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Approximate Solution of Fuzzy Hammerstein Integral Equation by Using Fuzzy B-Spline Series

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Abstract: In this paper, numerical solution of nonlinear fuzzy Hammerstein integral equations is studied by fuzzy B-spline series. An error bound for the method is found based on modulus of continuity and it is proved that the proposed algorithm is numerically stable. Finally, theoretical results are illustrated by some numerical examples.

Keywords: Fuzzy numbers, Fuzzy B-spline series, Fuzzy Hammerstein integral equation, Numerical methods

1 Introduction

Mathematical modeling of physical phenomena, in the most cases, is resulted in differential or integral equations. These equations possess some numerical parameters that often are referred to the physical properties and geometrical specifications of the phenomenon that their magnitude would not be deterministically known. So, the resulted equation has some fuzzy parameters that impose a kind of fuzzy behavior to the equation. Also, in some other cases, initial or boundary conditions are not crisp quantities and should be presented in fuzzy form. Considering these issues can reveal the great importance of fuzzy differential equations (FDE) and fuzzy integral equations (FIE) topics. A large amount of investigation has been dedicated to this topic especially in recent years from both theoretical and numerical points of view ([1]-[8]).

One of the important cases is the fuzzy Hammerstein integral equation which has the form:

$$u(x) = f(x) + \lambda \int_0^1 k(x,t)\phi(t,u(t))dt, \quad x \in [0,1], \quad (1.1)$$

where $f \in C([0,1],\mathbb{R}_{\mathscr{F}})$, $k \in C([0,1]\times[0,1],\mathbb{R})$ and $\mathbb{R}_{\mathscr{F}}$ denotes the set of fuzzy numbers. Existence and uniqueness of solution for this problem have been investigated by Bica et al in [4], where the authors used Lipschitz conditions to guarantee the existence and uniqueness result. Furthermore the authors used an iterative method which comes from Banach fixed point theorem and in [9], they developed this method to the fuzzy Hammerstein Volterra delay integral equations. In [10], an iterative method was used to solve nonlinear fuzzy integral equations based on quadrature rules. Successive approximations method was used for solving two-dimensional nonlinear fuzzy integral equations in [11]. The method of successive approximations in terms of a hybrid of Taylor series and a block-pulse function for solving nonlinear fuzzy Fredholm integral equation was used in [12].

Although numerical solution of the fuzzy Hammerstein integral equation has been done previously (as it is mentioned in previous paragraph), the solution approach proposed in this paper has some advantages which are mostly originated from the following essential properties of the fuzzy B-spline approximation:

– Having nonnegative values which is a major feature in fuzzy calculus.
– Having compact supports which leads to low computational cost and stability in numerical results.

These properties give us a strong motivation to design a method of solution for nonlinear integral equations based on these functions. The fuzzy B-spline series introduced by Anile et al ([13]) and then studied in details in [14], where, the authors found an efficient error bound of fuzzy B-spline approximation in terms of modulus of continuity and also investigated an uncertainty diminishing property. The interpolation of fuzzy data based on B-spline functions introduced and discussed by Zeinali et al. [15]

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(Other kind of fuzzy piecewise cubic interpolation can be found in [16]). Additional advantages of the proposed method can be summarized as follow:

– Instead of discrete values, results of the numerical solution are presented by some functions (Eq. (3.1)) that are continuous in the whole range of the solution domain.
– Weak conditions for convergence: The only condition to establish convergence of the method is continuity of the kernel.
– Weak conditions for stability: There is no need to the Lipschitz condition (as an strong condition) on f and k functions in order to grantee the stability in the proposed numerical solution

This paper is organized as follows: after a preliminary section, fuzzy B-spline series and it's properties are introduced in section 3. In section 4, the construction of the method is presented. Section 5 is devoted to the convergence of the method. The numerical stability is investigated in section 6 and finally, the efficiency of the method will be examined by some numerical examples in section 7.

2 Preliminaries

In this section, the necessary theorems and definitions are stated that are used later.

Definition 2.1. The function $\mu:\mathbb{R}\to[0,1]$ is called a fuzzy number if:
(i) μ is normal (i.e. $\exists x_0\in\mathbb{R}$ with $\mu(x_0)=1$);
(ii) μ is convex, i.e. $\forall t\in[0,1], x,y\in\mathbb{R}$

$$\mu(tx+(1-t)y)\geq min\{\mu(x),\mu(y)\};$$

(iii) μ is upper semicontinuous on $\mathbb{R}$;
(iv) $\overline{\{x\in\mathbb{R};\mu(x)>0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

The set of all fuzzy real numbers is denoted by $\mathbb{R}_{\mathscr{F}}$. Obviously $\mathbb{R}\subset\mathbb{R}_{\mathscr{F}}$. Here $\mathbb{R}\subset\mathbb{R}_{\mathscr{F}}$ is understood as $\mathbb{R}=\{\chi_x;\ x\ is\ a\ usual\ real\ number\}$.
For $0\leq r\leq 1$, the r-cut of fuzzy number μ is defined by

$$[\mu]^r=\begin{cases}\{x\in R;\mu(x)\geq r\} & 0<r\leq 1\\ \overline{\{x\in R;\mu(x)>0\}} & r=0.\end{cases}$$

Then it is easily shown that μ is a fuzzy number if and only if $[\mu]^r$ is a closed and bounded interval for each $r\in[0,1]$, and $[\mu]^1\neq\emptyset$ (see e.g. [7]).
For $u,v\in\mathbb{R}_{\mathscr{F}}$, and $\lambda\in\mathbb{R}$, the r-cuts of $u+v$ and $\lambda.u$ are defined by $[u+v]^r=[u]^r+[v]^r$ and $[\lambda.u]^r=\lambda[u]^r$, $\forall r\in[0,1]$.

Let $D:\mathbb{R}_{\mathscr{F}}\times\mathbb{R}_{\mathscr{F}}\to\mathbb{R}_+\bigcup\{0\}$, be the Hausdorff distance $D(u,v)=sup_{r\in[0,1]}max\{|u_-^r-v_-^r|,|u_+^r-v_+^r|\}$, where $[u]^r=[u_-^r,u_+^r]$ and $[v]^r=[v_-^r,v_+^r]$. Define $\|.\|=D(.,\tilde{0})$, where $\tilde{0}\in\mathbb{R}_{\mathscr{F}}$, $\tilde{0}=\chi_{\{0\}}$. Then the following properties are satisfied (see [17]):
(i) $(\mathbb{R}_{\mathscr{F}},D)$ is a complete metric space,
(ii) $D(u+v,u+\omega)=D(v,\omega)$,
(iii) $D(k.u,k.v)=|k|D(u,v)$,
(iv) $D(u+v,\omega+e)\leq D(u,\omega)+D(v,e)$.

Definition 2.2. For $f:[a,b]\to\mathbb{R}_{\mathscr{F}}$, the function $\omega(f,.):\mathbb{R}_+\to\mathbb{R}$ given by

$$\omega(f,\delta)=\sup\{D(f(x),f(y))|x,y\in[a,b],|x-y|\leq\delta\}$$

is called the modulus of continuity of f.

Definition 2.3. ([17]) Let $f:[a,b]\to\mathbb{R}_{\mathscr{F}}$, $\delta:[a,b]\to\mathbb{R}_+$, and $\Delta_n: a=x_0<x_1<\ldots<x_n=b$ be a partition of the interval $[a,b]$ with the intermediate points $\psi_i\in[x_{i-1},x_i]$. The partition $P=\{([x_{i-1},x_i];\psi_i);i=1,\ldots,n\}$ denoted by $P=(\Delta_n,\psi)$ is called δ-fine iff $[x_{i-1},x_i]\subseteq(\psi_i-\delta(\psi_i),\psi_i+\delta(\psi_i))$.

Definition 2.4. ([17]) The function f is called Henstock integrable if for every $\varepsilon>0$, there exists a function $\delta:[a,b]\to\mathbb{R}_+$ such that for any δ-fine partition P, we have $D(\sum_{i=1}^n(x_i-x_{i-1}).f(\psi_i),A)\leq\varepsilon$ for some $A\in\mathbb{R}_{\mathscr{F}}$. Then A is called the Henstock integral of f and denoted by $(FH)\int_a^b f(t)dt$.

The integrals used in this paper are in the sense of fuzzy Riemann integral which is a particular case of the fuzzy Henstock integral.

Lemma 2.5. ([18]) **(i)** Let f and g be Henstock integrable functions and let $D(f(t),g(t))$ be Lebesgue integrable. Then

$$D\left((FH)\int_a^b f(t)dt,(FH)\int_a^b g(t)dt\right)\leq L\int_a^b D(f(t),g(t))dt.$$

(ii) Let the function $f:[a,b]\to\mathbb{R}_{\mathscr{F}}$ be a Henstock integrable and bounded function. Then for every fixed point $u\in[a,b]$, the function $\phi_u:[a,b]\to\mathbb{R}_+$ defined by $\phi_u(t)=D(f(u),f(t))$ is Lebesgue integrable on $[a,b]$.

3 Approximation by fuzzy B-spline series

Let $\pi: 0=t_0<t_1<\ldots<t_n=1$ be a strictly increasing nodes on $[0,1]$ and $S_3(\pi)$ denotes the space of polynomial splines of order 4 on this partition. Here, B-spline bases for this space is introduced. Let $t_i=\frac{i}{n}$ and introduce 6 additional knots as $t_{-3}<t_{-2}<t_{-1}<t_0$ and $t_{n+3}>t_{n+2}>t_{n+1}>t_n$. Then, the functions $B_i(t)$ defined by

$$B_i(t)=\begin{cases}\frac{(t-t_{i-2})^3}{6h^3} & t\in[t_{i-2},t_{i-1}]\\ \frac{1}{6}+\frac{1}{2h}(t-t_{i-1})+\frac{1}{2h^2}(t-t_{i-1})^2 \\ -\frac{1}{2h^3}(t-t_{i-1})^3 & t\in[t_{i-1},t_i]\\ \frac{1}{6}+\frac{1}{2h}(t_{i+1}-t)+\frac{1}{2h^2}(t_{i+1}-t)^2 \\ -\frac{1}{2h^3}(t_{i+1}-t)^3 & t\in[t_i,t_{i+1}]\\ \frac{(t_{i+2}-t)^3}{6h^3} & t\in[t_{i+1},t_{i+2}]\\ 0 & otherwise\end{cases}$$

for $i=-1,...,n+1$ is called the B-spline functions of order 4.

Theorem 3.1. ([19]) Dim $S_3(\pi)=n+3$ and $\{B_{-1},B_{-2},...,B_{n+1}\}$ constitute a basis for $S_3(\pi)$.

Definition 3.2. ([14]) Let $\xi_j\in[0,1]\cap suppB_j$, $j=-1,...,n+1$. Then the fuzzy B-spline series for the function f will be

$$S(f,x)=\sum_{j=-1}^{n+1}B_j(x)f(\xi_j). \tag{3.1}$$

Theorem 3.3. ([14]) For $f:[0,1]\to R_{\mathscr{F}}$ continuous we have

$$D(f(x),S(f,x))\le 4\omega(f,\delta),$$

where $\delta=\max_{0\le j\le n}(t_{j+1}-t_j)$ and $\omega(f,\delta)$ is the modulus of continuity of the function f.

4 The New Method

I rewrite Eq.(1.1) as

$$u(x)=f(x)+\lambda\int_0^1 k(x,t)L(t)dt, \tag{4.1}$$

with

$$L(t)=\phi(t,u(t)),$$

which yields

$$L(t)=\phi\left(t,f(t)+\lambda\int_0^1 k(t,s)L(s)ds\right). \tag{4.2}$$

Approximating $L(s)$ by (3.1), I get

$$L(s)\simeq S(L,s)=\sum_{j=-1}^{n+1}L(\xi_j)B_j(s), \tag{4.3}$$

where $\xi_j\in[0,1]\cap suppB_j$. Then, by substituting $S(L,s)$ in (4.1) the approximate solution for the Eq. (1.1) will be obtained from

$$u_n(x)=f(x)+\lambda\sum_{j=-1}^{n+1}L(\xi_j)\int_0^1 k(x,t)B_j(t)dt.$$

Therefore, it suffices to determine $L(\xi_j)$. Let

$$\xi_j=\begin{cases}t_j, & j=0,...,n\\ t_0, & j=-1\\ t_n, & j=n+1\end{cases}$$

which is belong to $[0,1]\cap suppB_j$. From (4.2) and (4.3), I have

$$L(t)\simeq\phi\left(t,f(t)+\lambda\sum_{i=-1}^{n+1}L_i\int_0^1 k(t,s)B_i(s)ds\right), \tag{4.4}$$

where $L_i=L(t_i), i=0,...,n$, $L_{-1}=L_0$ and $L_{n+1}=L_n$. Setting $t=t_k$ for $k=0,...,n$, in (4.4), a nonlinear system obtains for $L_j=L(t_j)$ as

$$L_k=\phi\left(t_k,f(t_k)+\lambda\sum_{i=-1}^{n+1}L_i\int_0^1 k(t_k,t)B_i(t)dt\right), \quad k=0,..n. \tag{4.5}$$

I should now prove that the system (4.5) has a unique solution.

Definition 4.1. We denote by

$$\mathbb{R}^n_{\mathscr{F}}=\{(x_1,x_2,...,x_n);x_i\in\mathbb{R}_{\mathscr{F}},i=1,2,...,n\}$$

the n-dimensional fuzzy space equipped with the distance $\mathscr{D}:\mathbb{R}^n_{\mathscr{F}}\times\mathbb{R}^n_{\mathscr{F}}\to\mathbb{R}^+\cup\{0\}$, defined by

$$\mathscr{D}(X,Y)=\max_{1\le i\le n}D(x_i,y_i),$$

where $X=(x_1,x_2,...,x_n)$ and $Y=(y_1,y_2,...,y_n)$.

Obviously $(\mathbb{R}^n_{\mathscr{F}},\mathscr{D})$ is a complete metric space.

Theorem 4.2. Suppose that ϕ is a Lipschitz function with respect to the second variable with Lipschitz constant L_ϕ, k is a continuous function and

$$M_k=\max\{|k(s,t)|:(s,t)\in[0,1]\times[0,1]\}.$$

Then the system (4.5) has a unique solution.

Proof. Define the operator $T:\mathbb{R}^{n+1}_{\mathscr{F}}\to\mathbb{R}^{n+1}_{\mathscr{F}}$ by

$$T(L)=(\phi_0(L),...,\phi_n(L)),$$

where

$$\phi_k(L)=\phi(t_k,f(t_k))+\lambda\sum_{i=-1}^{n+1}L_i\int_0^1 k(t_k,t)B_i(t)dt,$$

in which I set $L_{-1}:=L_0$ and $L_{n+1}:=L_n$. I claim that T is a contractive mapping. To prove this, let $L^1,L^2\in\mathbb{R}^{n+1}_{\mathscr{F}}$, where

$L^1=(L^1_0,L^1_1,...,L^1_n)$
$L^2=(L^2_0,L^2_1,...,L^2_n)$

and define

$$L^1_{-1}:=L^1_0,\quad L^1_{n+1}:=L^1_n,$$
$$L^2_{-1}:=L^2_0,\quad L^2_{n+1}:=L^2_n.$$

Then,

$$
\begin{aligned}
\mathscr{D}(T(L^1),T(L^2)) &= \max_{0\le k\le n} D\left(\phi_k(L^1),\phi_k(L^2)\right)\\
&= \max_{0\le k\le n} D\Bigg(\phi(t_k,f(t_k)+\lambda\sum_{i=-1}^{n+1}L_i^1\int_0^1 k(t_k,t)B_i(t)dt),\\
&\qquad \phi(t_k,f(t_k)+\lambda\sum_{i=-1}^{n+1}L_i^2\int_0^1 k(t_k,t)B_i(t)dt)\Bigg)\\
&\le \max_{0\le k\le n} L_\phi D\Bigg(f(t_k)+\lambda\sum_{i=-1}^{n+1}L_i^1\int_0^1 k(t_k,t)B_i(t)dt,\\
&\qquad f(t_k)+\lambda\sum_{i=-1}^{n+1}L_i^2\int_0^1 k(t_k,t)B_i(t)dt\Bigg)\\
&= \max_{0\le k\le n} L_\phi|\lambda|D\Bigg(\sum_{i=-1}^{n+1}L_i^1\int_0^1 k(t_k,t)B_i(t)dt,\\
&\qquad \sum_{i=-1}^{n+1}L_i^2\int_0^1 k(t_k,t)B_i(t)dt\Bigg)\\
&\le \max_{0\le k\le n} L_\phi|\lambda|\sum_{i=-1}^{n+1}D\Bigg(L_i^1\int_0^1 k(t_k,t)B_i(t)dt,\\
&\qquad L_i^2\int_0^1 k(t_k,t)B_i(t)dt\Bigg)\\
&= \max_{0\le k\le n} L_\phi|\lambda|\sum_{i=-1}^{n+1}|\int_0^1 k(t_k,t)B_i(t)dt|D\left(L_i^1,L_i^2\right)\\
&\le L_\phi|\lambda|\max_{0\le k\le n}\sum_{i=-1}^{n+1}\int_0^1 |k(t_k,t)|B_i(t)dt\,\mathscr{D}(L^1,L^2)\\
&\le \mathscr{D}(L^1,L^2)L_\phi M_k|\lambda|\sum_{i=-1}^{n+1}\int_0^1 B_i(t)dt\\
&= \mathscr{D}(L^1,L^2)L_\phi M_k|\lambda|6h^4(n+3),
\end{aligned}
\tag{4.6}
$$

since $\int_0^1 B_i(t)dt \le 6h^4$.
On the other hand, for $n\ge 3$, I have

$$
\begin{aligned}
6|\lambda|L_\phi M_k h^4(n+3) &= 6|\lambda|L_\phi M_k\frac{n+3}{n^4}\\
&\le 6|\lambda|L_\phi M_k\frac{2n}{n^4}\\
&= \frac{12|\lambda|L_\phi M_k}{n^3}\\
&= 12|\lambda|L_\phi M_k h^3.
\end{aligned}
\tag{4.7}
$$

From (4.6) and (4.7), I obtain

$$
\mathscr{D}(T(L^1),T(L^2)) \le 12|\lambda|L_\phi M_k h^3\mathscr{D}(L^1,L^2) \tag{4.8}
$$

So, for $n>\sqrt[3]{12|\lambda|L_\phi M_k}$, T is a contraction mapping and the Banach fixed point theorem completes the proof.

Furthermore, the Banach fixed point theorem offers an iteration method to find successive approximations, i.e.

$$
\begin{aligned}
L_k^l = \phi\Bigg(& t_k, f(t_k)+\\
&\lambda(L_0^{l-1}(\int_0^1 k(t_k,t)B_{-1}(t)dt+\int_0^1 k(t_k,t)B_0(t)dt)\\
&+\sum_{i=1}^{n-1}L_i^{l-1}\int_0^1 k(t_k,t)B_i(t)dt\\
&+L_n^{l-1}(\int_0^1 k(t_k,t)B_n(t)dt+\int_0^1 k(t_k,t)B_{n+1}(t)dt))\Bigg)
\end{aligned}
\tag{4.9}
$$

for $l=1,2,\ldots$ and $L^0=(L_1^0,L_2^0,\ldots,L_n^0)$. Let $\tilde{L}:=L^m=(\tilde{L}_0,\ldots,\tilde{L}_n)$ be the approximation obtained from m-th iteration of (4.9) and define $\tilde{L}_{-1}:=\tilde{L}_0$ and $\tilde{L}_{n+1}:=\tilde{L}_n$. Then, the approximate solution of Eq. (1.1) takes the form

$$
\tilde{u}_n(x)=f(x)+\lambda\sum_{i=-1}^{n+1}\tilde{L}_i\int_0^1 k(x,t)B_i(t)dt. \tag{4.10}
$$

5 Convergence

In this section, I try to find an upper bound for the distance

$$E_n = D(u,\tilde{u}_n)$$

where u and $\tilde{u}_n$ denote the exact and the approximate solutions of Eq. (1.1), respectively.

Theorem 5.1. Let k and M_k be as in Theorem 4.2. Then

$$
D(u,\tilde{u}_n)\le|\lambda|M_k\Bigg(4\omega(L,h)+\frac{(12|\lambda|M_k h^3)^m}{1-12|\lambda|M_k h^3}\mathscr{D}(L^0,L^1)6h^4(n+3)\Bigg).
$$

Proof.

$$
E_n = D(u,\tilde{u}_n)\le D(u,u_n)+D(u_n,\tilde{u}_n), \tag{5.1}
$$

where

$$
\begin{aligned}
D(u,u_n) &= \sup_{x\in[0,1]} D\Big(u(x),u_n(x)\Big)\\
&= \sup_{x\in[0,1]} D\Bigg(f(x)+\lambda\int_0^1 k(x,t)L(t)dt, f(x)\\
&\quad +\lambda\int_0^1 k(x,t)S(L,t)dt\Bigg)\\
&\le |\lambda|\sup_{x\in[0,1]} D\Bigg(\int_0^1 k(x,t)L(t)dt, \int_0^1 k(x,t)S(L,t)dt\Bigg)\\
&\le |\lambda|\sup_{x\in[0,1]}\int_0^1 D\Big(k(x,t)L(t),k(x,t)S(L,t)\Big)dt\\
&\le |\lambda|\sup_{x\in[0,1]}\int_0^1 |k(x,t)|D\Big(L(t),S(L,t)\Big)dt.
\end{aligned}
$$

By Theorem 3.3, I get

$$D(u,u_n) \le |\lambda| M_k 4\omega(L,h). \tag{5.2}$$

On the other hand,

$$\begin{aligned}
D(u_n,\tilde{u}_n) &= \sup_{x\in[0,1]} D(u_n(x),\tilde{u}_n(x))\\
&= \sup_{x\in[0,1]} D\Big(f(x)+\lambda \sum_{j=-1}^{n+1} L_j \int_0^1 k(x,t)B_j(t)dt,\\
&\quad f(x)+\lambda \sum_{j=-1}^{n+1} \tilde{L}_j \int_0^1 k(x,t)B_j(t)dt\Big)\\
&\le |\lambda| \sum_{j=-1}^{n+1} \sup_{x\in[0,1]} D\Big(L_j \int_0^1 k(x,t)B_j(t)dt,\\
&\qquad \tilde{L}_j \int_0^1 k(x,t)B_j(t)dt\Big)\\
&\le |\lambda| \sum_{j=-1}^{n+1} \sup_{x\in[0,1]} |\int_0^1 k(x,t)B_j(t)dt| D(L_j,\tilde{L}_j)\\
&\le |\lambda| M_k \sum_{j=-1}^{n+1} D(L_j,\tilde{L}_j) \int_0^1 B_j(t)dt. \qquad (5.3)
\end{aligned}$$

Since $\tilde{L} = L^m$, the Banach fixed point theorem and inequality (4.8) yield

$$\mathscr{D}(L,\tilde{L}) = \mathscr{D}(L,L^m) \le \frac{(12|\lambda|M_k h^3)^m}{1-12|\lambda|M_k h^3}\mathscr{D}(L^0,L^1),$$

thus

$$D(L_j,\tilde{L}_j) \le \mathscr{D}(L,\tilde{L}) \le \frac{(12|\lambda|M_k h^3)^m}{1-12|\lambda|M_k h^3}\mathscr{D}(L^0,L^1) \tag{5.4}$$

and so from (5.1), (5.2), (5.3) and (5.4) I get

$$\begin{aligned}
D(u,\tilde{u}_n) &\le |\lambda| M_k \Big(4\omega(L,h)\\
&\quad + \frac{(12|\lambda|M_k h^3)^m}{1-12|\lambda|M_k h^3}\mathscr{D}(L^0,L^1) \sum_{j=-1}^{n+1} \int_0^1 B_j(t)dt\Big)\\
&\le |\lambda| M_k \Big(4\omega(L,h)\\
&\quad + \frac{(12|\lambda|M_k h^3)^m}{1-12|\lambda|M_k h^3}\mathscr{D}(L^0,L^1) \sum_{j=-1}^{n+1} 6h^4\Big)\\
&= |\lambda| M_k \Big(4\omega(L,h)\\
&\quad + \frac{(12|\lambda|M_k h^3)^m}{1-12|\lambda|M_k h^3}\mathscr{D}(L^0,L^1) 6h^4(n+3)\Big).
\end{aligned}$$

6 Numerical stability

To validate a numerical method, it is needed to investigate it's numerical stability. A general technique for establishing the numerical stability of an algorithm, has been used in [4], in which the authors investigate the influence of small perturbation of the first iteration in the final approximation. The same technique is used in this section. For this purpose, suppose that Γ^0 is another starting value for iterations (4.9), where $D(L^0,\Gamma^0) \le \varepsilon$ and according to this starting value, I have

$$\begin{aligned}
\Gamma_k^l = \phi\Big(& t_k, f(t_k) +\\
& \lambda(\Gamma_0^{l-1}(\int_0^1 k(t_k,t)B_{-1}(t)dt + \int_0^1 k(t_k,t)B_0(t)dt)\\
& + \sum_{i=1}^{n-1} \Gamma_i^{l-1} \int_0^1 k(t_k,t)B_i(t)dt\\
& + \Gamma_n^{l-1}(\int_0^1 k(t_k,t)B_n(t)dt + \int_0^1 k(t_k,t)B_{n+1}(t)dt))\Big)
\end{aligned}$$

for $k = 0,..n$. Then the approximate solution corresponding to $\tilde{\Gamma}$ will be

$$\hat{u}_n(x) = f(x) + \lambda \sum_{i=-1}^{n+1} \tilde{\Gamma}_i \int_0^1 k(x,t)B_i(t)dt,$$

where $\tilde{\Gamma}_{n+1} := \tilde{\Gamma}_n$ and $\tilde{\Gamma}_{-1} := \tilde{\Gamma}_0$. By the following theorem, I prove stability of the iterative method (4.9).

Theorem 6.1. Let L^0 and Γ^0 be two different starting values for the iterative method (4.9), such that $D(L^0,\Gamma^0) \le \varepsilon$. Then

$$D(\tilde{u},\hat{u}) \le \frac{(12L_\phi|\lambda|M_k h^3)^{m+1}}{L_\phi}\varepsilon.$$

Proof. According to (4.9), for $j = 0,...,n$, I have

$$L_j^1 = \phi(t_j, f(t_j) + \lambda \sum_{i=-1}^{n+1} L_i^0 \int_0^1 k(t_j,t)B_i(t)dt),$$

$$\Gamma_j^1 = \phi(t_j, f(t_j) + \lambda \sum_{i=-1}^{n+1} \Gamma_i^0 \int_0^1 k(t_j,t)B_i(t)dt).$$

Hence

$$\begin{aligned}
D(L_j^1,\Gamma_j^1) &= D\Big(\phi(t_j, f(t_j) + \lambda \sum_{i=-1}^{n+1} L_i^0 \int_0^1 k(t_j,t)B_i(t)dt)\\
&\quad ,\phi(t_j, f(t_j) + \lambda \sum_{i=-1}^{n+1} \Gamma_i^0 \int_0^1 k(t_j,t)B_i(t)dt)\Big)\\
&\le L_\phi |\lambda| \sum_{i=-1}^{n+1} |\int_0^1 k(t_j,t)B_i(t)dt| D(L_i^0,\Gamma_i^0)\\
&\le L_\phi |\lambda| \sum_{i=-1}^{n+1} M_k 6h^4 D(L^0,\Gamma^0)\\
&= L_\phi |\lambda| M_k 6h^4 D(L^0,\Gamma^0)(n+3)\\
&\le 12 L_\phi |\lambda| M_k h^3 \varepsilon
\end{aligned}$$

and so

$$D(L^1,\Gamma^1) \leq 12L_\phi|\lambda|M_k h^3\varepsilon,$$

which implies (by induction) that

$$D(L_j^m,\Gamma_j^m) \leq (12L_\phi|\lambda|M_k h^3)^m\varepsilon. \tag{6.1}$$

Now, I should conclude the upper bound for

$$D(\tilde{u},\hat{u}) = \sup_{s\in[0,1]} D(\tilde{u}(s),\hat{u}(s)). \tag{6.2}$$

I have

$$\begin{aligned}
D(\tilde{u}(s),\hat{u}(s)) &= D\Big(f(s)+\lambda\sum_{i=-1}^{n+1}\tilde{L}_i\int_0^1 k(s,t)B_i(t)dt \\
&\qquad ,f(s)+\lambda\sum_{i=-1}^{n+1}\tilde{\Gamma}_i\int_0^1 k(s,t)B_i(t)dt\Big)\\
&\leq |\lambda|\sum_{i=-1}^{n+1} D\Big(\tilde{L}_i\int_0^1 k(s,t)B_i(t)dt \\
&\qquad ,\tilde{\Gamma}_i\int_0^1 k(s,t)B_i(t)dt\Big)\\
&\leq |\lambda|\sum_{i=-1}^{n+1}|\int_0^1 k(s,t)B_i(t)dt|D(\tilde{L}_i,\tilde{\Gamma}_i)\\
&\leq |\lambda|M_k 6h^4\sum_{i=-1}^{n+1} D(\tilde{L}_i,\tilde{\Gamma}_i)\\
&\leq |\lambda|M_k 6h^4\sum_{i=-1}^{n+1} D(\tilde{L},\tilde{\Gamma})\\
&= |\lambda|M_k 6h^4(n+3)D(\tilde{L},\tilde{\Gamma}).
\end{aligned}$$

For $n\geq 3$, I have

$$D(\tilde{u}(s),\hat{u}(s)) \leq 12|\lambda|M_k D(\tilde{L},\tilde{\Gamma})h^3. \tag{6.3}$$

If $\tilde{L}=L^m$ and $\tilde{\Gamma}=\Gamma^m$, then from (6.2), (6.3) and (6.1), I get

$$D(\tilde{u},\hat{u}) \leq \frac{(12L_\phi|\lambda|M_k h^3)^{m+1}}{L_\phi}\varepsilon,$$

which completes the proof.

Corollary 6.2. According to the Theorem 6.1, the method will be numerical stable if $h<\frac{1}{\sqrt[3]{12L_\phi|\lambda|M_k}}$.

7 Examples

In this section, some numerical examples are given to show the efficiency of proposed method. I choose the points $x_j; j=1,2,\ldots,N$ arbitrary in $[a,b]$ and report the errors in these points. The results are shown in tables 1,2, where

$$E_-^r = \max_{0\leq j\leq N}|u_-^r(x_j)-\tilde{u}_-^r(x_j)|$$

$$E_+^r = \max_{0\leq j\leq N}|u_+^r(x_j)-\tilde{u}_+^r(x_j)|.$$

Example 1. Consider the linear fuzzy Fredholm integral equation

$$u(x) = f(x)+\int_0^1 k(x,t)u(t)dt, \qquad x\in[0,1],$$

where

$$f^r(x) = [(\frac{r+1}{3})x,(\frac{3-r}{3})x],$$
$$k(x,t) = xt,$$

with the exact fuzzy solution $u^r(x)=[(\frac{r+1}{2})x,(\frac{3-r}{2})x]$.

Table 1: Numerical results of Example 1 for N=10.

n	E_-^0	E_+^0	$E_-^{1/2}$	$E_+^{1/2}$	E_-^1
10	$2.99e^{-4}$	$8.99e^{-4}$	$4.49e^{-4}$	$7.49e^{-4}$	$5.99e^{-4}$
20	$7.65e^{-5}$	$2.29e^{-4}$	$1.14e^{-4}$	$1.91e^{-4}$	$1.53e^{-4}$
30	$3.42e^{-5}$	$1.02e^{-4}$	$5.13e^{-5}$	$8.56e^{-5}$	$6.85e^{-5}$
40	$1.93e^{-5}$	$5.80e^{-5}$	$2.90e^{-5}$	$4.83e^{-5}$	$3.86e^{-5}$
50	$1.23e^{-5}$	$3.71e^{-5}$	$1.85e^{-5}$	$3.10e^{-5}$	$2.47e^{-5}$

Example 2. Consider the fuzzy Hammerstein integral equation

$$u(x) = f(x)+\int_0^1 k(x,t)u^2(t)dt, \qquad x\in[0,1],$$

where

$$\begin{aligned}
f^r(x) &= \Big[e^{x-0.1(1-r)}-e^{x-0.2(1-r)}(\frac{e-1}{4})\\
&\qquad ,e^{x+0.1(1-r)}-e^{x+0.2(1-r)}(\frac{e-1}{4})\Big],\\
k(x,t) &= \frac{1}{4}e^{x-t},
\end{aligned}$$

with the exact solution $u^r(x)=[e^{x-0.1(1-r)},e^{x+0.1(1-r)}]$.

Table 2: Numerical results of Example 2 for N=10.

n	E_-^0	E_+^0	$E_-^{1/2}$	$E_+^{1/2}$	E_-^1
10	$2.51e^{-2}$	$1.89e^{-1}$	$3.44e^{-2}$	$8.86e^{-2}$	$5.12e^{-2}$
20	$6.23e^{-3}$	$2.45e^{-2}$	$8.39e^{-3}$	$1.88e^{-2}$	$1.19e^{-2}$
30	$2.60e^{-3}$	$3.57e^{-3}$	$3.64e^{-3}$	$7.25e^{-3}$	$4.71e^{-3}$
40	$1.50e^{-3}$	$2.23e^{-3}$	$2.42e^{-3}$	$5.32e^{-3}$	$1.58e^{-3}$
50	$7.47e^{-4}$	$6.53e^{-3}$	$1.85e^{-4}$	$1.70e^{-4}$	$4.56e^{-4}$

8 Conclusion and open problem

In this paper, approximate solution of fuzzy Hammerstein integral equation has been studied based on fuzzy

B-spline series along with an error bound for the constructed method. I have proved that the method is numerically stable. The error bound of the method is independent of the Lipschitz constant which is a strong condition and only continuity of the kernel is sufficient. In our future research, I will try to use the proposed algorithm for the system of fuzzy Hammerstein integral equations.

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