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# Existence and Uniqueness of Solutions for a Class of Fractional Differential Coupled System with Integral Boundary Conditions

*Jackie Harjani, Juan Rocha and Kishin Sadarangani*[*]

Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain.

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**Abstract:** In this paper, we prove the existence and uniqueness of solutions for a coupled system of fractional differential equations with integral boundary conditions. Our analysis relies on a generalized coupled fixed point theorem in the space of the continuous functions defined on $[0,1]$. An example is also presented to illustrate the obtained results.

**Keywords:** Fixed point, fractional boundary value problem, fractional differential coupled system.

## 1 Introduction

Fractional differential equations have recently been studied by a lot of number of researchers due to the fact that they are valuable tools in the mathematical modelling of many phenomena appearing in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, economics, control theory, signal and image processing, etc... For detail see [1,2,3,4,5] and the references therein.
Integral boundary conditions have various applications in thermoelasticity, chemical engineering, population dynamics, etc.. For a detailed description of the integral boundary conditions we refer to the reader to some recent papers [6,7,8,9,10].
Recently, in [7] the authors investigated the existence of positive solutions for the following fractional boundary value problem

$$\begin{cases} {}^cD^{\alpha}u(t)+f(t,u(t))=0\,, \quad 0<t<1 \\ u(0)=u''(0)=0, u(1)=\lambda\int_0^1 u(s)\mathrm{d}s\,, \end{cases} \tag{1}$$

where $2<\alpha<3$, $0<\lambda<2$, ${}^cD^{\alpha}$ is the Caputo fractional derivative and $f\colon[0,1]\times[0,\infty)\to[0,\infty)$ under certain assumptions. The main tool used in [7] is the well known Guo-Krasnoselkii fixed point theorem. In [7], the question of uniqueness of solutions is not studied.
In [9], the authors studied Problem (1) by using a fixed point theorem in partially ordered metric spaces and they obtained uniqueness of solutions for Problem (1). In this paper, we study the existence and uniqueness of solutions of the following coupled system of fractional differential equations with integral boundary conditions

$$\begin{cases} {}^cD^{\alpha}u(t)+f(t,u(t),v(t))=0\,, \quad 0<t<1 \\ {}^cD^{\alpha}v(t)+f(t,u(\rho t),v(\rho t))=0\,, \quad 0<t<1 \\ u(0)=u''(0)=0, u(1)=\lambda\int_0^1 u(s)\mathrm{d}s\,, \\ v(0)=v''(0)=0, v(1)=\lambda\int_0^1 v(s)\mathrm{d}s\,, \end{cases} \tag{2}$$

where $2<\alpha<3$, $0<\lambda<2$, and $0<\rho<1$.
The study of a coupled system of fractional differential equations is also very significant because this kind of system can often occur in applications and problem in connection with the real world. For more details, the reader is referred to the paper [11,12,13,14].
The main tool in our study is a coupled fixes point theorem for weakly contractive mappings, due to Rhoades [15].

## 2 Preliminaries and basic facts

For the convenience of the reader, we present in this section some notations and results which will be used in the proofs of our results.

[*] Corresponding author e-mail: ksadaran@dma.ulpgc.es

**Definition 1.** *The Caputo derivative of fractional order $\alpha > 0$ of a function $f\colon [0,\infty) \to \mathbb{R}$ is defined by*

$$^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{n)}(s)\mathrm{d}s,$$

*where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of $\alpha$, provided that the integral exists.*

**Definition 2.** *The Riemman-Liouville fractional integral of order $\alpha > 0$ of a function $f\colon (0,\infty) \to \mathbb{R}$ is defined by*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s,$$

*provided that such integral exists.*

For more details, see [1,2,3]. The next theorem appears in [7].

**Theorem 1.** *[7] Let $2 < \alpha < 3$ and $\lambda \neq 2$. Suppose that $f \in \mathscr{C}[0,1]$ then the unique solution of*

$$\begin{cases} ^{c}D^{\alpha}u(t) + f(t) = 0\,, \quad 0 < t < 1 \\ u(0) = u''(0) = 0, u(1) = \lambda \displaystyle\int_0^1 u(s)\mathrm{d}s\,, \end{cases}$$

*is*

$$u(t) = \int_0^1 G(t,s) f(s)\mathrm{d}s,$$

*where*

$$G(t,s) = \frac{1}{(2-\lambda)\Gamma(\alpha+1)} \begin{cases} 2t(1-s)^{\alpha-1}(\alpha-\lambda+\lambda) \\ \quad -(2-\lambda)\alpha(t-s)^{\alpha-1}, \\ \quad \text{if } 0 \le s \le t \le 1 \\ 2t(1-s)^{\alpha-1}(\alpha-\lambda+\lambda), \\ \quad \text{if } 0 \le t \le s \le 1. \end{cases}$$

*Moreover, the Green's function $G(t,s)$ satisfies*

*(i) $G(t,s) > 0$ for any $t,s \in (0,1)$ if and only if $\lambda \in [0,2)$.*
*(ii) $G(t,s) \le \frac{2}{(2-\lambda)\Gamma(\alpha)}$ for any $t,s \in [0,1]$ and $\lambda \in [0,2)$.*
*(iii) $G(t,s)$ is continuous function on $[0,1]\times[0,1]$, for $2 < \alpha < 3$ and $\lambda \neq 2$.*

The following fixed point theorem which appears in [15] will be a crucial tool in our study.

**Theorem 2.** *[15] Let $(X,d)$ be a complete metric space and $T\colon X \to X$ a mapping satisfying*

$$d(Tx,Ty) \le d(x,y) - \varphi(d(x,y))\,, \quad \text{for any } x,y \in X\,,$$

*where $\varphi\colon [0,\infty) \to [0,\infty)$ is nondecreasing and $\varphi(t) = 0$ if and only if $t = 0$.*
*Then $T$ has a unique fixed point.*

*Remark.* In [15], the condition about the continuity of $\varphi$ is considered, but it is easily seen that such condition is superfluous [9].

*Remark.* In the sequel, we will denote by $\mathscr{A}$ the class of functions $\varphi\colon [0,\infty) \to [0,\infty)$ which is nondecreasing and satisfies $\varphi(t) = 0$ if and only if $t = 0$.

## 3 Main results

The solutions of Problem (2) are in the space $\mathscr{C}[0,1]$ of the continuous functions defined on $[0,1]$ and with real values. In this space we consider the classical distance given by

$$d(x,y) = \sup\{|x(t) - y(t)| : t \in [0,1]\}\,,$$

and it is a known fact that $(\mathscr{C}[0,1],d)$ is a complete metric space.
Next, we consider $\varphi\colon [0,1] \to [0,1]$, where $\varphi$ is a continuous function.
For $x \in \mathscr{C}[0,1]$, we denote by $\tilde{x}$ the function $\tilde{x} = x(\varphi(t))$ for $t \in [0,1]$. It is clear that $\tilde{x} \in \mathscr{C}[0,1]$.

**Definition 3.** *An element $(x,y) \in \mathscr{C}[0,1]\times\mathscr{C}[0,1]$ is said to be a $\varphi$-coupled fixed point of a mapping $G\colon \mathscr{C}[0,1]\times\mathscr{C}[0,1] \to \mathscr{C}[0,1]$ if $G(x,y) = x$ and $G(\tilde{x},\tilde{y}) = y$.*

For our study, we need to introduce the class of functions $\mathscr{B}$ defined by those functions $\phi\colon [0,\infty) \to [0,\infty)$ which are nondecreasing and such that $I - \phi \in \mathscr{A}$, where $I$ denotes the identity mapping on $[0,\infty)$ and $\mathscr{A}$ is the class of function defined in Remark 2.
The next theorem is very important in our study.

**Theorem 3.** *Let $G\colon \mathscr{C}[0,1]\times\mathscr{C}[0,1] \to \mathscr{C}[0,1]$ be a mapping satisfying*

$$d(G(x_1,y_1),G(x_2,y_2)) \le \phi(\max(d(x_1,x_2),d(y_1,y_2)))\,, \tag{3}$$

*for any $x_1,x_2,y_1,y_2 \in \mathscr{C}[0,1]$, where $\phi \in \mathscr{B}$.*
*Then $G$ has a unique $\varphi$-coupled fixed point, where $\varphi\colon [0,1] \to [0,1]$ is a continuous function.*

*Proof.* Consider the Cartesian product $\mathscr{C}[0,1]\times\mathscr{C}[0,1]$ endowed with the distance

$$\tilde{d}((x_1,y_1),(x_2,y_2)) = \max(d(x_1,x_2),d(y_1,y_2))\,.$$

It is easily seen that $(\mathscr{C}[0,1]\times\mathscr{C}[0,1],\tilde{d})$ is a complete metric space.
Now, we consider the mapping $\tilde{G}\colon \mathscr{C}[0,1]\times\mathscr{C}[0,1] \to \mathscr{C}[0,1]\times\mathscr{C}[0,1]$ defined by

$$\tilde{G}(x,y) = (G(x,y),G(\tilde{x},\tilde{y}))\,,$$

for any $(x,y) \in \mathscr{C}[0,1]\times\mathscr{C}[0,1]$ . Next, we check that $\tilde{G}$ satisfies assumptions of Theorem 2 in the complete metric space $\mathscr{C}[0,1]\times\mathscr{C}[0,1]$.
In fact, taking into account (3), for any $x_1,x_2,y_1,y_2 \in \mathscr{C}[0,1]$ we have

$$\begin{aligned} &\tilde{d}(\tilde{G}(x_1,y_1),\tilde{G}(x_2,y_2)) \\ &= \tilde{d}((G(x_1,y_1),G(\tilde{x}_1,\tilde{y}_1)),(G(x_2,y_2),G(\tilde{x}_2,\tilde{y}_2)) \\ &= \max\left[d(G(x_1,y_1),G(x_2,y_2)),d(G(\tilde{x}_1,\tilde{y}_1),G(\tilde{x}_2,\tilde{y}_2))\right] \\ &\le \max\left[\phi(\max(d(x_1,x_2),d(y_1,y_2))),\right. \\ &\qquad \left.\phi(\max(d(\tilde{x}_1,\tilde{x}_2),d(\tilde{y}_1,\tilde{y}_2)))\right]. \end{aligned}$$

Taking into account that

$$\begin{aligned} d(\tilde{x}_1,\tilde{x}_2) &= \sup\{|\tilde{x}_1(t)-\tilde{x}_2(t)| : t\in[0,1]\} \\ &= \sup\{|x_1(\varphi(t))-x_2(\varphi(t))| : t\in[0,1]\} \\ &\le d(x_1,x_2) \end{aligned}$$

and, analogously, $d(\tilde{y}_1,\tilde{y}_2)\le d(y_1,y_2)$. Since $\phi$ is nondecreasing it follows

$$\begin{aligned} &\tilde{d}(\tilde{G}(x_1,y_1),\tilde{G}(x_2,y_2)) \le \phi(\max(d(x_1,x_2),d(y_1,y_2))) \\ &= \phi(\tilde{d}((x_1,y_1),(x_2,y_2))) \\ &= \tilde{d}((x_1,y_1),(x_2,y_2)) - (\tilde{d}((x_1,y_1),(x_2,y_2)) \\ &\quad -\phi(\tilde{d}((x_1,y_1),(x_2,y_2))). \end{aligned}$$

Finally, since $\phi\in\mathscr{B}$ and, therefore, $I-\phi\in\mathscr{A}$, $\tilde{G}$ satisfies the contractive condition appearing in Theorem 2. Therefore, by Theorem 2, there exist a unique $(x_0,y_0)\in\mathscr{C}[0,1]\times\mathscr{C}[0,1]$ such that $\tilde{G}(x_0,y_0)=(x_0,y_0)$. This means that

$$\begin{aligned} G(x_0,y_0) &= x_0 \\ G(\tilde{x}_0,\tilde{y}_0) &= y_0\,, \end{aligned}$$

or, equivalently, $(x_0,y_0)$ is a $\varphi$-coupled fixed point of $G$. this complete the proof.

Problem (2) will be studied under the following assumptions

H1$2<\alpha<3, 0<\lambda<2$ and $0<\rho<1$.
H2$f\in\mathscr{C}([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$.
H3$f$ satisfies

$$|f(t,x,y)-f(t,u,v)|\le\gamma\phi(\max(|x-u|,|y-v|))\,,$$

for any$t\in[0,1]$ and $x,u,y,v\in\mathbb{R}$, where $0<\gamma\le\frac{(2-\lambda)\Gamma(\alpha)}{2}$ and $\phi\in\mathscr{B}$.

**Theorem 4.***Under assumptions H1-H3, Problem (2) has a unique solution in $\mathscr{C}[0,1]\times\mathscr{C}[0,1]$.*

*Proof.*In $\mathscr{C}[0,1]\times\mathscr{C}[0,1]$ we define the operator $H$ by

$$H(x,y)=\int_0^1 G(t,s)f(s,x(s),y(s))\mathrm{d}s\,,$$

for any $(x,y)\in\mathscr{C}[0,1]\times\mathscr{C}[0,1]$ and $t\in[0,1]$, where $G(t,s)$ is the Green's function considered in Theorem 1.
By Theorem 1 and H2, for any $(x,y)\in\mathscr{C}[0,1]\times\mathscr{C}[0,1]$ we have $H(x,y)\in\mathscr{C}[0,1]$.
Notice that, in virtue of Theorem 1, a solution $(x,y)\in\mathscr{C}[0,1]\times\mathscr{C}[0,1]$ of Problem (1) is a $\varphi$-coupled fixed point of the function

$$H\colon \mathscr{C}[0,1]\times\mathscr{C}[0,1]\to\mathscr{C}[0,1]\,,$$

where $\varphi\colon[0,1]\to[0,1]$ is the continuous function defined by $\varphi(t)=\rho t$.
In the sequel, we will prove that $H$ satisfies assumption of Theorem 3.
In fact, taking into account our assumptions H1-H3, for $x_1,y_1,x_2,y_2\in\mathscr{C}[0,1]$ and $t\in[0,1]$, we have

$$\begin{aligned} &d(H(x_1,y_1),H(x_2,y_2)) = \sup\{|H(x_1,y_1)(t)-H(x_2,y_2)(t)|\} \\ &= \sup\{|\int_0^1 G(t,s)f(s,x_1(s),y_1(s))\mathrm{d}s \\ &\quad -\int_0^1 G(t,s)f(s,x_2(s),y_2(s))\mathrm{d}s|\} \\ &\le \sup\{\int_0^1 G(t,s)|f(s,x_1(s),y_1(s))\mathrm{d}s - f(s,x_2(s),y_2(s))\mathrm{d}s|\} \\ &\le \sup\{\int_0^1 G(t,s)\gamma\phi(\max(|x_1(s)-x_2(s)|,|y_1(s)-y_2(s)|))\mathrm{d}s\} \\ &\le \sup\{\int_0^1 G(t,s)\gamma\phi(\max(d(x_1,x_2),d(y_1,y_2)))\mathrm{d}s\} \\ &\le \gamma\phi(\max(d(x_1,x_2),d(y_1,y_2)))\sup\{\int_0^1 G(t,s)\mathrm{d}s\} \\ &\le \gamma\phi(\max(d(x_1,x_2),d(y_1,y_2)))\cdot\frac{2}{(2-\lambda)\Gamma(\alpha)} \\ &\le \phi(\max(d(x_1,x_2),d(y_1,y_2)))\,, \end{aligned}$$

where we have used the facts that $\phi$ is nondecreasing and $0<\gamma\le\frac{(2-\lambda)\Gamma(\alpha)}{2}$.
Therefore, $H$ satisfies assumptions of Theorem 3 and, consequently, $H$ has a unique $\varphi$-coupled fixed point.
Thus, the proof is complete.

## 4 Example

Finally, we present an example illustrating our results.

*Example 1.*Consider the following coupled system of fractional differential equations with integral boundary conditions

$$\begin{cases} {}^cD^\alpha u(t)+\sqrt{t}+\dfrac{\mu|u(t)|}{1+|u(t)|}+\dfrac{\beta|v(t)|}{1+|v(t)|}=0\,, \\ {}^cD^\alpha v(t)+\sqrt{t}+\dfrac{\mu|u(\frac{1}{3}t)|}{1+|u(\frac{1}{3}t)|}+\dfrac{\beta|v(\frac{1}{3}t)|}{1+|v(\frac{1}{3}t)|}=0\,, \\ u(0)=u''(0)=0, u(1)=\dfrac{\sin 1}{1-\cos 1}\displaystyle\int_0^1 u(s)\mathrm{d}s\,, \\ v(0)=v''(0)=0, v(1)=\dfrac{\sin 1}{1-\cos 1}\displaystyle\int_0^1 v(s)\mathrm{d}s\,, \end{cases} \tag{4}$$

where $2<\alpha<3$, $0<t<1$, and $\mu,\beta$ are positive constants.
In this case, $f(t,u,v)=\sqrt{t}+\frac{\mu|u(t)|}{1+|u(t)|}+\frac{\beta|v(t)|}{1+|v(t)|}$, $0<\lambda=\frac{\sin 1}{1-\cos 1}\simeq 1.83048<2$ and $0<\rho=\frac{1}{3}<1$.
It is clear that $f\in\mathscr{C}([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$. Moreover, for

$t \in [0,1]$ and $x,y,u,v \in \mathbb{R}$, we have

$$
\begin{aligned}
&|f(t,x,y) - f(t,u,v)| \\
&\quad = \left|\frac{\mu|x|}{1+|x|} + \frac{\beta|y|}{1+|y|} - \frac{\mu|u|}{1+|u|} - \frac{\beta|v|}{1+|v|}\right| \\
&\leq \mu\left|\frac{|x|}{1+|x|} - \frac{|u|}{1+|u|}\right| + \beta\left|\frac{|y|}{1+|y|} - \frac{|v|}{1+|v|}\right| \\
&\leq \mu\frac{|x-u|}{(1+|x|)(1+|u|)} + \beta\frac{|y-v|}{(1+|y|)(1+|v|)} \\
&\leq \mu\frac{|x-u|}{1+|x-u|} + \beta\frac{|y-v|}{1+|y-v|} \\
&\leq 2\max\left(\frac{\mu|x-u|}{1+|x-u|}, \frac{\beta|y-v|}{1+|y-v|}\right) \\
&\leq 2\max(\mu,\beta)\max\left(\frac{|x-u|}{1+|x-u|}, \frac{|y-v|}{1+|y-v|}\right) \\
&= 2\max(\mu,\beta)\max(\phi(|x-u|), \phi(|y-v|)) \\
&= 2\max(\mu,\beta)\phi(\max(|x-u|,|y-v|)),
\end{aligned}
$$

where $\phi\colon [0,\infty] \to [0,\infty]$ is given by $\phi(t) = \frac{t}{1+t}$.
It is easily checked that $\phi \in \mathscr{B}$. Moreover, in the last equality, we have used the fact that $\phi(\max(t,s)) = \max(\phi(t),\phi(s))$ for $t,x \in [0,\infty)$ when $\phi$ is nondecreasing.
Therefore, if

$$
\begin{aligned}
2\max(\mu,\beta) &\leq (2-\lambda)\frac{\Gamma(\alpha)}{2} \\
&= (2 - \frac{\sin 1}{1-\cos 1})\frac{\Gamma(\alpha)}{2} \simeq 0.08475\,\Gamma(\alpha),
\end{aligned}
$$

then, in virtue of Theorem 4, Problem (2) has a unique solution $(u,v) \in \mathscr{C}([0,1] \times \mathscr{C}([0,1]$.

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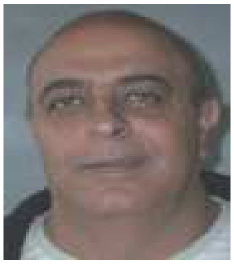
**Jackie Harjani** Received the PhD degree in Mathematics at University of Las Palmas de Gran Canaria (ULPGC). His research interests are in fixed point theory, and its applications to boundary value problems (classical and fractional type). He has published research articles in reputed international journals of mathematical.

**Juan Rocha** PhD, associate professor, applied mathematics at the Department of Mathematics of the University of Las Palmas de Gran Canaria (ULPGC).
Member of Institute of Intelligent Systems and Numerical Applications in Engineering (ULPGC). Secretary of the Department of Mathematics at ULPGC. His main research interest is related to fixed point theorems, integral equations and functional equations.



**Kishin Sadarangani** Senior lecturer at the Department of Mathematics, University of Las Palmas de Gran Canaria (ULPGC). His research interest areas are geometry of Banach spaces, Theory of measures of noncompactness and fixed point theory. He has published research articles in reputed international journals of mathematical. He is referee and editor of mathematical journals.