Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090308

Weighted Exponential Distribution: Different Methods of Estimations

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Received: 9 Jul. 2014, Revised: 10 Oct. 2014, Accepted: 11 Oct. 2014
Published online: 1 May 2015

Abstract: The aim of this paper is to compare through Monte Carlo simulations the finite sample properties of the estimates of the parameters of the weighted exponential distribution obtained by five estimation methods: maximum likelihood, moments, L-moments, ordinary least-squares, and weighted least-squares. The bias and mean-squared error are used as the criterion for comparison. The simulation study concludes that the last four estimation methods perform well and are highly competitive with the maximum likelihood method in small and large samples. This conclusion is also supported with the analysis of two real data sets.

Keywords: Weighted exponential distribution, maximum likelihood, method of moments, L-moments, ordinary least-squares, weighted least-squares

1 Introduction

In the past few years, several statistical distributions have been proposed to model lifetime data which exhibit non-constant failure rate functions. One of such distributions is the two-parameter weighted exponential distribution introduced by Gupta and Kundu (2009). Its probability density function (p.d.f.) is given by

$$f(x|\alpha,\beta)=\frac{\alpha+1}{\alpha}\,\beta\,e^{-\beta x}(1-e^{-\alpha\beta x}),\quad x,\alpha,\beta>0. \quad (1)$$

Note that $f(x|0,\beta)$ converges to the gamma distribution with shape parameter 2 and scale parameter β and $f(x|\infty,\beta)$ converges to the exponential distribution with scale parameter β.

The corresponding cumulative distribution function (c.d.f.) and hazard rate function (h.r.f.), respectively, are given by

$$F(x|\alpha,\beta)=1-\frac{1}{\alpha}\,e^{-\beta x}(\alpha+1-\,e^{-\alpha\beta x}), \qquad (2)$$

and

$$h(x|\alpha,\beta)=(\alpha+1)\beta\,\frac{1-e^{-\alpha\beta x}}{\alpha+1-\,e^{-\alpha\beta x}}. \qquad (3)$$

For all values $\alpha,\beta>0$, the p.d.f. $f(x|\alpha,\beta)$ is a unimodal function in x with mode at $x_0=\frac{1}{\alpha\beta}\ln(\alpha+1)$ and the h.r.f. $h(x|\alpha,\beta)$ is an increasing function in x with $h(0|\alpha,\beta)=0$ and $h(\infty|\alpha,\beta)=\beta$.

Farahani and Khorram (2014) considered the Bayesian statistical inference for the weighted exponential distribution. Extension of the weighted exponential distribution is given in Shakhatreh (2012). Roy and Adnan (2012) introduced a class of non-symmetric circular distributions by wrapping an asymmetric weighted exponential distribution around the circumference of a unit circle. Extension of the weighted exponential distribution to the bivariate and multivariate cases are investigated by Al-Mutairi *et al.* (2011).

As far as the estimation of the parameters of the weighted exponential distribution, Gupta and Kundu (2009) considered only the maximum likelihood estimation (MLE) and method of moments estimation (MME). It is of interest to compare these methods with other estimation methods such as the L-moments estimation (LME), ordinary least-squares estimation (OLSE) and weighted least-squares estimation (WLSE).

The main aim of this paper is to compare the above different estimation methods via intensive simulation studies. Similar studies for other distributions can be found in, for example, Shawky and Bakoban (2012) for the exponentiated gamma distribution, Teimouri *et al.*

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(2013) for the Weibull distribution, and Usta (2013) for the extended Burr XII distribution.

In Section 2 we discuss the five estimation methods considered in this paper. The comparison of these methods in terms of bias and mean-squared error is presented in Section 3. The five estimation methods are used in fitting two real data sets in Section 4. Some concluding remarks are presented in Section 5.

2 Methods of Estimations

In this section we describe the five considered estimation methods to obtain the estimates of the parameters α and β of the weighted exponential distribution.

2.1 Maximum Likelihood

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the weighted exponential distribution with parameters α and β with p.d.f. (1).

The maximum likelihood estimates $\widehat{\alpha}_{MLE}$ and $\widehat{\beta}_{MLE}$, of α and β are obtained by maximizing the log-likelihood function

$$\ell(\alpha,\beta) = n\ln\left[\frac{(\alpha+1)\beta}{\alpha}\right] - \beta n\bar{x} + \sum_{i=1}^{n} \ln\left(1-e^{-\alpha\beta x_i}\right),$$

where $\bar{x}$ is the sample mean.

These estimates can also be obtained by solving the non-linear equations:

$$\frac{-n}{(\alpha+1)\alpha} + \sum_{i=1}^{n} \frac{\beta x_i\, e^{-\alpha\beta x_i}}{1-e^{-\alpha\beta x_i}} = 0,$$
$$\frac{n}{\beta} - n\bar{x} + \sum_{i=1}^{n} \frac{\alpha x_i\, e^{-\alpha\beta x_i}}{1-e^{-\alpha\beta x_i}} = 0.$$

It follows that $\widehat{\alpha}_{MLE}$ is the solution of the non-linear equation

$$\frac{n\bar{x}}{\alpha(\alpha+2)} - \sum_{i=1}^{n} \frac{x_i\, \exp[-\frac{\alpha(\alpha+2)}{(\alpha+1)\,\bar{x}}\, x_i]}{1-\exp[-\frac{\alpha(\alpha+2)}{(\alpha+1)\,\bar{x}}\, x_i]} = 0, \qquad (4)$$

and

$$\widehat{\beta}_{MLE} = \frac{\widehat{\alpha}_{MLE}+2}{(\widehat{\alpha}_{MLE}+1)\,\bar{x}}. \qquad (5)$$

2.2 Method of Moments

The method of moments is another technique commonly used in parameter estimation. For the weighted exponential distribution, the first two raw moments, respectively, are

$$E(X|\alpha,\beta) = \frac{\alpha+2}{(\alpha+1)\beta},$$
$$E(X^2|\alpha,\beta) = \frac{2(\alpha^2+3\alpha+2)}{(\alpha+1)^2\beta^2}.$$

The method of moments estimates $\widehat{\alpha}_{MME}$ and $\widehat{\beta}_{MME}$ for α and β, are obtained by solving the equations:

$$E(X|\widehat{\alpha}_{MME},\widehat{\beta}_{MME}) = m_1, \qquad E(X^2|\widehat{\alpha}_{MME},\widehat{\beta}_{MME}) = m_2,$$

where

$$m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}, \qquad m_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2,$$

are the first two sample moments, respectively.

It follows that

$$\widehat{\alpha}_{MME} = \frac{2M-3+\sqrt{2M-3}}{2-M}, \qquad (6)$$

$$\widehat{\beta}_{MME} = \frac{\widehat{\alpha}_{MME}+2}{(\widehat{\alpha}_{MME}+1)\,\bar{x}}, \qquad (7)$$

provided that

$$\frac{3}{2} < M = \frac{m_2}{m_1^2} < 2. \qquad (8)$$

Note that $M-1 = \frac{m_2-m_1^2}{m_1^2} = \frac{s^2}{\bar{x}^2}$, where $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$ is the (biased) sample variance. This means that condition (8) is equivalent to

$$0.7071 = \frac{1}{\sqrt{2}} < \frac{s}{\bar{x}} < 1, \qquad (9)$$

i.e. the value of the sample coefficient of variation is in the interval $(\frac{1}{\sqrt{2}}, 1)$.

2.3 Method of L-Moments

The method of L-moments was proposed by Hosking (1990). This method is defined in terms of linear functions of population order statistics and their sample counterparts.

For the weighted exponential distribution, the first two population L-moments, respectively, are given by

$$l_1(\alpha,\beta) = E(X_{1:1}|\alpha,\beta) = E(X|\alpha,\beta),$$
$$l_2(\alpha,\beta) = \frac{1}{2}E\left(X_{2:2}-X_{1:2}|\alpha,\beta\right)$$
$$= E(X|\alpha,\beta) - E(X_{1:2}|\alpha,\beta),$$

where $X_{r:m}$ is the rth order statistic from a random sample of size m, $E(X|\alpha,\beta)$ is the population mean and, using the substitution $y=e^{-\beta x}$,

$$\begin{aligned} E(X_{1:2}|\alpha,\beta) &= \int_0^\infty [1-F(x|\alpha,\beta)]^2\,dx \\ &= \frac{1}{\alpha^2\beta}\int_0^1 y(\alpha+1-y^\alpha)^2\;dy \\ &= \frac{\alpha^2+5\alpha+5}{2(\alpha+1)(\alpha+2)\beta}. \end{aligned}$$

The L-moments estimators $\widehat{\alpha}_{LME}$ and $\widehat{\beta}_{LME}$ of α and β are obtained by solving the equations:

$$l_1(\widehat{\alpha}_{LME},\widehat{\beta}_{LME}) = l_1, \qquad l_2(\widehat{\alpha}_{LME},\widehat{\beta}_{LME}) = l_2,$$

where

$$l_1 = \overline{x}, \qquad l_2 = \left\{\frac{2}{n(n-1)}\sum_{i=0}^{n}(i-1)x_{i:n}\right\} - \overline{x},$$

are the first two sample L-moments, respectively.

It follows that

$$\widehat{\alpha}_{LME} = \frac{2L-3+\sqrt{2L-3}}{2-L}, \tag{10}$$

$$\widehat{\beta}_{LME} = \frac{\widehat{\alpha}_{LME}+2}{(\widehat{\alpha}_{LME}+1)\,\overline{x}}, \tag{11}$$

provided that

$$\frac{3}{2} < L = \frac{4l_2}{l_1} < 2. \tag{12}$$

2.4 Ordinary and Weighted Least-Squares

Let $x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{n:n}$ be the order statistics of a random sample of size n from a distribution with c.d.f. $F(x)$. It is well known that:

$$E\left[F(x_{i:n})\right] = \frac{i}{n+1}, \qquad Var\left[F(x_{i:n})\right] = \frac{i(n-i+1)}{(n+1)^2(n+2)}.$$

For the weighted exponential distribution, the ordinary least squares estimates $\widehat{\alpha}_{OLSE}$ and $\widehat{\beta}_{OLSE}$ of the parameters α and β are obtained by minimizing the function

$$S(\alpha,\beta) = \sum_{i=1}^{n}\left[F(x_{i:n}\mid\alpha,\beta) - \frac{i}{n+1}\right]^2.$$

These estimates can also be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}\left[F(x_{i:n}\mid\alpha,\beta) - \frac{i}{n+1}\right]\Delta_1(x_{i:n}\mid\alpha,\beta) = 0, \tag{13}$$

$$\sum_{i=1}^{n}\left[F(x_{i:n}\mid\alpha,\beta) - \frac{i}{n+1}\right]\Delta_2(x_{i:n}\mid\alpha,\beta) = 0, \tag{14}$$

where

$$\Delta_1(x_{i:n}|\alpha,\beta) = \left[1-(\alpha\beta\,x_{i:n}+1)e^{-\alpha\beta x_{i:n}}\right]e^{-\beta x_{i:n}}, \tag{15}$$

$$\Delta_2(x_{i:n}|\alpha,\beta) = \left(1-e^{-\alpha\beta x_{i:n}}\right)x_{i:n}\,e^{-\beta x_{i:n}}. \tag{16}$$

The weighted least-squares estimates $\widehat{\alpha}_{WLSE}$ and $\widehat{\beta}_{WLSE}$ of the parameters α and β are obtained by minimizing the function

$$W(\alpha,\beta) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}\mid\alpha,\beta) - \frac{i}{n+1}\right]^2.$$

These estimates can also be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}\frac{\Delta_1(x_{i:n}\mid\alpha,\beta)}{i(n-i+1)}\left[F(x_{i:n}\mid\alpha,\beta) - \frac{i}{n+1}\right] = 0, \tag{17}$$

$$\sum_{i=1}^{n}\frac{\Delta_2(x_{i:n}\mid\alpha,\beta)}{i(n-i+1)}\left[F(x_{i:n}\mid\alpha,\beta) - \frac{i}{n+1}\right] = 0, \tag{18}$$

where $\Delta_1(x_{i:n}\mid\alpha,\beta)$ and $\Delta_2(x_{i:n}\mid\alpha,\beta)$, are given by (15) and (16), respectively.

3 Simulations

In this section we present results of some numerical experiments to compare the performance of the five estimators discussed in the previous section. We have taken sample sizes $n = 25, 50, \ldots, 200$, and parameter values $(\alpha,\beta) : (0.5,0.5), (0.5,2), (1,4), (2,0.5)$.

For each combination (n,α,β), we have generated $N = 10,000$ pseudo-random samples from the weighted exponential distribution using the fact that $X = Y + Z$ where Y and Z are independent exponential random variables with scale parameters β and $(\alpha+1)\beta$, respectively. The generated sample values satisfy the conditions (8) and (12) in order to make the comparison between all the considered five estimations methods.

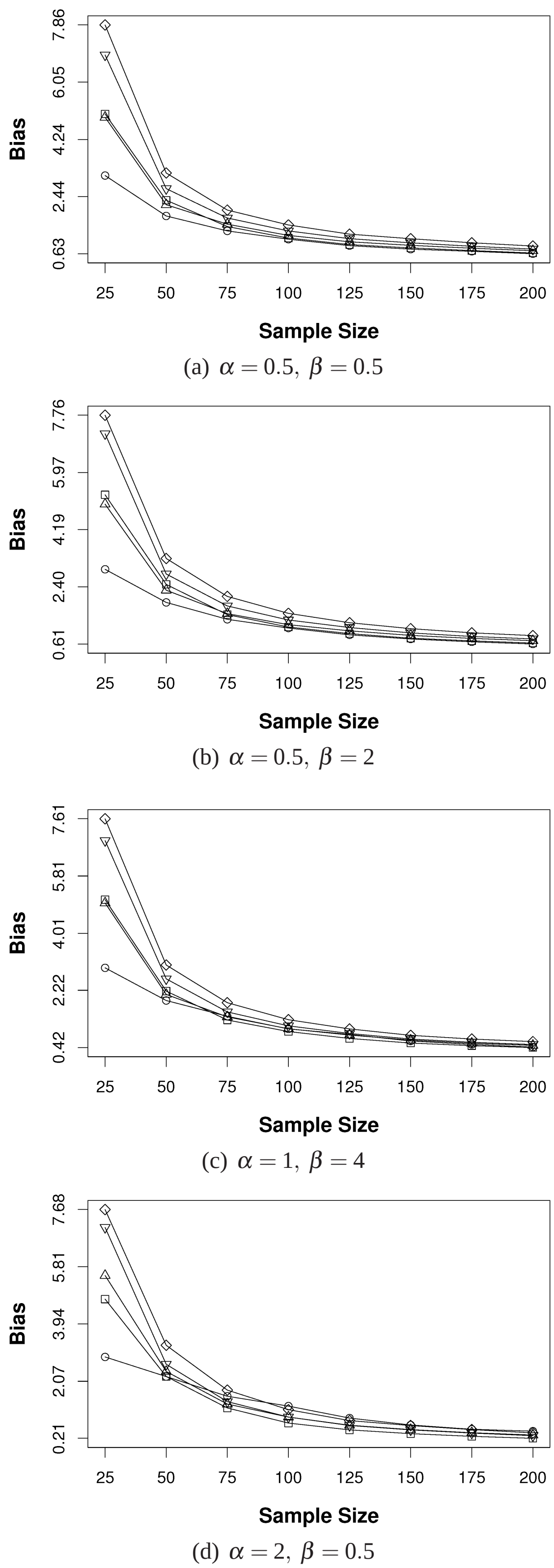
All calculations were performed using the R statistical software Version 3.0.0., R Core Team (2013). The estimates using MLE, OLSE and WLSE methods were obtained using the `optim` function. To assess the performance of the methods, we calculated the bias and the mean-squared error for the simulated estimates of $\theta = \alpha, \beta$:

$$Bias(\widehat{\theta}) = \tfrac{1}{N}\sum_{i=1}^{N}(\widehat{\theta}_i - \theta), \qquad MSE(\widehat{\theta}) = \tfrac{1}{N}\sum_{i=1}^{N}(\widehat{\theta}_i - \theta)^2.$$

Figures 1-2 show, respectively, the bias of the simulated estimates of α and β. From these two figures, we observe that

(i) all the estimators of the parameter α (β) are *positively biased* (*positively and/or negatively biased*),

(ii) the biases of the estimators of the parameters α and β tend to zero for large n, i.e. the estimators are asymptotically unbiased for the parameters,



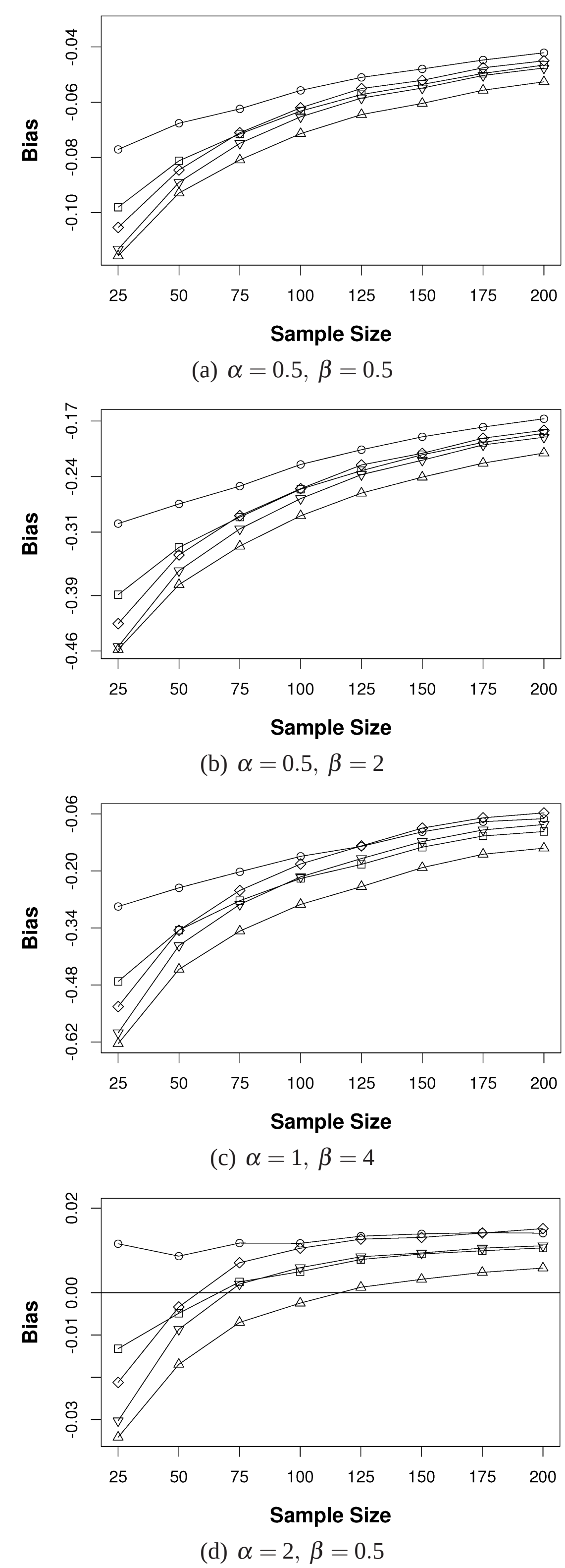
(a) $\alpha = 0.5,\ \beta = 0.5$

(b) $\alpha = 0.5,\ \beta = 2$

(c) $\alpha = 1,\ \beta = 4$

(d) $\alpha = 2,\ \beta = 0.5$

Fig. 1: Bias of $\widehat{\alpha}$ ($\square$: MLE, $\bigcirc$: MME, $\triangle$: LME, $\lozenge$: OLSE, $\triangledown$: WLSE).



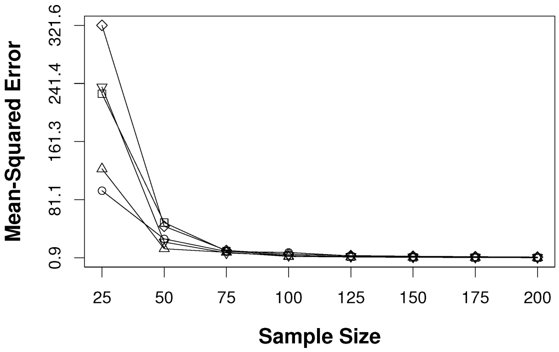
(a) $\alpha = 0.5,\ \beta = 0.5$

(b) $\alpha = 0.5,\ \beta = 2$

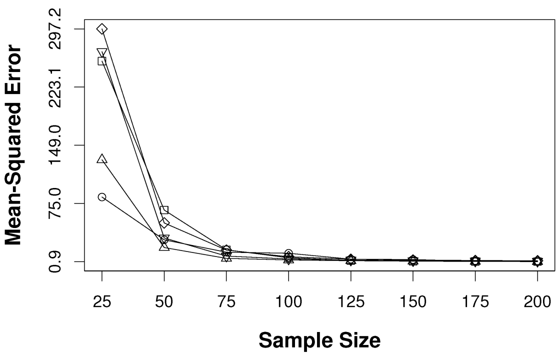
(c) $\alpha = 1,\ \beta = 4$

(d) $\alpha = 2,\ \beta = 0.5$

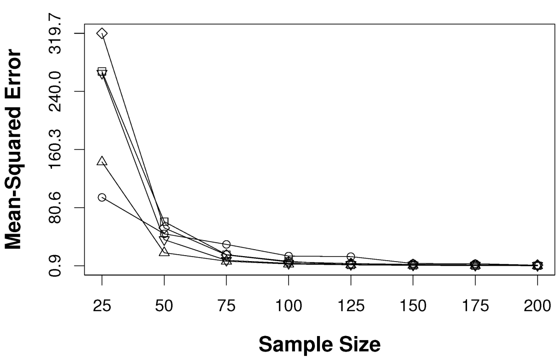
Fig. 2: Bias of $\widehat{\beta}$ ($\square$: MLE, $\bigcirc$: MME, $\triangle$: LME, $\lozenge$: OLSE, $\triangledown$: WLSE).



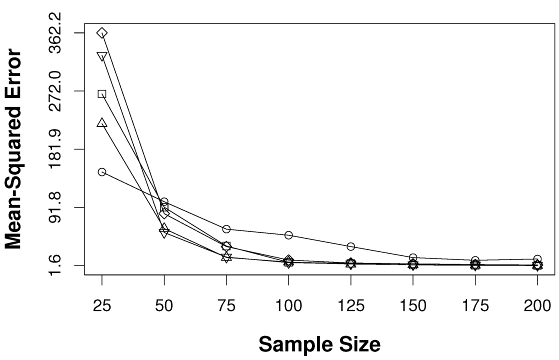
(a) $\alpha = 0.5,\ \beta = 0.5$



(b) $\alpha = 0.5,\ \beta = 2.0$

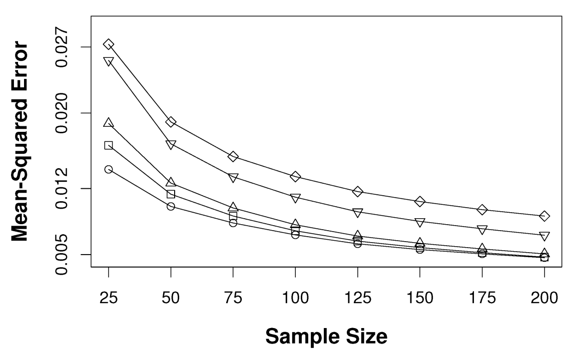


(c) $\alpha = 1,\ \beta = 4$

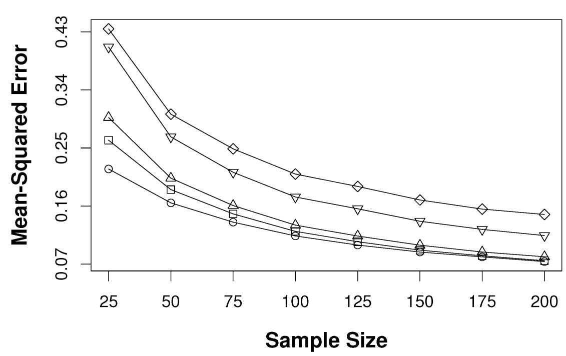


(d) $\alpha = 2,\ \beta = 0.5$

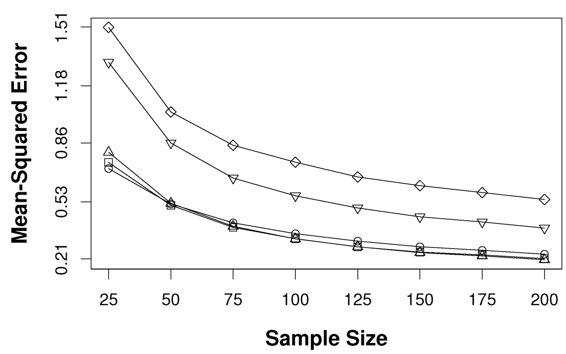
Fig. 3: MSE of $\widehat{\alpha}$ ($\square$: MLE, $\bigcirc$: MME, $\triangle$: LME, $\lozenge$: OLSE, $\triangledown$: WLSE).



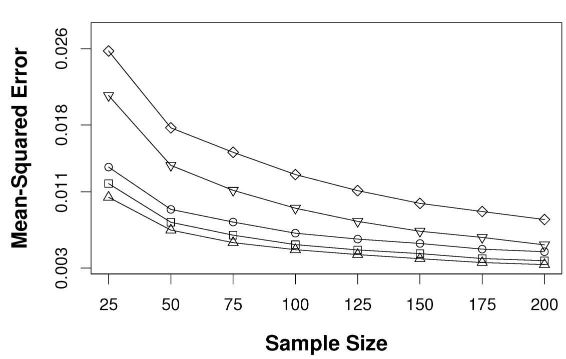
(a) $\alpha = 0.5,\ \beta = 0.5$



(b) $\alpha = 0.5,\ \beta = 2$



(c) $\alpha = 1$ and $\beta = 4$



(d) $\alpha = 2,\ \beta = 0.5$

Fig. 4: MSE of $\widehat{\beta}$ ($\square$: MLE, $\bigcirc$: MME, $\triangle$: LME, $\lozenge$: OLSE, $\triangledown$: WLSE).

(iii) the MME of the parameter α has smaller positive bias compared to other estimators when the sample size is small ($n < 50$), otherwise, all estimators biases are very close,

(iv) the MME of the parameter β has a smaller absolute bias compared to other estimators except the case $\alpha = 2,\ \beta = 0.5$ where the MLE, OLSE, WLE and WLE have smaller absolute bias.

Figures 3-4 show, respectively, the MSE of the simulated estimates of α and β. From these two figures, we observe that

(i) the MSE of all estimators of the parameter α (β) tend to zero for large n, i.e. the estimators are consistent for the parameter,

(ii) the MME of the parameter α has smaller MSE compared to other estimators when the sample size is small ($n < 50$), otherwise, LME or WLSE have smaller MSE,

(iii) the MME or LME of the parameter β has smaller MSE compared to other estimators.

4 Data anaylsis

In this section we analyze two real data sets for comparing the considered five estimation methods for the weighted exponential distribution.

Data set 1: (Gupta and Kundu 2009)

This data set represents the marks in Mathematics for 48 students in the slow pace programme in the year 2003: 29, 25, 50, 15, 13, 27, 15, 18, 7, 7, 8, 19, 12, 18, 5, 21, 15, 86, 21, 15, 14, 39, 15, 14, 70, 44, 6, 23, 58, 19, 50, 23, 11, 6, 34, 18, 28, 34, 12, 37, 4, 60, 20, 23, 40, 65, 19, 31.

Data set 2: (Ghitany *et al.* 2008)

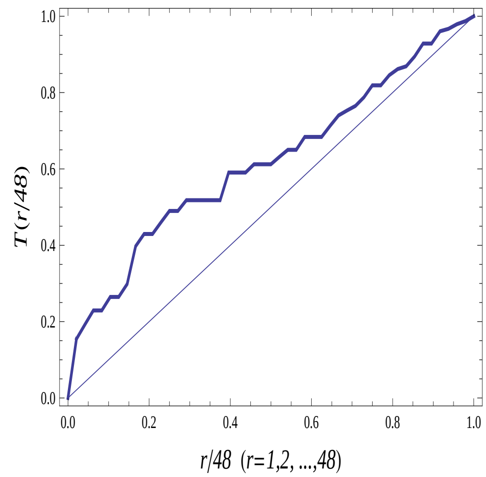
This data set represents the waiting times (in minutes) before service of 100 bank customers:

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

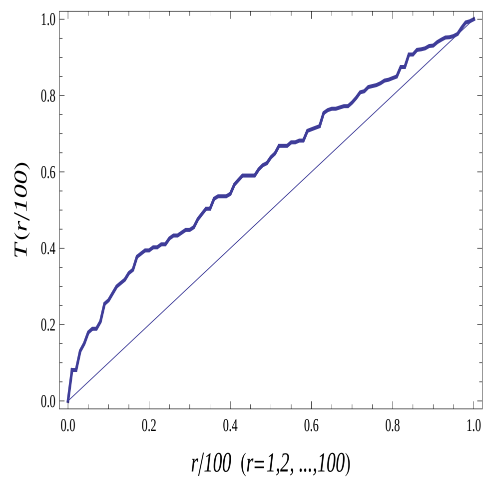
Table 1 shows that the conditions for the existence of MME and LME for the considered data sets are satisfied.

Table 1: Existence of MME and LME for data sets 1 and 2.

Data set	M	$s/\bar{x}$	L
1	1.5054	0.7109	1.5209
2	1.5315	0.7290	1.5416



(a) Data set 1



(b) Data set 2

Fig. 5: Empirical scaled total time on test plots for data sets 1 and 2.

Figure 5 shows the empirical scaled total time on test (TTT)-transform (Barlow and Campo 1975) where

$$T(r/n) = \frac{\sum_{i=1}^{r} x_{i:n} + (n-r)x_{r:n}}{\sum_{i=1}^{n} x_{i:n}}, \qquad r = 1, 2, \ldots, n.$$

Inspection of Figure 5 shows concave behavior above the diagonal line, indicating that each of the considered data sets is drawn from a population with an increasing failure rate (IFR).

Tables 2-3 show the estimates of the parameters α and β under the considered five estimation methods. These tables also show the corresponding Crámer-von Mises goodness-of-fit test statistic:

$$\text{CvM} = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{i:n}|\hat{\alpha}, \hat{\beta}) - \frac{2i-1}{2n} \right]^2,$$

Table 2: Parameters estimates, Crámer-von Mises test and SSR for data set 1.

Method	$\widehat{\alpha}$	$\widehat{\beta}$	CvM	p-value	SSR
MLE	0.2797	0.0688	0.0639	0.7892	0.0972
MME	0.2322	0.0700	0.0642	0.7873	0.0983
LME	0.5139	0.0641	0.0624	0.7988	0.0906
OLSE	0.5668	0.0645	0.0588	0.8218	0.0690
WLSE	0.8410	0.0592	0.0631	0.7946	0.0879

Table 3: Parameters estimates, Crámer-von Mises test and SSR for data set 2.

Method	$\widehat{\alpha}$	$\widehat{\beta}$	CvM	p-value	SSR
MLE	0.7033	0.1607	0.0220	0.9947	0.0298
MME	0.6702	0.1619	0.0222	0.9944	0.0305
LME	0.8103	0.1572	0.0217	0.9951	0.0277
OLSE	0.8888	0.1552	0.0216	0.9953	0.0248
WLSE	0.8860	0.1545	0.0221	0.9946	0.0382

and its p-value as well as the sum of squares of the residuals (SSR):

$$\mathrm{SSR} = \sum_{i=1}^{n}\left[F(x_{i:n}|\hat{\alpha},\hat{\beta}) - F_n(x_{i:n})\right]^2,$$

where $F_n(x) = \frac{1}{n}(\text{number of } x_i's \leq x)$ is the empirical c.d.f.

Tables 2-3 show that the OLSE method has the smallest test statistic and highest p-values of Crámer-von Mises test as well as smallest sum of squares of the residuals. Hence, for each of the given data sets, the OLSE is the most suitable estimation method among the five considered methods.

5 Conclusions

In this paper we compared, via intensive simulation experiments, the estimation of the parameters of the weighted exponential distribution using five well known estimation methods, namely the maximum likelihood, method of moments, method of L-moments, ordinary least-squares, and weighted least-squares. The simulation study concludes that the last four estimation methods perform well and are highly competitive with the maximum likelihood method in small and large sample sizes. This conclusion is also supported with the analysis of two real data sets.

6 Acknowledgements

The authors are grateful to Professor Debasis Kundu for helpful discussions during his visit to the Department of Statistics and Operations Research at Kuwait University.

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