1

and its advantages.

From table 2, Proposed method is so easier and simpler, which its advantage.

*Example 3.*Consider the data used in Sadi-Nezhad et al. [27] i.e. the two points in a two dimensional space $A = \big((2,3,4),(3,4,5)\big)$ and $B = \big((7,7.5,9),(7,8,9)\big)$ as shown in fig. 2. The distance results by sadi-nezhad's approach [27] same with our method:

$$Dist_{AB} = Dist^1_{AB} + Dist^2_{AB} = (3,4.5,7) + (2,4,6) = (5,8.5,13)$$

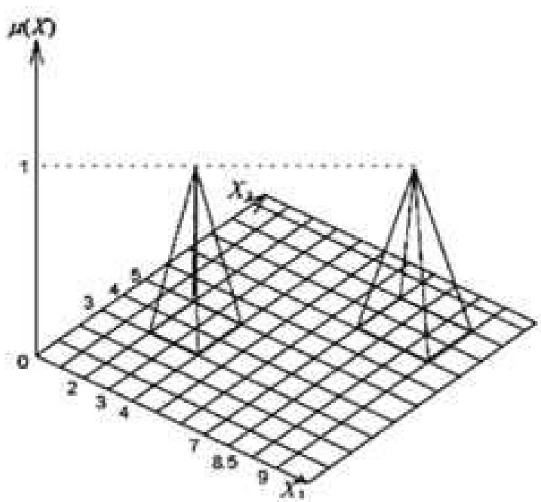

Fig. 2: Two-dimensional fuzzy numbers A and B in Example 4

*Example 4.*Consider the data used in Sadi-Nezhad et al. [27] i.e.

Table 1: Triangular fuzzy numbers A_i of Example 2

i	A_i
1	(0,2,3)
2	(-1,0,1)
3	(-3,-2,-1)
4	(-5,-4,0)

Table 2: Comparative results of Example 2

Case number		$D_{Sadi-Nezhad}$ [27]	D_{Voxman} [12]	$D_{Guhaetal.}$ [13]	$D_{Proposed}$
1	(0,2,3)	(0,2,3)	(0,2,3)	(0,2,3)	(0,2,3)
2	(-1,0,1)	(0,2,3)	(0,2,3)	(0,2,3)	(0,2,3)
3	(-3,-2,-1)	(0,2,3)	(0,2,3)	(0,2,3)	(0,2,3)
4	(-5,-4,0)	(0,2,3)	(0,2,3)	(0,2,3)	(0,2,3)
1	(0,2,3)	(0,2,3)	(0,2,3)	(0,2,3)	(0,2,3)
2	(-1,0,1)	(0,2,3)	(0,2,3)	(0,2,3)	(0,2,3)
3	(-3,-2,-1)	(0,2,3)	(0,2,3)	(0,2,3)	(0,2,3)
4	(-5,-4,0)	(0,2,3)	(0,2,3)	(0,2,3)	(0,2,3)
3	(-3,-2,-1)	(0,2,3)	(0,2,3)	(0,2,3)	(0,2,3)
4	(-5,-4,0)	(0,2,3)	(0,2,3)	(0,2,3)	(0,2,3)

Table 3: Comparison of the proposed method with the other methods in Example 5

Methods	Results
Chen [30]	0.9
Chen and Chen [31]	0.81
Guha and Chakraborty [13]	(0.8,0.9,1)
Adabitabar Firozja et al. [29]	[0.91,0.91]
Proposed Method	(0.1,0.1,0.3)

Let $A = ((1,2,2.5),(2,3.5,4),(-1,-0.5,0),(-2,-1.5,-1))$ and $B = ((1,3,3.5),(2,4,5),(0,0.8,1),(4,5,5.5))$ be two point in a fourth dimensional space. The distance between the two triangular fuzzy numbers in fourth dimensional as following:

The distance results by Sadi Nejad et al. [27] and new method are the same as following:

$$\begin{aligned} Dist_{AB} &= Dist^1_{AB} + Dist^2_{AB} + Dist^3_{AB} + Dist^4_{AB} \\ &= (0,1,2.5) + (0,0.5,3) + (0,1.3,2) \\ &+ (5,6.5,7.5) = (5,9.3,15). \end{aligned}$$

*Example 5.*Consider the two triangular fuzzy numbers as follows: $A = (0.1,0.2,0.3)$ and $B = (0.2,0.3,0.4)$ as in [29]. Table 3. gives a comparative of proposed distance method with some other approaches.

6 Conclusion

Fuzzy distance measure play an important role in image processing under impression, as well as it can be widely used in data mining and pattern recognition. In this paper, we review on some fuzzy distance methods, then we discussed about a distance for the two triangular fuzzy numbers that unlike to the existing methods. We used the left point, centre and right point. The advantage of the proposed method is the calculation is easier than previous one and overcome to drawback of some other methods.

Acknowledgement

This work was partly supported by Islamic Azad University, FiroozKooh Branch.
The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] I. Bloch, On fuzzy distances and their use in image processing under imprecision, Pattern Recognition, vol.32, pp.1873-1895, (1999).
[2] S. H. Chen and C. H. Hsieh, Representation, Ranking, Distance, and Similarity of L-R type fuzzy number and Application, Australia Journal of Intelligent Information Processing Systems, vol.6, no.4, pp.217-229, Australia, (2000).
[3] S. Heilpern, Representation and application of fuzzy numbers, Fuzzy sets and Systems, vol.91, no.2, pp.259-268, (1997).
[4] C.H. Hsieh and S.H. Chen, Graded mean representation distance of generalized fuzzy number, Proceeding of sixth Conference on Fuzzy Theory and its Applications (CD ROM), filename 032.wdl, pp. 1-5, Chinese Fuzzy Systems Association, Taiwan, (1998).
[5] A. Kaufmann and M.M. Gupta, Introduction to fuzzy arithmetic theory and applications, Van Nostrand Reinhold (1991).
[6] X. Liu, Entropy, distance measure and similarity measure of fuzzy sets and their relations, Fuzzy Sets and Systems, vol.52, no.3, pp.305-318, (1992).
[7] W. Pedrycz, Collaborative and knowledge-based fuzzy clustering, International journal of Innovating computing, Information and Control, Vol.3, no.1, pp.1-12, (2007).
[8] P.K. Saha, F. W. Wehrli and B. R. Gomberg, Fuzzy Distance Transform: Theory, Algorithms and Applications, Computer Vision and Image Understanding, vol. 86, pp.171-190, (2002).
[9] L. Tran and L. Duckstein, Comparison of fuzzy numbers using fuzzy distance measure, Fuzzy Sets and Systems, vol.130, pp.231-341, (2002).
[10] C. Chakraborty and D. Chakraborty, A theoretical development on a fuzzy distance measure for fuzzy numbers, Mathematical and Computer Modeling, vol.43, pp.254-261, (2006).
[11] M. A. Jahantigh and S. Hajighasemi, Ranking of generalized fuzzy nimbers using distance measure ans similarity measure, International Journal of Industrial Mathematics, 4, 405-416 (2012).
[12] W. Voxman, Some remarks on distances between fuzzy numbers, Fuzzy Sets and Systems, vol.100, pp.353-365, (1998).
[13] D. Guha and D. Chakraborty, A new approach to fuzzy distance measure and similarity measure between two generalized fuzzy numbers, Applied Soft Computing, vol.10, pp.90-99, (2010).
[14] T. Hajjari, Fuzzy Euclidean Distance for Fuzzy Data, In: 8th Iranian Conference on Fuzzy Systems, Iran, pp. 127-132 (2008).
[15] S. Abbasbandy and S. Hajighasemi, A fuzzy distance between two fuzzy numbers, in Information Processing and Management of Uncertainty in Knowledge-Based Systems, vol. 81 of Applications Communications in Computer and Information Science, pp. 376 - 382 (2010).
[16] T. Hajjari, New Approach for Distance Measure of Fuzzy Numbers, International Conference on Operations Research and Optimization, IPM, Tehran, Iran (2011).
[17] T. Hajjari, and S. Abbasbandy, ”A note on The revised method of ranking LR fuzzy number based on deviation degree”, Expert Syst with Applications, pp. 13491-13492 (2011).
[18] T. Hajjari, Fuzzy Distance Measure for Generalized Fuzzy Numbers, The 11th International Conference on Intelligent Technologies (INTECH), Bangkok, Thailan, Dec. (2010).
[19] H. Rouhparvar, A. Panahi and A. Noorafkan Zanjani Fuzzy Distance Measure for Fuzzy Numbers. Australian Journal of Basic and Applied Sciences, 5(6): p. 258-265 (2011).
[20] D. Dubois and H. Prade , Operation on fuzzy numbers, International Journal of Systems Science, vol.9, pp.613-626 (1978).
[21] L. Fachao, S. Lianqing, Y. Xiangdong and Q. Jiqing, The absolute value of fuzzy number ans its basic properties, The Journal of Fuzzy Mathematics 9, pp403-412 (2001).
[22] C. H. Cheng, A new approach for ranking fuzzy numbers by distance method, Fuzzy Sets and Syst 95, pp. 307-317 (1998).
[23] Y. M. Wang, Yang, J-B., Xu, D-L., and Chin, K-S., On the centroids fuzzy numbers, Fuzzy Sets and Syst. 157, 919 (2006).
[24] SH-H. Chen and Ch-CH. Wang, Fuzzy distance Using Fuzzy Absolute Value, in: Proceeding of the eighth International Conference on Machine Learning and Cybernetics, Baoding, 12-15 (2007).
[25] SH-H. Chen, and Ch-CH. Wang, Fuzzy distance of trapezoidal fuzzy numbers, in: Proceeding of International Conference on Machine Learning and Cyberneting (2007).
[26] S. Abbasbandy and T. Hajjari, An improvement for ranking fuzzy numbers, J. Sci. I.A.U (JSIAU), Vol 20, No. 78/2 (2011).
[27] S. Sadi-Nezhad, A. Noroozi and A. Makui, Fuzzy distance of triangular fuzzy numbers, Journal of Intelligent and Fuzzy Systems (2012).
[28] A. Kumar, N. Bahtia and M. Kuar, A new approach for solving fuzzy maximal flow problems rough sets, fuzzy sets, data mining and granular computing, H. Sakai, et al., Editors, springer Berlin/ Heidelberg, pp. 278-286 (2009).
[29] M. Adabitabar Firozja, G. H. Fath-Tabar, and Z. Eslampia, The similarity measure of generalized fuzzy numbers based on interval distance, Applied Mathematics Letters 25, 1528-1534 (2012).

[30] S. M. Chen, New methods for subjective mental workload assessment and fuzzy risk analysis, Cybernetics and Systems 27, 449-472 (1996).
[31] S. J. Chen, S. M. Chen, Fuzzy risk analysis based on similarity measures of generalized fuzzy numbers, IEEE Transactions on Fuzzy Systems 11, 45-56 (2003).
[32] S. Abbasbandy, and T. Hajjari, A new approach for ranking of trapezoidal fuzzy numbers, Comput. Math. Appl. 57, pp. 413-419 (2009).
[33] T. Hajjari, Ranking Fuzzy numbers by Based on Ambiguity Degree. In: Third International Conference in Mathematical Applications in Engineering, Kuala Lumpur, Malaysia, pp. 9-13 (2010).
[34] T. Hajjari, Ranking of Fuzzy Numbers by Middle of Expected Interval, In: First International Conference in Mathematics and Statistics, Sharjah (2010).
[35] T. Hajjari, Comparison of Fuzzy numbers by Modified Centroid Point Method. In: Third International Conference in Mathematical Sciences Dubai, pp. 1139-1145 (2008).
[36] T. Hajjari, Ranking Fuzzy Numbers by Sign Length, In: 7th Iranian Conference on Fuzzy Systems , Iran, pp. 297-301 (2007).
[37] L. A. Zadeh, Fuzzy set, Information and Control, vol.8, pp.338-353 (1965).

Mostafa Ali Beigi is studying M. Sc. degree in financial mathematics at Central Tehran branch of IAU (Iran). He is referee and Editor of several international journals in the frame of applied mathematics, management, applied economics and art. His research interests are in the areas both of pure and applied mathematics. Moreover, he has published some papers in the areas of fuzzy mathematics, financial mathematics, relationship between mathematics and art, and diving in conferences and journal. He is member of young researchers and elite club..

Tayebeh Hajjari is assistant professor of applied mathematics in Firoozkooh branch of IAU (Iran). She received her PhD in applied mathematics-numerical analysis. Her main research interests are in the areas of applied mathematics such as numerical analysis, fuzzy systems and its applications, neural networks, fuzzy risk analysis. She is referee of several international journals in the frame of applied mathematics and fuzzy systems.

Erfan Ghasem Khani received his BSc degree in Industrial Engineering at Qazvin branch of IAU (Iran). His research interests are: industrial engineering, applied mathematics: fuzzy mathematics. He has published some papers in areas of fuzzy systems, art and economics.