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Solitons, Shock Waves, Conservation Laws and Bifurcation Analysis of Boussinesq Equation with Power Law Nonlinearity and Dual Dispersion

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Abstract: This paper obtains the soliton solutions to the Boussinesq equation with the effect of surface tension being taken into account. The power law nonlinearity is considered. Three integration tools are adopted in order to extract the soliton solutions. They are the traveling wave hypothesis, ansatz method and the semi-inverse variational principle. Finally, the Lie symmetry approach is adopted to extract the conservation laws of this equation.

Keywords: Boussinesq equation, soliton solutions, variational principle

1 Introduction

The dynamics of shallow water waves along ocean and sea shores are modeled by several forms of nonlinear evolution equations (NLEEs) [1,2,3,4,5,6,7,8,9,10,11, 12,13,14,15,16,17,18,19,20,21,22]. A few of the well-known models are Korteweg-de Vries (KdV) equation, Peregrine equation, Benjamin-Bona-Mahoney equation. On the other had two-layered shallow water waves are typically modeled by the Bona-Chen equation, Gear-Grimshaw model, coupled KdV equation and many such coupled equations. This paper will focus on one such NLEE that models shallow water wave dynamics along with the effect of surface tension. This is the Boussinesq equation (BE). The effect of surface tension will introduce a second dispersion term and hence the name dual-dispersion. For a generalized setting, BE will be studied with power law nonlinearity.

The main focus of this paper is going to be the integrability aspect of this model. While there are several approaches to the integrability aspect of NLEEs, one needs to exercise caution while administering the integration process. Some of the commonly applied integration architectures are the exp-function approach, G'/G-expansion method, simplest equation method, Lie symmetry approach, differential transformation technique, variational iteration method, Adomian decomposition method and quite a few others. However, this paper will apply the classic travel wave hypothesis where a wave of permanent form will be revealed. The ansatz approach will be utilized to extract the solitary wave and singular soliton solutions to the BE. The semi-inverse variational method will also be applied as a third tool to solve the BE. Subsequently, the multiplier method using Lie symmetry approach will lead to the conservation laws of the equation. Finally, the bifurcation analysis will for the BE will be seen.

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2 TRAVELING WAVE SOLUTION

The dimensionless form of the BE with power law nonlinearity and dual-dispersion is given by [18]

$$q_{tt} - k^2 q_{xx} + a\left(q^{2n}\right)_{xx} + b_1 q_{xxxx} + b_2 q_{xxtt} = 0 \tag{1}$$

In (1), the dependent variable $q(x,t)$ is the wave profile while x and t are the spatial and temporal variables respectively. Also, k, a and b_j for $j = 1,2$ are real-valued constants. The first two terms constitute the wave operator, where the first term is the linear evolution term. The coefficient of a represents the nonlinear term where the parameter n is the power law nonlinearity parameters. Typically, the case $n = 1$ is studied in the literature of Ocean Dynamics. However, this paper will leave n as an arbitrary power law parameter in order to give BE a generalized flavor. Then, the coefficients of b_j are the two dispersion terms. The coefficient of b_1 is the regular dispersion, while the second dispersion term that is given by the coefficient of b_2 accounts for the surface tension [18]. The soliton solutions to (1) will be the only issue of this paper that can be formulated only when a delicate balance between dispersion and nonlinearity is in place.

In order to seek traveling wave solution to (1), the hypothesis [8]

$$q(x,t) = g(x - vt) \tag{2}$$

where $g(s)$ represents the wave of permanent form with

$$s = x - vt \tag{3}$$

and v represents the velocity of the wave. Substituting the hypothesis (2) into (1) leads to the ordinary differential equation (ODE) that is given by

$$\left(v^2 - k^2\right) g'' + a\left(g^{2n}\right)'' + \left(b_1 + b_2 v^2\right) g'''' = 0 \tag{4}$$

Integrating (4) twice, leads to

$$\left(v^2 - k^2\right) g + a g^{2n} + \left(b_1 + b_2 v^2\right) g'' = 0 \tag{5}$$

after choosing the integration constant to be zero, since the search is for a soliton solution. Multiplying both sides of (5) by g' and integrating one more time, again after taking the integration constant to be zero implies

$$\frac{dg}{ds} = \frac{g}{\sqrt{b_1 + b_2 v^2}} \sqrt{k^2 - v^2 - \frac{2a g^{2n-1}}{2n+1}} \tag{6}$$

After separating variables and integrating yields

$$g(s) = g(x - vt) = q(x,t) = A \, \mathrm{sech}^{\frac{2}{2n-1}}\left[B(x - vt)\right] \tag{7}$$

where the amplitude A of the soliton is

$$A = \left[\frac{(2n+1)\left(k^2 - v^2\right)}{2a}\right]^{\frac{1}{2n-1}} \tag{8}$$

while the inverse width of the soliton is

$$B = \frac{2n-1}{2} \sqrt{\frac{k^2 - v^2}{b_1 + b_2 v^2}} \tag{9}$$

The soliton width given by (9) provokes a constraint condition that is given by

$$\left(k^2 - v^2\right)\left(b_1 + b_2 v^2\right) > 0 \tag{10}$$

Upon setting $b_2 = 0$ in (1) and (9) the equation and consequently the corresponding result collapses to the regular BE with power law nonlinearity that has been studied earlier [8].

3 ANSATZ METHOD

This section will focus on another approach to integrating the BE. This is the ansatz method. Using a judicious hypothesis, a soliton solution will be obtained to the BE. While the traveling wave hypothesis revealed a solitary wave solution, this approach, however, is unable to obtain singular or topological soliton solution to this equation. The appealing ansatz method will therefore lead to a better picture for the BE. The hypothesis will be based on the type of soliton solution that is being sought for. Therefore the study of ansatz approach will be split into the following three subsections.

3.1 SOLITARY WAVES

In order to solve equation (1) for the solitary waves, the starting hypothesis is given by [15]

$$q(x,t) = A \, \mathrm{sech}^p \tau \tag{11}$$

Here in (11), A and B represent the amplitude and the inverse width of the soliton. The definition of τ is given by

$$\tau = B(x - vt) \tag{12}$$

Substituting this hypothesis into (1) and simplifying leads to

$$
\begin{aligned}
& pA\left\{v^2 - k^2 + \left(b_1 + b_2 v^2\right) p^2 B^2\right\} \mathrm{sech}^p \tau \\
& - (p+1)A\left(v^2 - k^2\right) AB\mathrm{sech}^{p+2} \\
& + \left\{p^2 + (p+2)^2\right\}\left(b_1 + b_2 v^2\right) AB^3 \mathrm{sech}^{p+2}\tau \\
& + AB^2(p+1)(p+2)(p+3)\left(b_1 + b_2 v^2\right) \mathrm{sech}^{p+4}\tau \\
& + 4an^2 pA^{2n}\mathrm{sech}^{2np}\tau \\
& - 2an(2np+1)A^{2n}\mathrm{sech}^{2np+2}\tau = 0
\end{aligned}
\tag{13}
$$

From (13) equating the exponents $2np$ and $p+2$ leads to

$$2np = p + 2 \tag{14}$$

so that

$$p = \frac{2}{2n-1} \tag{15}$$

One important observation is that the same value of the unknown exponent p is obtained upon equating the exponents $2np + 2$ and $p + 4$. Now, from (13), setting the

coefficients of the linearly independent functions $\mathrm{sech}^{p+j}\tau$ to zero, for $j = 0, 2, 4$, leads to (9) as well as

$$A = \left[\frac{(2n+1)\left\{ \left(v^2 - k^2\right)(2n-1)^2 + \eta \right\}}{8n^2(2n-1)^2 a} \right]^{\frac{1}{2n-1}} \qquad (16)$$

where $\eta = 4\left(4n^2 + 1\right)\left(b_1 + b_2 v^2\right)B^2$, and

$$A = \left[\frac{2\left(2n+1\right)\left(b_1 + b_2 v^2\right)B^2}{(2n-1)^2 a} \right]^{\frac{1}{2n-1}}. \qquad (17)$$

Next, upon equating the two values of the amplitude A from (16) and (17) leads to (9). Thus the system of relations from the coefficients of the linearly independent functions, leads to a closure. Hence, the 1-soliton solution to the BE is given by (7) with the amplitude (A) given by (16) or (17) while the width (B) is in (9) along with the constraint condition that is given by (10).

The following figure shows the profile of a solitary wave for $a = 1$, $b_1 = 1$, $b_2 = 1$, $k = 1$ and $n = 1$.
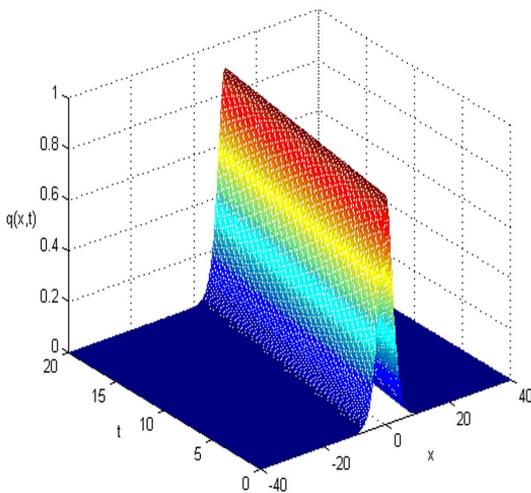
Figure 1: Solitary wave q(x,t) with n=1, a=1, b1=1, b2=1, k=1, v=0.5, -40<x<40, 0<t<20

3.2 SHOCK WAVES

Shock waves, that are also known as topological solitons or kinks are studied in the context of fluid dynamics. Thus, for topological soliton solution, the starting hypothesis is taken to be [13]

$$q(x,t) = A\tanh^p \tau \qquad (18)$$

In this case, the parameters A and B are free parameters. Therefore substituting this hypothesis into (1) leads to

$$\begin{aligned} &\left(v^2 - k^2\right)AB\left\{(p-1)\tanh^{p-2}\tau - 2p\tanh^p \tau\right\} \\ &+ (p+1)\tanh^{p+2}\tau + 2anA^{2n}B(2pn-1)\tanh^{2pn-2}\tau \\ &- 2anA^{2n}B\left\{4pn\tanh^{2pn}\tau - (2pn+1)\tanh^{2pn+2}\tau\right\} \\ &+ \left(b_1 + b_2 v^2\right)AB^3\left[(p-1)(p-2)(p-3)\tanh^{p-4}\tau \right. \\ &+ (p+1)(p+2)(p+3)\tanh^{p+4}\tau \\ &- 2(p-1)\left\{p^2 + (p-2)^2\right\}\tanh^{p-2}\tau \\ &- 2(p+1)\left\{p^2 + (p+2)^2\right\}\tanh^{p+2}\tau \\ &+ 4p^3\tanh^p + (p-1)^2(p-2)\tanh^p \\ &+ \left.(p+1)^2(p+2)\tanh^p \tau\right] = 0 \end{aligned} \qquad (19)$$

In this case from coefficient of the stand-alone linearly independent function is $\tanh^{p-4}\tau$ and the coefficient of the linearly independent function $\tanh^{p-2}\tau$ leads to

$$p = 1 \qquad (20)$$

Then again, equating the exponents $2np$ and $p+2$ as in the case of solitary waves in the previous section leads to (15) which therefore implies

$$n = \frac{3}{2} \qquad (21)$$

Hence the power law BE that supports topological soliton solution is given by

$$q_{tt} - k^2 q_{xx} + a\left(q^3\right)_{xx} + b_1 q_{xxxx} + b_2 q_{xxtt} = 0 \qquad (22)$$

Now, from (17) setting the coefficients of the linearly independent functions $\tanh^{p+j}\tau$, for $j = 0, 2, 4$, to zero leads to

$$B = \frac{A}{2}\sqrt{-\frac{a}{b_1 + b_2 v^2}} \qquad (23)$$

$$3aA^2 + 8\left(b_1 + b_2 v^2\right)B^2 = v^2 - k^2 \qquad (24)$$

and

$$9aA^2 + 20\left(b_1 + b_2 v^2\right)B^2 = v^2 - k^2 \qquad (25)$$

From (24) and (25) subtracting one from the other, the same relation between the free parameters as in (23) is revealed. This free parameters' relation kicks in a constraint condition given by

$$a\left(b_1 + b_2 v^2\right) < 0 \qquad (26)$$

in order for the shock waves to exist.

So, finally, the shock solution to the BE with power law nonlinearity and dual-dispersion exists only when the power law nonlinearity collapses to cubic nonlinearity. This is being observed for the very first time in this paper. In this case, the shock wave solution is given by

$$q(x,t) = A\tanh[B(x - vt)] \qquad (27)$$

where the relation between the free parameters is depicted in (21). Additionally, the constraint relation given by (24) must hold in order for these kinks to exist.

The following figure shows the profile of a shock wave for $p = 1$, $a = 1$, $b_1 = b_2 = 1$, $k = 1$, $A = 1$.
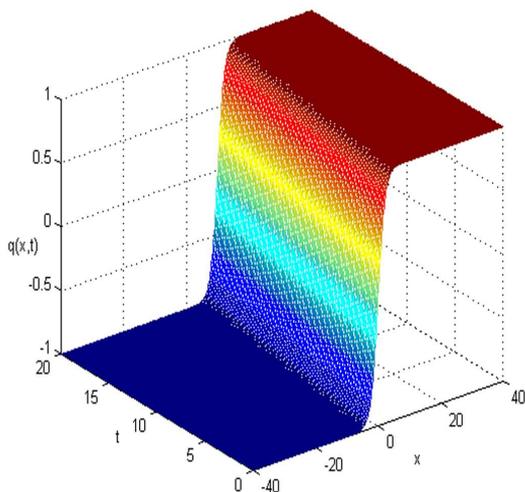
Figure 2. Shock wave q(x,t) with p=1, a=-1, b1=1, b2=1, k=1, v=0.5, -4<x<40, 0<t<20.

3.3 SINGULAR SOLITON

Singular solitons are the third kind of solitons that provide a possible analytical explanation to the formation of rogue waves. In order to obtain the singular soliton solution, the starting hypothesis is given by

$$q(x,t) = A \operatorname{csch}^p \tau \qquad (28)$$

where the definition of τ stays the same as in (12). However, the parameters A and B is this case are free parameters. Substituting this hypothesis into (1) and simplifying leads to

$$pA \left\{ v^2 - k^2 + \left(b_1 + b_2 v^2 \right) p^2 B^2 \right\} \operatorname{csch}^p \tau$$
$$- (p+1)A \left(v^2 - k^2 \right) \operatorname{csch}^{p+2} \tau$$
$$+ (p+1)A \left[\left\{ p^2 + (p+2)^2 \right\} \left(b_1 + b_2 v^2 \right) B^2 \right] \operatorname{csch}^{p+2} \tau$$
$$+ AB^2 (p+1)(p+2)(p+3) \left(b_1 + b_2 v^2 \right) \operatorname{csch}^{p+4} \tau$$
$$+ 4an^2 pA^{2n} \operatorname{csch}^{2np} \tau$$
$$- 2an(2np+1)A^{2n} \operatorname{csch}^{2np+2} \tau = 0 \qquad (29)$$

The balancing principle again gives (15). Similarly, as in the case of solitary waves, the coefficients of the linearly independent functions $\operatorname{csch}^{p+j} \tau$ for $j = 0, 2, 4$ leads to the relations (9) as well as

$$A = \left[-\frac{(2n+1) \left\{ \left(v^2 - k^2 \right) (2n-1)^2 + \theta \right\}}{8n^2 (2n-1)^2 a} \right]^{\frac{1}{2n-1}} \qquad (30)$$

where $\theta = 4 \left(4n^2 + 1 \right) \left(b_1 + b_2 v^2 \right) B^2$, and

$$A = \left[-\frac{2 (2n+1) \left(b_1 + b_2 v^2 \right) B^2}{(2n-1)^2 a} \right]^{\frac{1}{2n-1}}. \qquad (31)$$

Again, upon equating the two values of the parameter A from (30) and (31), leads to the same value of B as in (9). Thus, the singular 1-soliton solution to (1) is

$$q(x,t) = A \operatorname{csch}^{\frac{2}{2n-1}} [B(x - vt)] \qquad (32)$$

where the free paremter A is given by (30) or (31) while the parameter B is in (9) with the constraint condition (10).

4 OBSERVATION

This section will discuss an interesting connection between the solitary wave solution and the singular soliton solution. For the special case when $n = 1$, equation (1) modifies to

$$q_{tt} - k^2 q_{xx} + a \left(q^2 \right)_{xx} + b_1 q_{xxxx} + b_2 q_{xxtt} = 0 \qquad (33)$$

whose solitary wave solution can be written as

$$q(x,t) = A \operatorname{sech}^2 [B(x - vt) - x_0], \qquad (34)$$

where x_0 is the center position of the soliton, and the singular soliton solution can be written as

$$q(x,t) = A \operatorname{csch}^2 [B(x - vt)] \qquad (35)$$

after the respective parameters A and B are modified with respect to the specific value $n = 1$. It can be easily observed that the singular soliton solution (34) can be easily re-casted into (35) by implementing the transformation

$$e^{2x_0} = -1. \qquad (36)$$

Therefore, the singular solitons can be recovered from solitary waves using this exponential transformation with regards to the center position.

5 SEMI-INVERSE VARIATIONAL PRINCIPLE

This semi-inverse variational principle (SVP) is the third alternative scheme to solve the governing equation. One must note that the special case of the governing equation was solved earlier by the aid of SVP [6]. The starting hypothesis to solve (1) is the traveling wave hypothesis that is given by (2). Next, multiplying both sides of (5) by g' and integrating both sides with respect to s, gives

$$\left(v^2 - k^2 \right) g^2 + \left(b_1 + b_2 v^2 \right) \left(g' \right)^2 + \frac{2a}{2n+1} g^{2n+1} = K \qquad (37)$$

where K is an integration constant. The stationary integral J is then defined as [6]

$$J = \int_{-\infty}^{\infty} K ds \qquad (38)$$

which is therefore

$$J = \int_{-\infty}^{\infty} \left[\left(v^2 - k^2 \right) g^2 + \left(b_1 + b_2 v^2 \right) \left(g' \right)^2 + \vartheta \right] ds \qquad (39)$$

where $\vartheta = \frac{2a}{2n+1} g^{2n+1}$. The 1-soliton solution assumption to (1) is [6]

$$g(s) = A \operatorname{sech}^{\frac{2}{2n-1}}(Bs).\tag{40}$$

Inserting (40) into (39) and carrying out the integrations yield

$$J = \left\{ (v^2 - k^2) \frac{A^2}{B} + \frac{4(b_1 + b_2 v^2)A^2 B}{(2n-1)(2n+3)}\right.$$
$$\left. + \frac{8a}{(2n+1)(2n+3)} \frac{A^{2n+1}}{B} \right\} \frac{\Gamma\left(\frac{2}{2n-1}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{2n-1}+\frac{1}{2}\right)}\tag{41}$$

after integration, where $\Gamma(x)$ is the Euler's gamma function. The variational principle states that the parameters A and B are so chosen that [6]

$$\frac{\partial J}{\partial A} = 0\tag{42}$$

and

$$\frac{\partial J}{\partial B} = 0.\tag{43}$$

Thus from (41), equations (42) and (43), after simplification, respectively are

$$v^2 - k^2 + \frac{4(b_1 + b_2 v^2)B^2}{(2n-1)(2n+3)} + \frac{4aA^{2n-1}}{2n+3} = 0\tag{44}$$

and

$$v^2 - k^2 - \frac{4(b_1 + b_2 v^2)B^2}{(2n-1)(2n+3)} + \frac{8aA^{2n-1}}{(2n+1)(2n+3)} = 0\tag{45}$$

Equations (44) and (45), after uncoupling, gives the soliton amplitude (A) as given by (8) and the width (B) as

$$B = \frac{2n-1}{2}\sqrt{\frac{v^2 - k^2}{b_1 + b_2 v^2}}\tag{46}$$

which induces the constraint condition

$$(b_1 + b_2 v^2)(v^2 - k^2) > 0.\tag{47}$$

Also, equations (8) and (46) produces the amplitude-width relationship as

$$B = (2n-1)\left[-\frac{aA^{2n-1}}{(2n+1)(b_1 + b_2 v^2)}\right]^{\frac{1}{2}}\tag{48}$$

which poses the condition

$$a(b_1 + b_2 v^2) < 0\tag{49}$$

in order for the solitons to exist. Once again, it needs to be noted that the results of this section is a generalized version of the results that was reported in 2013 [6].

6 CONSERVATION LAWS

In order to determine conserved densities and fluxes, we resort to the invariance and multiplier approach based on the well known result that the Euler-Lagrange operator annihilates a total divergence (see [10]). Firstly, if (T^t, T^x) is a conserved vector corresponding to a conservation law, then

$$D_t T^t + D_x T^x = 0$$

along the solutions of the differential equation $(G(t, x, q, q_t, q_x, \ldots) = 0)$.

Moreover, if there exists a nontrivial differential function Q, called a 'multiplier', such that

$$E_q[QG] = 0,$$

then $QG)$ is a total divergence, i.e.,

$$QG = D_t T^t + D_x T^x,$$

for some (conserved) vector (T^t, T^x) and E_q is the respective Euler-Lagrange operator. Thus, a knowledge of each multiplier Q leads to a conserved vector determined by, inter alia, a Homotopy operator. See details and references in [10,12]. Below, we list the nontrivial conserved densities T_i^t corresponding to the multipliers Q^i. It turns out that there are no derivative dependent multipliers.

(i) $Q^1 = 1$,
$$T_1^t = q_t + \tfrac{1}{2} b_2 q_{xxt}$$

(ii) $Q^2 = x$,
$$T_2^t = xq_t + \tfrac{1}{6} b_2 (-2q_{xt} + 3xq_{xxt})$$

(iii) $Q^3 = t$,
$$T_3^t = -q + tq_t - \tfrac{1}{6} b_2 (q_{xx} - 3tq_{xxt})$$

(iv) $Q^4 = xt$,
$$T_4^t = \tfrac{1}{6}(-6xq + 6txq_t + b_2(2q_x - 2tq_{xt} - xq_{xx} + 3txq_{xxt}))$$

The corresponding conserved quantities are therefore given by

$$I_1 = \int_{-\infty}^{\infty} T_1^t dx = \int_{-\infty}^{\infty}\left(q_t + \frac{b_2}{2}q_{xxt}\right)dx = 0$$

$$I_2 = \int_{-\infty}^{\infty} T_2^t dx = \int_{-\infty}^{\infty}\left(xq_t - \frac{b_2}{3}q_{xt} + \frac{b_2}{2}xq_{xxt}\right)dx$$
$$= \frac{vA}{B}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2n-1}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2n-1}\right)}$$

$$I_3 = \int_{-\infty}^{\infty} T_3^t dx = \int_{-\infty}^{\infty}\left(-q + tq_t - \frac{b_2}{6}q_{xx} + \frac{b_2}{2}tq_{xxt}\right)dx$$
$$= -\frac{A}{B}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2n-1}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2n-1}\right)}$$

$$I_4 = \int_{-\infty}^{\infty} T_3^t dx = \int_{-\infty}^{\infty} \omega dx$$

$$= \frac{tvA}{B} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2n-1}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2n-1}\right)} = 0$$

where
$\omega = -xq + xtq_t - \frac{b_2}{3}q_x - \frac{b_2}{3}tq_{xt} - \frac{b_2}{6}xq_{xx} + \frac{b_2}{2}xtq_{xxt}$. The fourth conserved quantity is zero since this conservation law is valid at $v = 0$, namely when the soliton is stationary.

7 BIFURCATION ANALYSIS

This section will carry out the bifurcation analysis of the Boussinesq equation with power law nonlinearity. Initially, the phase portraits will be obtained and the corresponding qualitative analysis will be discussed. Several interesting properties of the solution structure will be obtained based on the parameter regimes. Subsequently, the traveling wave solutions will be discussed from the bifurcation analysis.

7.1 BIFURCATION PHASE PORTRAITS AND QUALITATIVE ANALYSIS

To relate conveniently, let

$$\alpha = \frac{a}{b_1 + b_2 v^2}, \tag{50}$$

and

$$\beta = \frac{k^2 - v^2}{b_1 + b_2 v^2}. \tag{51}$$

Setting $g' = y$, then via (5), (50) and (51) we get the following planar system

$$\begin{cases} \frac{dg}{d\xi} = y, \\ \frac{dy}{d\xi} = -\alpha g^{2n} + \beta g. \end{cases} \tag{52}$$

Obviously, the above system (52) is a Hamiltonian system with Hamiltonian function

$$H(g,y) = y^2 + \frac{2\alpha}{2n+1}g^{2n+1} - \beta g^2. \tag{53}$$

In order to investigate the phase portrait of (52), set

$$f(g) = -\alpha g^{2n} + \beta g. \tag{54}$$

Obviously, $f(g)$ has two zero points, g_0 and g_*, which are given as follows

$$g_0 = 0, \qquad g_* = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2n-1}}. \tag{55}$$

Letting $(g_i, 0)$ be one of the singular points of system (52), then the characteristic values of the linearized system of system (52) at the singular points $(g_i, 0)$ are

$$\lambda_\pm = \pm\sqrt{f'(g_i)}. \tag{56}$$

From the qualitative theory of dynamical systems, we know that
 (i) If $f'(g_i) > 0$, $(g_i, 0)$ is a saddle point.
 (ii) If $f'(g_i) < 0$, $(g_i, 0)$ is a center point.
 (iii) If $f'(g_i) = 0$, $(g_i, 0)$ is a degenerate saddle point.
Therefore, we obtain the bifurcation phase portraits of system (52) in Fig.3.
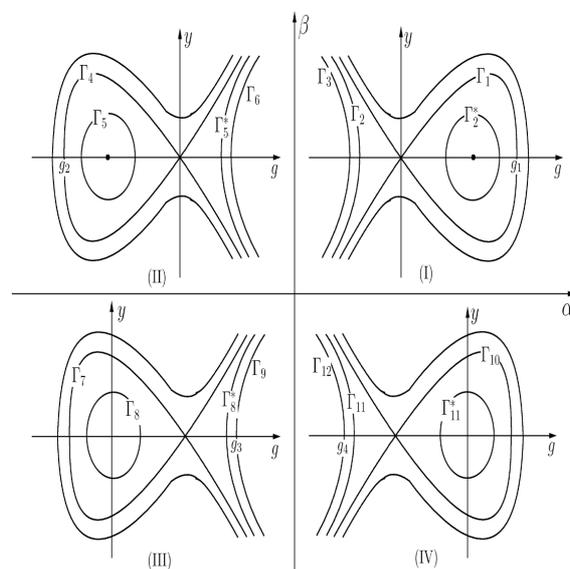
Fig. 3. The bifurcation phase portraits of system (52), (I) $\alpha > 0$, $\beta > 0$, (II) $\alpha < 0$, $\beta > 0$, (III) $\alpha < 0$, $\beta < 0$, (IV) $\alpha > 0$, $\beta < 0$

Let

$$H(g,y) = h, \tag{57}$$

where h is Hamiltonian.
Next, we consider the relations between the orbits of (52) and the Hamiltonian h.
 Set

$$h^* = |H(g_*, 0)|. \tag{58}$$

Proposition 1. When $\alpha > 0$ and $\beta > 0$, we have
(i) When $h = 0$, system (52) has a homoclinic orbit Γ_1.
(ii) When $-h^* < h < 0$, system (52) has two periodic orbits Γ_2 and Γ_2^*.
(iii) When $h = -h^*$, system (52) has a periodic orbit Γ_3.
(iv) When $h < -h^*$ or $h > 0$, system (52) does not any closed orbit.
Proposition 2. When $\alpha < 0$ and $\beta > 0$, we have
(i) When $h = 0$, system (52) has a homoclinic orbit Γ_4.
(ii) When $-h^* < h < 0$, system (52) has two periodic orbits Γ_5 and Γ_5^*.
(iii) When $h = -h^*$, system (52) has a periodic orbit Γ_6.

(iv) When $h < -h^*$ or $h > 0$, system (52) does not any closed orbit.

Proposition 3. When $\alpha < 0$ and $\beta < 0$, we have

(i) When $h = h^*$, system (52) has a homoclinic orbit Γ_7.

(ii) When $0 < h < h^*$, system (52) has two periodic orbits Γ_8 and Γ_8^*.

(iii) When $h = 0$, system (52) has a periodic orbits Γ_9.

(iv) When $h < 0$ or $h > h^*$, system (52) does not any closed orbit.

Proposition 4. When $\alpha > 0$ and $\beta < 0$, we have

(i) When $h = h^*$, system (52) has a homoclinic orbit Γ_{10}.

(ii) When $0 < h < h^*$, system (52) has two periodic orbits Γ_{11} and Γ_{11}^*.

(iii) When $h = 0$, system (52) has a periodic orbits Γ_{12}.

(iv) When $h < 0$ or $h > h^*$, system (52) does not any closed orbit.

From the qualitative theory of dynamical systems, we know that a smooth solitary wave solution of a partial differential system corresponds to a smooth homoclinic orbit of a traveling wave equation. A periodic orbit of a traveling wave equation corresponds to a periodic traveling wave solution of a partial differential system. According to above analysis, we have the following propositions.

Proposition 5. When $\alpha > 0$ and $\beta > 0$, we have

(i) When $h = 0$, (1) has a solitary wave solution(corresponding to the homoclinic orbit Γ_1 in Fig.3).

(ii) When $-h^* < h < 0$, (1) has a periodic wave solution and two periodic singular wave solutions(corresponding to the periodic orbits Γ_2 and Γ_2^* in Fig.3).

(iii) When $h = -h^*$, (1) has two periodic singular wave solutions(corresponding to the periodic orbit Γ_3 in Fig.3).

Proposition 6. When $\alpha < 0$ and $\beta > 0$, we have

(i) When $h = 0$, (1) has a solitary wave solution(corresponding to the homoclinic orbit Γ_4 in Fig.3).

(ii) When $-h^* < h < 0$, (1) has a periodic wave solution and two periodic singular wave solutions(corresponding to the periodic orbits Γ_5 and Γ_5^* in Fig.3).

(iii) When $h = -h^*$, (1) has two periodic singular wave solutions(corresponding to the periodic orbit Γ_6 in Fig.3).

Proposition 7. When $\alpha < 0$ and $\beta < 0$, we have

(i) When $h = h^*$, (1) has a solitary wave solution(corresponding to the homoclinic orbit Γ_7 in Fig.3).

(ii) When $0 < h < h^*$, (1) has a periodic wave solution and two periodic singular wave solutions(corresponding to the periodic orbits Γ_8 and Γ_8^* in Fig.3).

(iii) When $h = 0$, (1) has two periodic singular wave solutions(corresponding to the periodic orbit Γ_9 in Fig.3).

Proposition 8. When $\alpha > 0$ and $\beta < 0$, we have

(i) When $h = h^*$, (1) has a solitary wave solution(corresponding to the homoclinic orbit Γ_{10} in Fig.3).

(ii) When $0 < h < h^*$, (1) has a periodic wave solution and two periodic singular wave solutions(corresponding to the periodic orbits Γ_{11} and Γ_{11}^* in Fig.3).

(iii) When $h = 0$, (1) has two periodic singular wave solutions(corresponding to the periodic orbit Γ_{12} in Fig.3).

7.2 TRAVELING WAVE SOLUTIONS

Firstly, we will obtain the explicit expressions of traveling wave solutions for the (i) when $\alpha > 0$ and $\beta > 0$(or $\alpha < 0$ and $\beta > 0$). From the phase portrait, we see that there is a homoclinic orbit Γ_1 (or Γ_2). In (g, y)-plane the expressions of the homoclinic orbit is given as

$$y = \pm \sqrt{-\frac{2\alpha}{2n+1} g^{2n+1} + \beta g^2}. \tag{59}$$

Substituting (59) into $\frac{dg}{ds} = y$ and integrating them along the orbits Γ_1 and Γ_2, we have

$$\pm \int_{g_1}^{g} \frac{1}{\sqrt{-\frac{2\alpha}{2n+1} z^{2n+1} + \beta z^2}} \, dz = \int_0^s dz, \tag{60}$$

$$\pm \int_{g_2}^{g} \frac{1}{\sqrt{-\frac{2\alpha}{2n+1} z^{2n+1} + \beta z^2}} \, dz = \int_0^s dz, \tag{61}$$

Completing above integrals we obtain

$$g = \pm \left(\frac{(2n+1)\beta}{\alpha(1 - \cosh((2n-1)\sqrt{\beta}s))} \right)^{\frac{1}{2n-1}}. \tag{62}$$

Using the notations of (2) and (3), we get the following singular solitary wave solutions

$$q_{1\pm}(x,t) = \pm \left(\frac{(2n+1)\beta}{\alpha(1 - \cosh((2n-1)\sqrt{\beta}(x - vt)))} \right)^{\frac{1}{2n-1}}, \tag{63}$$

where the parameters α and β are given by (50) and (51) respectively.

Secondly, we will obtain the explicit expressions of traveling wave solutions for the (ii) when $\alpha < 0$ and $\beta < 0$(or $\alpha > 0$ and $\beta > 0$). From the phase portrait, we note that there is a special orbit Γ_9 (or Γ_{12}). In (g, y)-plane the expressions of the orbit are given as

$$y = \pm \sqrt{-\frac{2\alpha}{2n+1} g^{2n+1} + \beta g^2}. \tag{64}$$

Substituting (64) into $\frac{dg}{d\xi} = y$ and integrating them along the two orbits Γ_9 and Γ_{12}, it follows that

$$\pm \int_{g_3}^{g} \frac{1}{\sqrt{-\frac{2\alpha}{2n+1} z^{2n+1} + \beta z^2}} \, dz = \int_0^s dz, \tag{65}$$

$$\pm \int_{g_4}^{g} \frac{1}{\sqrt{-\frac{2\alpha}{2n+1} z^{2n+1} + \beta z^2}} \, dz = \int_0^s dz, \tag{66}$$

$$\pm \int_g^\infty \frac{1}{\sqrt{-\frac{2\alpha}{2n+1}s^{2n+1} + \beta z^2}} \mathrm{d}z = \int_0^s \mathrm{d}z. \qquad (67)$$

Completing above integrals we obtain

$$g = \pm \left(\frac{(2n+1)\beta}{2\alpha} \sec^2 \left(\frac{(2n-1)\sqrt{-\beta}}{2} s \right) \right)^{\frac{1}{2n-1}}, \quad (68)$$

$$g = \pm \left(\frac{(2n+1)\beta}{2\alpha} \csc^2 \left(\frac{(2n-1)\sqrt{-\beta}}{2} s \right) \right)^{\frac{1}{2n-1}}. \quad (69)$$

From the notations of (2) and (3), we get the following periodic singular wave solutions

$$q_{2\pm}(x,t) = \pm \left(\mu \sec^2 \nu (x - vt) \right)^{\frac{1}{2n-1}}, \qquad (70)$$

$$q_{3\pm}(x,t) = \pm \left(\mu \csc^2 \nu (x - vt) \right)^{\frac{1}{2n-1}}, \qquad (71)$$

where $\mu = \frac{(2n+1)\beta}{2\alpha}, \nu = \frac{(2n-1)\sqrt{-\beta}}{2}$, the parameters α and β are given by (50) and (51) respectively.

8 CONCLUSIONS

This paper studied the BE with power law nonlinearity in presence of dual dispersion. There are three integration schemes that were applied to obtain soliton solutions. These are the traveling wave hypothesis, ansatz approach and finally the SVP. The ansatz method additionally produced shock waves and singular soliton solutions to BE. The wave profiles are also displayed numerically. The connection between singular solitons and solitary waves were also established. The conserved quantities were also obtained by the aid of multiplier method in Lie symmetry. Finally, the bifurcation analysis was carried out and phase portraits are also given. These results are going to be extremely useful in further future study of the equation in the context of shallow water waves.

The soliton perturbation theory will be established and addressed in future. The perturbation terms will also lead to the quasi-stationary soliton solution. Additional integration architectures will be employed in future to extract several other solutions. A few of them are the exp-function method, G'/G-expansion method, simplest equation approach, Lie symmetry analysis and several others. These results will be reported in future.

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