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On Generalized Statistically Convergent Functions via Ideals

Ekrem Savas[*]

Department of Mathematics, Istanbul Ticaret University, Sutluce-Istanbul, Turkey

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Abstract: In this paper, we introduce the notions of $[V,\lambda]$-summability and $\lambda-$statistical convergence of function by taking a nonnegative real-valued Lebesque measurable function in the interval $(1,\infty)$ and introduce new notions, namely, $[V,\lambda](\mathscr{I})-$summability, and $\mathscr{I}_\lambda-$statistical convergence of function. We mainly examine the relation between these two new methods.

Keywords: Ideal, filter, $\mathscr{I}$- statistical convergence, $\mathscr{I}_\lambda-$ statistical convergence,real-valued function.

1 Introduction

The idea of convergence of a real sequence had been extended to statistical convergence by Fast [1](see, also Schoenberg [2]). Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from the sequence space point of view and linked with summability theory by Fridy [3], Connor [4], Connor and Savas [5], Šalát [6], Cakalli [7] and many others. In the recent years, generalization of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on Stone-Čech compactification of the natural numbers. Moreover statistical convergence is closely related to the concept of convergence in probability.

The notion of statistical convergence depends on the density of subsets of $\mathbf{N}$. A subset E of $\mathbf{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_E(k) \ exists.$$

Note that if $K \subset \mathbf{N}$ is finite set, then $\delta(K) = 0$, and for any set $K \subset \mathbf{N}, \delta(K^C) = 1 - \delta(K)$.
A sequence $x = (x_k)$ is said to be *statistically convergent* to L if for every $\varepsilon > 0$

$$\delta(\{k \in \mathbf{N} : |x_k - L| \geq \varepsilon\}) = 0.$$

We write $st - \lim x_k = L$ in case $x = (x_k)$ is $st-$ statistically convergent to L, (see, Fridy [3]).

The concept of $\mathscr{I}-$convergence was introduced by Kostyrko et al. in a metric space, (see,[8]). Later on it was further studied by Dems [9], Das, Savas and Ghosal [10], Savas ([11], [12], [13], [14], [15], [16], [17] and [18]) and many others. $\mathscr{I}-$convergence is a generalization form of statistical convergence and that is based on the notion of an ideal of the subset of positive integers $\mathbb{N}$. More applications of ideals can be found in [19] and [20].

In another direction, the idea of $\lambda-$statistical convergence was introduced and studied by Mursaleen [21].

In this note we use ideals to introduce the concept of $\mathscr{I}_\lambda-$ statistical convergence of nonnegative real-valued Lebesque measurable function in the interval $(1,\infty)$ and investigate some of its convergence.

Throughout by function $x(t)$ we shall mean a nonnegative real-valued Lebesque measurable function in the interval $(1,\infty)$ and $\mathbb{N}$ will stand for the set of natural numbers.

2 Main Results

The following definitions and notions will be needed in the sequel.

[*] Corresponding author e-mail: ekremsavas@yahoo.com

Definition 1. *A family $\mathscr{I} \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if the following conditions hold:*
(a) $A, B \in I$ implies $A \cup B \in \mathscr{I}$,
(b) $A \in \mathscr{I}$, $B \subset A$ implies $B \in \mathscr{I}$,

Definition 2. *A non-empty family $F \subset 2^{\mathbb{N}}$ is said to be a filter of $\mathbb{N}$ if the following conditions hold:*
(a) $\phi \notin F$,
(b) $A, B \in F$ implies $A \cap B \in F$,
(c) $A \in F$, $A \subset B$ implies $B \in F$,

If $\mathscr{I}$ is a proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathscr{I}$), then the family of sets $F(\mathscr{I}) = \{M \subset \mathbb{N} : \exists A \in \mathscr{I} : M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal.

Definition 3. *A proper ideal $\mathscr{I}$ is said to be admissible if $\{n\} \in \mathscr{I}$ for each $n \in \mathbb{N}$.*

Throughout $\mathscr{I}$ will stand for a proper admissible ideal of $\mathbb{N}$.

Definition 4. *(See [8]). Let $\mathscr{I} \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$.*

(i) The sequence (x_n) of elements of $\mathbb{R}$ is said to be $\mathscr{I}$-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \mathscr{I}$.

(ii) The sequence (x_n) of elements of $\mathbb{R}$ is said to be $\mathscr{I}^*$-convergent to $L \in \mathbb{R}$ if there exists $M \in F(\mathscr{I})$ such that $\{x_n\}_{n \in M}$ converges to L.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$$

The collection of such sequence λ will be denoted by Δ.

The generalized de la Valée-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be $(V, \lambda)-$summable to a number L if $t_n(x) \to L$ as $n \to \infty$. If $\lambda_n = n$, then $(V, \lambda)-$ summability reduces to $(C, 1)$ summability.

In [15], Savaş and Das defined $\mathscr{I}$- statistical convergence and $\mathscr{I}_\lambda$- statistical convergence as follows:

Definition 5. *A sequence $x = (x_k)$ is said to be $\mathscr{I}-$statistically convergent to L, if for every $\varepsilon > 0$, and $\delta > 0$,*

$$\left\{ n \in N : \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| \geq \delta \right\} \in \mathscr{I}.$$

For $\mathscr{I} = \mathscr{I}_{fin} = \{A \subseteq \mathbf{N} : A$ is a finite subset $\}$, $\mathscr{I}-$statistical convergence coincides with statistical convergence.

Definition 6. *A sequence $x = (x_k)$ is said to be $\mathscr{I}_\lambda-$statistically convergent to L or $S_\lambda(\mathscr{I})$- convergent to L if for any $\varepsilon > 0$ and $\delta > 0$*

$$\{n \in \mathbb{N} : \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| \geq \delta\} \in \mathscr{I}.$$

In this case we write $x_k \to L(S_\lambda(\mathscr{I}))$. The class of all $\mathscr{I}_\lambda-$statistically convergent sequences will be denoted by $S_\lambda(\mathscr{I})$.

We now introduce our main definition.

Definition 7. *Let $\lambda \in \Delta$ and $x(t)$ be a nonnegative real-valued Lebesque measurable function in the interval $(1, \infty)$ if*

$$I - \lim_{n \to \infty} \frac{1}{\lambda_n} \int_{n-\lambda_n+1}^{n} |x(t) - L| \, dt = 0. \quad (1)$$

Then we say that the function $x(t)$ is $[V, \lambda](\mathscr{I})-$ summable to L. If $\mathscr{I} = \mathscr{I}_{fin} = \{A \subseteq \mathbf{N} : A$ is a finite subset $\}$, $[V, \lambda](\mathscr{I})-$ summability becomes $[V, \lambda]$ summability, which is defined as follows :

$$\lim_{n \to \infty} \frac{1}{\lambda_n} \int_{n-\lambda_n+1}^{n} |x(t) - L| \, dt = 0, (\text{see, [22]}). \quad (2)$$

Definition 8. *A nonnegative real-valued Lebesque measurable function $x(t)$ is said to be $\mathscr{I}_\lambda-$statistically convergent or $S_\lambda(\mathscr{I})$- convergent to L, if for every $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{ n \in N : \frac{1}{\lambda_n} |\{t \in I_n : |x(t) - L| \geq \varepsilon\}| \geq \delta \right\} \in \mathscr{I}.$$

In this case we write $S_\lambda(\mathscr{I}) - \lim x(t) = L$ or $x(t) \to L \ (S_\lambda(\mathscr{I}))$. We also write $\mathscr{I} - \lim x(t) = L$. For $\mathscr{I} = \mathscr{I}_{fin}, S_\lambda(\mathscr{I})$-convergence again coincides with $\lambda-$statistical convergence,[22].

We shall denote by $S(\mathscr{I}), S_\lambda(\mathscr{I})$ and $[V, \lambda](\mathscr{I})$ the collections of all $\mathscr{I}$-statistically convergent, $S_\lambda(\mathscr{I})$-convergent and $[V, \lambda](\mathscr{I})$ -convergent functions respectively.

Theorem 1. *Let $\lambda = (\lambda_n) \in \Delta$ and $x(t)$ be a nonnegative real-valued Lebesque measurable function in the interval $(1, \infty)$. Then*

(i) $x(t) \to L[V, \lambda](\mathscr{I}) \Rightarrow x(t) \to L(S_\lambda(\mathscr{I}))$ and the inclusion $[V, \lambda](\mathscr{I}) \subset S_\lambda(\mathscr{I})$ is proper for every ideal $\mathscr{I}$.

(ii) If $x(t)$ is bounded and $x_k \to L \ (S_\lambda(\mathscr{I}))$ then $x(t) \to L \ [V, \lambda](\mathscr{I})$.

Proof.(i) Let $\varepsilon > 0$ and $x(t) \to L[V,\lambda](\mathscr{I})$. We have

$$\int_{t\in I_n \& |x(t)-L|\geq\varepsilon} |x(t)-L| \geq \varepsilon |\{t \in I_n : |x(t)-L| \geq \varepsilon\}|.$$

So for a given $\delta > 0$,

$$\frac{1}{\lambda_n}|\{t \in I_n : |x(t)-L| \geq \varepsilon\}| \geq \delta \Rightarrow \frac{1}{\lambda_n}\int_{t\in I_n} |x(t)-L| \geq \varepsilon\delta$$

i.e. $\left\{n \in N : \frac{1}{\lambda_n}|\{t \in I_n : |x(t)-L| \geq \varepsilon\}| \geq \delta\right\}$
$\subset \left\{n \in N : \frac{1}{\lambda_n}\left\{\int_{t\in I_n} |x(t)-L| \geq \varepsilon\right\} \geq \varepsilon\delta\right\}$.

Since $x(t) \to L[V,\lambda](\mathscr{I})$, so the set on the right hand side belongs to $\mathscr{I}$ and so it follows that $x(t) \to L(S_\lambda(\mathscr{I}))$. This proves the result.

To show that $S_\lambda(\mathscr{I}) \subsetneq [V,\lambda](\mathscr{I})$, take a fixed $A \in \mathscr{I}$. Define a function x by

$$x(t) = \begin{cases} t & \text{for } n-[\sqrt{\lambda_n}]+1 \leq t \leq n, n \notin A \\ t & \text{for } n-\lambda_n+1 \leq t \leq n, n \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then for every $\varepsilon > 0 (0 < \varepsilon < 1)$ since

$$\frac{1}{\lambda_n}|\{t \in I_n : |x(t)-0| \geq \varepsilon\}| = \frac{[\sqrt{\lambda_n}]}{\lambda_n} \to 0$$

as $n \to \infty$ and $n \notin A$, so for every $\delta > 0$,

$$\left\{n \in N : \frac{1}{\lambda_n}|\{t \in I_n : |x(t)-L| \geq \varepsilon\}| \geq \delta\right\} \subset A \cup \{1,2,...,m\}$$

for some $m \in N$. Since $\mathscr{I}$ is admissible so it follows that $x(t) \to 0\ (S_\lambda(\mathscr{I}))$. Obviously

$$\frac{1}{\lambda_n}\int_{t\in I_n} |x(t)-L|dt \to \infty (n \to \infty)$$

i.e.$x(t) \nrightarrow 0[V,\lambda](\mathscr{I})$. Note that if $A \in \mathscr{I}$ is infinite then $x(t) \nrightarrow 0(S_\lambda)$. This example also shows that $\mathscr{I}_\lambda-$statistical convergence is more general than λ-statistical convergence. This completed the proof of (i)

(ii)Suppose that $x(t) \to L(S_\lambda(\mathscr{I}))$ and $x(t)$ is bounded by M. Let $\varepsilon > 0$ be given. Now

$$\begin{aligned} \frac{1}{\lambda_n}\int_{t\in I_n} |x(t)-L|dt &= \frac{1}{\lambda_n}\int_{t\in I_n \& |x(t)-L|\geq\varepsilon} |x(t)-L|dt \\ &+ \frac{1}{\lambda_n}\int_{t\in I_n \& |x(t)-L|<\varepsilon} |x(t)-L|dt \\ &\leq \frac{M}{\lambda_n}\{t \in I_n : |x(t)-L| \geq \varepsilon\} + \varepsilon. \end{aligned}$$

Note that $\left\{n \in N : \frac{1}{\lambda_n}|\{t \in I_n : |x(t)-L| \geq \varepsilon\}| \geq \frac{\varepsilon}{M}\right\} = A(\varepsilon)(say)\in \mathscr{I}$. If $n \in (A(\varepsilon))^c$ then

$$\frac{1}{\lambda_n}\sum_{t\in I_n} |x(t)-L| < 2\varepsilon.$$

Hence

$$\left\{n \in N : \frac{1}{\lambda_n}\sum_{t\in I_n} |x(t)-L| \geq 2\varepsilon\right\} \subset A(\varepsilon)$$

and so belongs to $\mathscr{I}$. This shows that $x(t) \to L[V,\lambda](\mathscr{I})$ and this completed the proof of (ii).

Theorem 2.*(i) $S(\mathscr{I}) \subset S_\lambda(\mathscr{I})$ if $\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0$.*

(ii) If $\liminf_{n\to\infty} \frac{\lambda_n}{n} = 0$, $\mathscr{I}$-strongly (by which we mean that $\exists$ a subsequence $(n(j))_{j=1}^{\infty}$, for which $\frac{\lambda_{n(j)}}{n(j)} < \frac{1}{j}\ \forall j$ and $\{n(j) : j \in N\} \notin \mathscr{I}$) then $S(\mathscr{I}) \subsetneq S_\lambda(\mathscr{I})$.

Proof.(i) For given $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{n}|\{t \leq n : |x(t)-L| \geq \varepsilon\}| &\geq \frac{1}{n}|\{t \in I_n : |x(t)-L| \geq \varepsilon\}| \\ &\geq \frac{\lambda_n}{n}\frac{1}{\lambda_n}|\{t \in I_n : |x(t)-L| \geq \varepsilon\}|. \end{aligned}$$

If $\liminf_{n\to\infty} \frac{\lambda_n}{n} = a$ then from definition $\left\{n \in N : \frac{\lambda_n}{n} < \frac{a}{2}\right\}$is finite. For $\delta > 0$,

$$\begin{aligned} &\left\{n \in N : \frac{1}{\lambda_n}|\{t \in I_n : |x(t)-L| \geq \varepsilon\}| \geq \delta\right\} \\ \subset &\left\{n \in N : \frac{1}{n}|\{t \in I_n : |x(t)-L| \geq \varepsilon\}| \geq \frac{a}{2}\delta\right\} \cup \\ &\left\{n \in N : \frac{\lambda_n}{n} < \frac{a}{2}\right\}. \end{aligned}$$

Since $\mathscr{I}$ is admissible, the set on the right hand side belongs to $\mathscr{I}$ and this completed the proof of (i).

(ii) Define a function $x(t)$ by

$$x(t) = \begin{cases} 1 & \text{if } t \in I_{n(j)}, j = 1,2,... \\ 0 & \text{otherwise.} \end{cases}$$

Then $x(t)$ is statistically convergent and so $x(t) \in S(\mathscr{I})$ (Since $\mathscr{I}$ is admissible). But $x(t) \notin [V,\lambda](\mathscr{I})$ and so by Theorem 2.1 (ii) $x(t) \notin S_\lambda(\mathscr{I})$. This completes the proof.

Theorem 3.*If $\lambda \in \triangle$ be such that $lim_n \frac{\lambda_n}{n} = 1$, then $S_\lambda(\mathscr{I}) \subset S(\mathscr{I})$.*

*Proof.*Let $\delta > 0$ be given. Since $lim_n \frac{\lambda_n}{n} = 1$, we can choose $m \in N$ such that $|\frac{\lambda_n}{n} - 1| < \frac{\delta}{2}$, for all $n \geq m$. Now observe that , for $\varepsilon > 0$

$$\begin{aligned} &\frac{1}{n}|\{t \leq n : |x(t)-L| \geq \varepsilon\}| = \frac{1}{n}|\{t \leq n-\lambda_n : |x(t)-L| \geq \varepsilon\}| \\ &+ \frac{1}{n}|\{t \in I_n : |x(t)-L| \geq \varepsilon\}| \\ &\leq \frac{n-\lambda_n}{n} + \frac{1}{n}|\{t \in I_n : |x(t)-L| \geq \varepsilon\}| \\ &\leq 1-(1-\frac{\delta}{2}) + \frac{1}{n}|\{t \in I_n : |x(t)-L| \geq \varepsilon\}| \\ &= \frac{\delta}{2} + \frac{1}{n}|\{t \in I_n : |x(t)-L| \geq \varepsilon\}|, \end{aligned}$$

for all $n \geq m$. Hence

$$\left\{n \in N : \frac{1}{n}\left|\{t \leq n : |x(t) - L| \geq \varepsilon\}\right| \geq \delta\right\}$$
$$\subset \left\{n \in N : \frac{1}{n}\left|\{t \in I_n : |x(t) - L| \geq \varepsilon\}\right| \geq \frac{\delta}{2}\right\} \cup \{1,2,3,\ldots,m\}.$$

If $S_\lambda(\mathscr{I}) - lim x(t) = L$ then the set on the right hand side belongs to $\mathscr{I}$ and so the set on the left hand side also belongs to $\mathscr{I}$. This shows that $x(t)$ is $\mathscr{I}$- statistically convergent to L.

*Remark.*We do not know whether the condition in Theorem 3 is necessary or not and leave it as an open problem.

3 Conclusion

The concept of ideal convergence is a generalization form of statistical convergence and any concept involving statistical convergence and λ-statistical convergence plays a vital role not only in mathematics but also in other branches of science and engineering involving mathematics, especially in information theory, computer science, biological science, dynamical systems and others. In this paper, we use a nonnegative real-valued Lebesque measurable function in the interval $(1,\infty)$ instead of sequences to define $\mathscr{I}_\lambda-$statistical convergence of function. $\mathscr{I}_\lambda-$statistical convergence of function has not been studied so far. Therefore the present paper is filled up a gab in the existing literature.

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Ekrem Savas is a professor of mathematics at the Istanbul Commerce University, Istanbul-Turkey. He received M.Sc. (1983) and Ph.D. (1986) degrees in mathematics. As soon as he completed Ph.D he went to Utkal University in India as visiting scholar under the indo- Turkey academic exchange programme for one year. In 1988 and 1993 he became associate and full professor of mathematics respectively. He has been several times at department of mathematics of Indiana University of USA under Fulbright programme and NATO grant to do joint works with Prof. Billy Rhoades. The research of Savas covers broadly the sequence spaces and summability. He has contributed numerous research papers in reputed journals and he has supervised several candidates for M. Sc. and Ph. D. degrees. He has actively participated in several national and international conferences. He became member of Turkish Academy of Science on June 2012. Now he has been working at Faculty of Arts and Sciences, Istanbul Commerce University.