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Some Remarks on Geometry of Fractional Calculus

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Abstract: In recent years, researchers interested in the field of fractals and related subjects have also begun using the concept of fractional calculus in some of their investigations. In this paper, some interesting aspects and features of fractional connection derivatives in differential manifold were discussed. In particular, transformation of Christoffel symbols for fractional connection, the torsion tensor of a fractional connection, and difference tensor of two fractional connection are presented.

Keywords: Fractional geometrical objects, Fractional manifold, Revised Riemann-Liouville fractional calculus on manifolds

1 Introduction

In mathematics, there are several special functions which appear many applications. One of them is well known as the Gamma function, see for example, [9]. The gamma function $\Gamma(x)$ plays some significant roles in the theory of integro-differential equations in particular fractional calculus. Thus we begin with some definitions, for the details we refer to [1] or [5].

The Gamma function of a positive integer n is again a positive integer, while the gamma function $\Gamma(-n)$ of a negative integer changes to infinities. The gamma function and As we known the Gamma function for any positive α value is defined as follows:

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt.$$

For $\Gamma(\alpha)$ can be demonstrated that:

$$\Gamma(1) = 1,\ \Gamma(\alpha+1) = \alpha\Gamma(\alpha),\ \alpha > 0$$

also, $\alpha! = \Gamma(\alpha+1)$.

For example the gamma functions $\Gamma\left(\frac{1}{2}+n\right)$ and $\Gamma\left(\frac{1}{2}-n\right)$ turn out to be multiples of $\sqrt{2}$ that is

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{2},$$
$$\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!\sqrt{2}}{4^n n!},$$
$$\Gamma\left(\frac{1}{2}-n\right) = \frac{(-4)^n n!\sqrt{2}}{(2n)!}.$$

In particular, some frequently encountered examples can be given as follows:

$$\left(\frac{5}{2}\right)! = \Gamma\left(\frac{5}{2}+1\right)! = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\Gamma\left(\frac{3}{2}+1\right) = \ldots = \frac{15}{8}\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}$$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}.$$

Two more useful properties of the gamma function are the reflection and the duplication as follows:

$$\Gamma(-x) = \frac{-\pi csc(\pi x)}{\Gamma(x+1)}$$
$$\Gamma(2x) = \frac{4^x\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)}{2\sqrt{2}}.$$

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The Gamma function $\Gamma(x)$ is considered as a generalization of the factorial and $\Gamma(x)$ is usually defined for $x > 0$ by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

In the classical sense since $\Gamma(0) = \frac{\Gamma(1)}{0}$, then it follows that $\Gamma(n)$ is not defined for integers $n \leq 0$. However the extension formula gives finite values for $\Gamma(z)$, for $\Re(z) \leq 0$ since $\Gamma(z)$ is analytic everywhere except at $z = 0, -1, -2, \ldots$, and the residue at $z = k$ is given by

$$\mathrm{Res}_{z=k}\Gamma(z) = \frac{(-1)^k}{k!}.$$

If we consider $x > 0$, then it follows that

$$\Gamma(x+1) = x\Gamma(x). \tag{1}$$

Now the equation (1) can then be used to define $\Gamma(x)$ for $x < 0$ and $x \neq -1, -2, \ldots$ and further this is one of the most important formulas that was satisfied by the Gamma function.

Even though the gamma function is defined as a locally summable function on the real line by [11]

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-x} dx, \qquad x > 0. \tag{2}$$

In the classical sense $\Gamma(x)$ function was not defined for the negative integer thus there was an open problem to give satisfactory definition. However, by using the neutrix limit, it has been shown in [12] that the gamma function (2) is defined as follows:

$$\Gamma(x) = N - \lim_{\varepsilon \to 0} \int_\varepsilon^\infty t^{x-1} e^{-t} dt, \tag{3}$$

for $x \neq 0, -1, -2, \ldots$, and this function is also defined by neutrix limit

$$\begin{aligned}\Gamma(-m) &= N - \lim_{\varepsilon\to 0}\int_\varepsilon^\infty t^{-m-1}e^{-t}dt \\ &= \int_1^\infty t^{-m-1}e^{-t}dt \\ &+ \int_0^1 t^{-m-1}\left[e^{-t} - \sum_{i=0}^{m}\frac{(-1)^i}{i!}t^i\right]dt - \sum_{i=0}^{m-1}\frac{(-1)^i}{i!(m-i)},\end{aligned} \tag{4}$$

for $m \in \mathbb{N}$. It was proven also in [13] the existence of rth derivative of the gamma function and defined it by the equation

$$\begin{aligned}\Gamma^{(r)}(0) &= N - \lim_{\varepsilon\to 0}\int_\varepsilon^\infty t^{-1}\ln^r t e^{-t}dt \\ &= \int_1^\infty t^{-1}\ln^r t e^{-t}dt + \int_0^1 t^{-1}\ln^r t\left[e^{-t}-1\right]dt, \\ \Gamma^{(r)}(-m) &= N - \lim_{\varepsilon\to 0}\int_\varepsilon^\infty t^{-m-1}\ln^r t e^{-t}dt \\ &= \int_1^\infty t^{-m-1}\ln^r t e^{-t}dt \\ &+ \int_0^1 t^{-m-1}\ln^r t\left[e^{-t} - \sum_{i=0}^{m}\frac{(-1)^i}{i!}t^i\right]dt \qquad (5) \\ &- \sum_{i=0}^{m-1}\frac{(-1)^i}{i!}r!(m-i)^{-r-1}, \qquad (6)\end{aligned}$$

for $r \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Further, it is proved that

$$\Gamma(-r) = \frac{(-1)^r}{r!}\phi(r) - \frac{(-1)^r}{r!}\gamma \tag{7}$$

for $r = 1, 2, \ldots$, where

$$\phi(r) = \sum_{i=1}^{r}\frac{1}{i}.$$

thus we can extend the definition to the whole real line where,

$$\Gamma(0) = \Gamma'(1) = -\gamma,$$

where γ denotes Euler's constant, see [14].

For a function $f : U \subset \mathbb{R} \to \mathbb{R}$ with $0 \in U$ the fractional derivative of order α is defined by

$$\frac{d^\alpha}{dt^\alpha}f(t) = \frac{1}{\Gamma(-\alpha)}\int_0^t \frac{f(s)-f(0)}{(t-s)^{1+\alpha}}ds, \ \ \alpha < 0 \tag{8}$$

$$\frac{d^\alpha}{dt^\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)-f(0)}{(t-s)^{\alpha-n+1}}ds, \ \ \alpha > 0 \tag{9}$$

where n is the first integer greater than or equal to α. The relation (8) gives a fractional integral and (9) gives a fractional derivative.

We express some of the operators of a fractional derivatives, see for example, [2,4,6,7,8,10].

1. $\frac{d^\alpha}{dt^\alpha}(t^\alpha) = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)}t^{\gamma-\alpha}$, $\alpha \in \mathbb{R}$ or $(\alpha \in \mathbb{C})$ and $1+\gamma \neq 0, -1, \ldots, -n$,
2. $\frac{d^n}{dt^n}\frac{d^\alpha}{dt^\alpha}f(t) = \frac{d^{n+\alpha}}{dt^{n+\alpha}}f(t)$, $n \in \mathbb{N}$.
3. $\frac{d^\alpha}{dt^\alpha}(f_1(t)+f_2(t)) = \frac{d^\alpha}{dt^\alpha}f_1(t) + \frac{d^\alpha}{dt^\alpha}f_2(t)$
4. $\frac{d^\alpha}{dt^\alpha}(Cf(t)) = C\frac{d^\alpha}{dt^\alpha}f(t)$, where C is a constant
5. $\frac{d^\alpha}{dt^\alpha}f(\beta t) = \beta^\alpha \frac{d^\alpha}{[d(\beta t)]^\alpha}f(\beta t)$.

2 Fractional Differential Calculus on Manifolds

Let M be an n-dimensional differential manifold (U,x_i) a local coordinate system on M and $U_0=\{x\in U: 0\le x_i\le b_i, i=1,2,...,n\}$, see [3].

For a function $f: U_0\to\mathbb{R}$ the fractional derivative with respect to x_i:

$$\partial_i^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)}$$

$$\partial_{x_i}^m \int_0^{x_i} \frac{f(x_1,...,x_{i-1},s,x_{i+1},...,x_n)-f(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)}{(x_i-s)^{\alpha-m+1}}ds$$

where

$$\partial_{x_i}^m = \frac{\partial}{\partial x_i}\circ\frac{\partial}{\partial x_i}\circ\dots\circ\frac{\partial}{\partial x_i}$$

(m times, i is fixed, $\alpha\ge 0$).

For $\alpha\in(0,1), \gamma>-1$,

$$\partial_i^\alpha (x_i)^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} \quad ;\partial_i^\alpha=\delta_i^j.$$

A fractional vector field $U\subset M$ is an object of the form $X^\alpha = X_i^\alpha\partial_i^\alpha$, where $X_i^\alpha\in\Im_U(M)\quad i=1,...,n$.

We denote by χ_U^α the fractional vector fields on $U.\chi_U^\alpha$ is generated by the operators $\partial_i^\alpha, i=1,2,...,n$. If $c: x=x(t), t\in I$ is a parametrized curve in U then the fractional tangent vector field of c is

$$x^\alpha(t) = \frac{1}{\Gamma(1+\alpha)}\partial_t^\alpha x_i(t)\partial_i^\alpha.$$

A fractional covariant derivative is given by

$$\bigtriangledown_{X^\alpha}^\alpha Y^\alpha = X_i^\alpha\left(\partial_i^\alpha Y_j^\alpha + \Gamma_{ik}^{\alpha j}Y_k^\alpha\right)\partial_j^\alpha \qquad (10)$$

where $X^\alpha, Y^\alpha\in\chi_U^\alpha$ and $\Gamma_{ik}^{\alpha j}$ the functions defining the coefficients of a fractional linear connection on M. They are determined by the relations

$$\bigtriangledown_{\partial_i^\alpha}^\alpha\partial_k^\alpha = \Gamma_{ik}^{\alpha j}\partial_j^\alpha.$$

Since it it is important to study fractional vector fields on a differentiable manifold M. For $\mathbb{R}^n$, there is an obvious way to do this. Recall that $\chi^\alpha(\mathbb{R}^n)$ denotes the space of fractional differentiable vector fields defined on $\mathbb{R}$. Examples are the fractional vector fields $\frac{\partial^\alpha}{\partial u_1^\alpha},...,\frac{\partial^\alpha}{\partial u_n^\alpha}$ determined by the natural coordinate functions $u_1,...,u_n$.

Now consider an arbitrary fractional vector field $Y^\alpha\in\chi^\alpha(\mathbb{R}^n)$; in terms of (10), we can write

$$Y^\alpha f = \sum_n^{i=1} Y_i^\alpha\frac{\partial^\alpha}{\partial u_i^\alpha}f = \frac{1}{(1-\alpha)!}\sum Y_i^\alpha f^{1-\alpha}\left(\frac{\partial f}{\partial u_i}\right)^\alpha$$

where $Y_i^\alpha\in\Im(\mathbb{R}^n)$.

Note that, in calculus the fractional vector field Y^α is frequently identified with the n-tuple Y_i^α, $1\le i\le n$. The obvious candidate for the fractional derivative of Y^α in the direction of a fractional vector X^α is then

$$\bigtriangledown_{X^\alpha}^\alpha Y^\alpha = (X^\alpha Y_1^\alpha,...,X^\alpha Y_n^\alpha).$$

For a general differentiable manifold, there is no automatic way to differentiate fractional vector fields. Instead, we must add a new element of structure:

Definition 1. *A fractional connection or fractional covariant derivative on a differentiable manifold M is a map*

$$\bigtriangledown^\alpha: \chi^\alpha(M)\times\chi^\alpha(M)\to\chi^\alpha(M)$$

which for $\chi^\alpha\in\chi^\alpha(M)$ also defines a map

$$\bigtriangledown_{X^\alpha}^\alpha: \chi^\alpha(M)\to\chi^\alpha(M).$$

Then we have the following properties:

1. $\bigtriangledown_{fX^\alpha+gY^\alpha}^\alpha = f\bigtriangledown_{X^\alpha}^\alpha + g\bigtriangledown_{Y^\alpha}^\alpha$,
2. $\bigtriangledown_{X^\alpha}^\alpha(Y^\alpha+Z^\alpha) = \bigtriangledown_{X^\alpha}^\alpha Y^\alpha + \bigtriangledown_{X^\alpha}^\alpha Z^\alpha$,
3. $\bigtriangledown_{X^\alpha}^\alpha(fY^\alpha) = (X^\alpha f)Y^\alpha + f\bigtriangledown_{X^\alpha}^\alpha Y^\alpha$,

for $X^\alpha, Y^\alpha, Z^\alpha\in\chi^\alpha(M)$ and $f,g\in\Im(M)$.

Remark. By using the properties in Definition (1) we easily can see that

$$\bigtriangledown_{X^\alpha}^\alpha(aY^\alpha+bZ^\alpha) = a\bigtriangledown_{X^\alpha}^\alpha Y^\alpha + b\bigtriangledown_{X^\alpha}^\alpha Z^\alpha$$

for $X^\alpha, Y^\alpha, Z^\alpha\in\chi^\alpha(M)$ and $a,b\in\mathbb{R}$.

Now given any two factional vector fields X^α, Y^α on M for $f\in\Im_U(M)$, then we can consider the functions $X^\alpha(Y^\alpha f)$ and $Y^\alpha(X^\alpha f)$. In general, such operations do not lead to factional vector fields, since they involve the derivatives of higher order rather than one. Nevertheless, we can affirm the following.

Lemma 1. *Given any differentiable manifold M, of dimension n, for any two factional vector fields X^α and Y^α on M, there is a unique factional vector field $[X^\alpha, Y^\alpha]$ such that for all $f\in\Im(M)$*

$$[X^\alpha, Y^\alpha](f) = X^\alpha(Y^\alpha f) - Y^\alpha(X^\alpha f).$$

Proof: First, we prove that if $[X^\alpha, Y^\alpha]$ exists, then it is unique. Let $m\in M$ and let $\varphi: U\to M$ be a parametrization at m, and let

$$X^\alpha = \sum X_i^\alpha\partial_i^\alpha, \text{ and } Y^\alpha = \sum Y_j^\alpha\partial_j^\alpha$$

where $X_i^\alpha, Y_j^\alpha\in\Im_U(M)$. Then for $f\in\Im(M)$ we have

$$X^\alpha(Y^\alpha f) = X_i^\alpha\partial_i^\alpha(Y_j^\alpha\partial_j^\alpha f) = X_i^\alpha(\partial_i^\alpha Y_j^\alpha)(\partial_j^\alpha f) + X_i^\alpha Y_j^\alpha\partial_i^\alpha\partial_j^\alpha f$$

$$Y^\alpha(X^\alpha f) = Y_j^\alpha\partial_j^\alpha(X_i^\alpha\partial_i^\alpha f) = Y_j^\alpha(\partial_j^\alpha X_i^\alpha)(\partial_i^\alpha f) + Y_j^\alpha X_i^\alpha\partial_j^\alpha\partial_i^\alpha f.$$

Since $\partial_i^\alpha\partial_j^\alpha f = \partial_j^\alpha\partial_i^\alpha f$, then we get

$$\begin{aligned} X^\alpha(Y^\alpha(f)) - Y^\alpha(X^\alpha(f)) &= X_i^\alpha(\partial_i^\alpha Y_j^\alpha)(\partial_j^\alpha f) + Y_j^\alpha(\partial_j^\alpha X_i^\alpha)(\partial_i^\alpha f) \\ &= [X_i^\alpha\partial_i^\alpha Y_j^\alpha - Y_i^\alpha\partial_i^\alpha X_j^\alpha]\partial_j^\alpha f. \end{aligned}$$

In order to show the existence, we use the above expression to define $[X^\alpha, Y^\alpha]_\beta$ for every chart (U_β, φ_β). By the uniqueness, $[X^\alpha, Y^\alpha]_\beta = [X^\alpha, Y^\alpha]_\gamma$ on $\varphi_\beta(U_\beta)\cap\varphi_\gamma(U_\gamma)$ which allows us to define $[X^\alpha, Y^\alpha]$ over entire manifold M.

The fractional vector field $[X^\alpha, Y^\alpha]$ given by above lemma is called the fractional bracket $[X^\alpha, Y^\alpha] = X^\alpha Y^\alpha - Y^\alpha X^\alpha$ of X^α and Y^α.

Proposition 1. *If X^α,Y^α, and Z^α are fractional vector fields on M, for all $f,g\in\Im(M)$, then*

1. (Anticommutativity) $[X^\alpha,Y^\alpha]=-[Y^\alpha,X^\alpha]$.
2. (Linearity) $[aX^\alpha+bY^\alpha,Z^\alpha]=a[X^\alpha,Z^\alpha]+b[Y^\alpha,Z^\alpha]$, $a,b\in\mathbb{R}$.
3.(Jacobi identity) $[[X^\alpha,Y^\alpha],Z^\alpha]+[[Y^\alpha,Z^\alpha],X^\alpha]+[[Z^\alpha,X^\alpha],Y^\alpha]=0$.
4.$[fX^\alpha,gY^\alpha]=fg[X^\alpha,Y^\alpha]+fX^\alpha(g)Y^\alpha-gY^\alpha(f)X$.

Proof. (1) and (2) are immediate.

(3)

$$\begin{aligned}[[X^\alpha,Y^\alpha],Z^\alpha] &= [X^\alpha Y^\alpha-Y^\alpha X^\alpha,Z^\alpha]\\ &= X^\alpha Y^\alpha Z^\alpha-Y^\alpha X^\alpha Z^\alpha-Z^\alpha X^\alpha Y^\alpha+Z^\alpha Y^\alpha X^\alpha\end{aligned}$$

$$\begin{aligned}[[Y^\alpha,Z^\alpha],X^\alpha] &= [Y^\alpha Z^\alpha-Z^\alpha Y^\alpha,X^\alpha]\\ &= Y^\alpha Z^\alpha X^\alpha-Z^\alpha Y^\alpha X^\alpha-X^\alpha Y^\alpha Z^\alpha+X^\alpha Z^\alpha Y^\alpha\end{aligned}$$

$$\begin{aligned}[[Z^\alpha,X^\alpha],Y^\alpha] &= [Z^\alpha X^\alpha-X^\alpha Z^\alpha,Y^\alpha]\\ &= Z^\alpha X^\alpha Y^\alpha-X^\alpha Z^\alpha Y^\alpha-Y^\alpha Z^\alpha X^\alpha+Y^\alpha X^\alpha Z^\alpha\end{aligned}$$

then

$$[[X^\alpha,Y^\alpha],Z^\alpha]+[[Y^\alpha,Z^\alpha],X^\alpha]+[[Z^\alpha,X^\alpha],Y^\alpha]=0.$$

(4)

$$\begin{aligned}[fX^\alpha,gY^\alpha] &= (fX^\alpha)(gY^\alpha)-(gY^\alpha)(fX^\alpha)\\ &= fgX^\alpha Y^\alpha+fX^\alpha(g)Y^\alpha-gY^\alpha(f)X^\alpha-gfY^\alpha X^\alpha\\ &= fg[X^\alpha,Y^\alpha]+fX^\alpha(g)Y^\alpha-gY^\alpha(f)X.\end{aligned}$$

Let $\bigtriangledown^\alpha$ be a fractional connection on manifold M. Let $\{\tilde{\Gamma}^j_{ik}\}$ be Christoffel symbols of this connection in given local coordinates $\{x_i\}$. Then we have

$$\bigtriangledown^\alpha_{X^\alpha}Y^\alpha=X^\alpha_i(\partial^\alpha_i Y^\alpha_j+\tilde{\Gamma}^j_{ik}Y^\alpha_k)\partial^\alpha_j$$

where

$$\tilde{\Gamma}^j_{ik}\partial^\alpha_j=\bigtriangledown^\alpha_{\partial^\alpha_i}\partial^\alpha_k.$$

We calculate Christoffel symbols in new coordinates $\{\grave{x}_i\}$ and we obtain

$$\tilde{\Gamma}^{\grave{j}}_{\grave{i}\grave{k}}\partial^\alpha_{\grave{j}}=\bigtriangledown^\alpha_{\partial^\alpha_{\grave{i}}}\partial^\alpha_{\grave{k}}$$

since

$$\partial^\alpha_{\grave{i}}=\frac{\partial^\alpha}{\partial x^\alpha_{\grave{i}}}\frac{\partial x^\alpha_i}{\partial x^\alpha_i}=\frac{\partial x^\alpha_i}{\partial x^\alpha_{\grave{i}}}\partial^\alpha_i$$

then

$$\begin{aligned}\tilde{\Gamma}^{\grave{j}}_{\grave{i}\grave{k}}\partial^\alpha_{\grave{j}} &= \bigtriangledown^\alpha_{\partial^\alpha_{\grave{i}}}\partial^\alpha_{\grave{k}}\\ &= \bigtriangledown^\alpha_{\partial^\alpha_{\grave{i}}}\Big(\frac{\partial x^\alpha_k}{\partial x^\alpha_{\grave{k}}}\partial^\alpha_k\Big)\\ &= \frac{\partial x^\alpha_k}{\partial x^\alpha_{\grave{k}}}\bigtriangledown^\alpha_{\partial^\alpha_{\grave{i}}}\partial^\alpha_k+\frac{\partial^\alpha}{\partial x^\alpha_{\grave{i}}}\Big(\frac{\partial x^\alpha_k}{\partial x^\alpha_{\grave{k}}}\Big)\partial^\alpha_k\\ &= \frac{\partial x^\alpha_k}{\partial x^\alpha_{\grave{k}}}\bigtriangledown^\alpha_{\frac{\partial x^\alpha_i}{\partial x^\alpha_{\grave{i}}}\partial^\alpha_i}\partial^\alpha_k+\frac{\partial^\alpha}{\partial x^\alpha_{\grave{i}}}\Big(\frac{\partial x^\alpha_k}{\partial x^\alpha_{\grave{k}}}\Big)\partial^\alpha_k\\ &= \frac{\partial x^\alpha_k}{\partial x^\alpha_{\grave{k}}}\frac{\partial x^\alpha_i}{\partial x^\alpha_{\grave{i}}}\bigtriangledown^\alpha_{\partial^\alpha_i}\partial^\alpha_k+\frac{\partial^\alpha}{\partial x^\alpha_{\grave{i}}}\Big(\frac{\partial x^\alpha_k}{\partial x^\alpha_{\grave{k}}}\Big)\partial^\alpha_k\\ &= \frac{\partial x^\alpha_k}{\partial x^\alpha_{\grave{k}}}\frac{\partial x^\alpha_i}{\partial x^\alpha_{\grave{i}}}\tilde{\Gamma}^j_{ik}\partial^\alpha_j+\frac{\partial^\alpha}{\partial x^\alpha_{\grave{i}}}\Big(\frac{\partial x^\alpha_k}{\partial x^\alpha_{\grave{k}}}\Big)\partial^\alpha_k.\end{aligned}$$

Now since

$$\partial^\alpha_j=\frac{\partial^\alpha}{\partial x^\alpha_j}\frac{\partial x^\alpha_{\grave{j}}}{\partial x^\alpha_{\grave{j}}}=\frac{\partial x^\alpha_{\grave{j}}}{\partial x^\alpha_j}\frac{\partial^\alpha}{\partial x^\alpha_{\grave{j}}}=\frac{\partial x^\alpha_{\grave{j}}}{\partial x^\alpha_j}\partial^\alpha_{\grave{j}}$$

and also, $\partial^\alpha_k=\frac{\partial x^\alpha_{\grave{j}}}{\partial x^\alpha_k}\partial^\alpha_{\grave{j}}$ then it follows that

$$\tilde{\Gamma}^{\grave{j}}_{\grave{i}\grave{k}}\partial^\alpha_{\grave{j}}=\frac{\partial x^\alpha_k}{\partial x^\alpha_{\grave{k}}}\frac{\partial x^\alpha_i}{\partial x^\alpha_{\grave{i}}}\tilde{\Gamma}^j_{ik}\frac{\partial x^\alpha_{\grave{j}}}{\partial x^\alpha_j}\partial^\alpha_{\grave{j}}+\frac{\partial^\alpha}{\partial x^\alpha_{\grave{i}}}\left(\frac{\partial x^\alpha_k}{\partial x^\alpha_{\grave{k}}}\right)\frac{\partial x^\alpha_{\grave{j}}}{\partial x^\alpha_k}\partial^\alpha_{\grave{j}}.$$

Then further we have

$$\tilde{\Gamma}^{\grave{j}}_{\grave{i}\grave{k}}=\frac{\partial x^\alpha_k}{\partial x^\alpha_{\grave{k}}}\frac{\partial x^\alpha_i}{\partial x^\alpha_{\grave{i}}}\frac{\partial x^\alpha_{\grave{j}}}{\partial x^\alpha_j}\tilde{\Gamma}^j_{ik}+\frac{\partial^\alpha}{\partial x^\alpha_{\grave{i}}}\left(\frac{\partial x^\alpha_k}{\partial x^\alpha_{\grave{k}}}\right)\frac{\partial x^\alpha_{\grave{j}}}{\partial x^\alpha_k}.$$

Then the transformation law is given by

$$\tilde{\Gamma}^{\grave{j}}_{\grave{i}\grave{k}}=\frac{\partial x^\alpha_k}{\partial x^\alpha_{\grave{k}}}\frac{\partial x^\alpha_i}{\partial x^\alpha_{\grave{i}}}\frac{\partial x^\alpha_{\grave{j}}}{\partial x^\alpha_j}\tilde{\Gamma}^j_{ik}+\frac{\partial^\alpha}{\partial x^\alpha_{\grave{i}}}\left(\frac{\partial x^\alpha_r}{\partial x^\alpha_{\grave{k}}}\right)\frac{\partial x^\alpha_{\grave{j}}}{\partial x^\alpha_r}\tag{11}$$

where $\{\tilde{\Gamma}^j_{ik}\}$ are Christoffel symbols in coordinates $\{x_i\}$ and $\{\tilde{\Gamma}^{\grave{j}}_{\grave{i}\grave{k}}\}$ are Christoffel symbols in new coordinates $\{\grave{x}_i\}$.

After define fractional connection we state the following proposition.

Proposition 2. *Christoffel symbols are equal to zero in these cartesian coordinates $\{x_i\}$*

$$\partial^\alpha_{e_i}e_j=\tilde{\Gamma}^k_{ij}e_k=0,\ \tilde{\Gamma}^k_{ij}=0.$$

Proof: Since $\frac{\partial^\alpha e_j}{\partial e^\alpha_i}=\tilde{\Gamma}^k_{ij}e_k$ then

$$\begin{aligned}\frac{\partial^\alpha e_j}{\partial e^\alpha_i}\frac{\partial e^\alpha_m}{\partial e^\alpha_m} &= \frac{\partial^\alpha e_j}{\partial e^\alpha_m}\frac{\partial e^\alpha_m}{\partial e^\alpha_i}\\ &= \frac{\partial^\alpha e_j}{\partial e^\alpha_m}(\partial_i e_m)^\alpha\\ &= \frac{\partial^\alpha e_j}{\partial e^\alpha_m}(\Gamma^k_{im}e_k)^\alpha\\ &= 0\end{aligned}$$

then $\tilde{\Gamma}^k_{ij}=0$.

Note that the relation (10) means that

$$\bigtriangledown^\alpha_{X^\alpha}Y^\alpha=X^\alpha_i(\partial^\alpha_i Y^\alpha_j)\partial^\alpha_j\tag{12}$$

in coordinates $\{x_i\}$.

Proposition 3. *In a coordinate system (U,x_i), the Christoffel symbols of fractional connection $\bigtriangledown^\alpha$ are symmetric if and only is*

$$[X^\alpha_i,X^\alpha_j]=\bigtriangledown^\alpha_{X^\alpha_i}X^\alpha_j-\bigtriangledown^\alpha_{X^\alpha_j}X^\alpha_i=0,\ \forall i,j=1,...,n.$$

Proof:

$$\begin{aligned}0 &= \bigtriangledown^\alpha_{X^\alpha_i}X^\alpha_j-\bigtriangledown^\alpha_{X^\alpha_j}X^\alpha_i\\ &= \tilde{\Gamma}^k_{ij}\partial^\alpha_k-\tilde{\Gamma}^k_{ji}\partial^\alpha_k\\ &= (\tilde{\Gamma}^k_{ij}-\tilde{\Gamma}^k_{ji})\partial^\alpha_k\end{aligned}$$

then $\tilde{\Gamma}^k_{ij} = \tilde{\Gamma}^k_{ji}$. Now we let $\tilde{\Gamma}^j_{ik} = \tilde{\Gamma}^j_{ki}$, we have

$$\tilde{\Gamma}^{\grave{j}}_{\grave{i}\grave{k}} = \frac{\partial x_k^\alpha}{\partial x_{\grave{k}}^\alpha}\frac{\partial x_i^\alpha}{\partial x_{\grave{i}}^\alpha}\frac{\partial x_{\grave{j}}^\alpha}{\partial x_j^\alpha}\tilde{\Gamma}^j_{ik} + \frac{\partial^\alpha}{\partial x_{\grave{i}}^\alpha}(\frac{\partial x_r^\alpha}{\partial x_{\grave{k}}^\alpha})\frac{\partial x_{\grave{j}}^\alpha}{\partial x_r^\alpha}$$

$$\tilde{\Gamma}^{\grave{j}}_{\grave{k}\grave{i}} = \frac{\partial x_i^\alpha}{\partial x_{\grave{i}}^\alpha}\frac{\partial x_k^\alpha}{\partial x_{\grave{k}}^\alpha}\frac{\partial x_{\grave{j}}^\alpha}{\partial x_j^\alpha}\tilde{\Gamma}^j_{ki} + \frac{\partial^\alpha}{\partial x_{\grave{k}}^\alpha}(\frac{\partial x_r^\alpha}{\partial x_{\grave{i}}^\alpha})\frac{\partial x_{\grave{j}}^\alpha}{\partial x_r^\alpha}$$

since

$$\frac{\partial^\alpha}{\partial x_{\grave{k}}^\alpha}(\frac{\partial x_r^\alpha}{\partial x_{\grave{i}}^\alpha}) \neq \frac{\partial^\alpha}{\partial x_{\grave{i}}^\alpha}(\frac{\partial x_r^\alpha}{\partial x_{\grave{k}}^\alpha})$$

then

$$\tilde{\Gamma}^{\grave{j}}_{\grave{i}\grave{k}} \neq \tilde{\Gamma}^{\grave{j}}_{\grave{k}\grave{i}}.$$

Note that if $\alpha = 1$ then

$$\tilde{\Gamma}^{\grave{j}}_{\grave{i}\grave{k}} = \tilde{\Gamma}^{\grave{j}}_{\grave{k}\grave{i}}.$$

*Example 1.*Calculate the fractional connection in polar coordinates r, φ

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi. \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \varphi = \arctan\frac{y}{x}. \end{cases}$$

$$\frac{\partial x}{\partial r} = \cos\varphi,\ \frac{\partial x}{\partial \varphi} = -r\sin\varphi$$
$$\frac{\partial y}{\partial r} = \sin\varphi,\ \frac{\partial y}{\partial \varphi} = r\cos\varphi$$
$$\frac{\partial r}{\partial x} = \cos\varphi,\ \frac{\partial r}{\partial y} = \sin\varphi$$
$$\frac{\partial \varphi}{\partial x} = -\frac{1}{r}\sin\varphi,\ \frac{\partial \varphi}{\partial y} = \frac{1}{r}\cos\varphi.$$

According (11) and since Christoffel symbols are equal to zero in cartesian coordinates, then

$$\tilde{\Gamma}^{\grave{j}}_{\grave{i}\grave{k}} = \frac{\partial^\alpha}{\partial x_{\grave{i}}^\alpha}(\frac{\partial x_r}{\partial x_{\grave{k}}})^\alpha\frac{\partial x_{\grave{j}}^\alpha}{\partial x_r^\alpha}$$

we have

$$\begin{aligned}\tilde{\Gamma}^r_{rr} &= \frac{\partial^\alpha}{\partial r^\alpha}(\frac{\partial x}{\partial r})^\alpha\frac{\partial r^\alpha}{\partial x^\alpha} + \frac{\partial^\alpha}{\partial r^\alpha}(\frac{\partial y}{\partial r})^\alpha\frac{\partial r^\alpha}{\partial y^\alpha} \\ &= \frac{1}{(1-\alpha)!}(\cos^\alpha\varphi)^{1-\alpha}.0.\cos^\alpha\varphi \\ &\quad + \frac{1}{(1-\alpha)!}(\sin^\alpha\varphi)^{1-\alpha}.0.\sin^\alpha\varphi \\ &= 0.\end{aligned}$$

$$\begin{aligned}\tilde{\Gamma}^r_{r\varphi} &= \frac{\partial^\alpha}{\partial r^\alpha}(\frac{\partial x}{\partial \varphi})^\alpha\frac{\partial r^\alpha}{\partial x^\alpha} + \frac{\partial^\alpha}{\partial r^\alpha}(\frac{\partial y}{\partial \varphi})^\alpha\frac{\partial r^\alpha}{\partial y^\alpha} \\ &= \frac{1}{(1-\alpha)!}(-1)^\alpha r^{\alpha-\alpha^2}\sin^\alpha\varphi\cos^\alpha\varphi \\ &\quad + \frac{1}{(1-\alpha)!}r^{\alpha-\alpha^2}\cos^\alpha\varphi\sin^\alpha\varphi \\ &= \frac{1}{(1-\alpha)!}r^{\alpha-\alpha^2}[(-1)^\alpha+1]\cos^\alpha\varphi\sin^\alpha\varphi \\ &= \frac{1}{(1-\alpha)!}r^{\alpha-\alpha^2}[e^{i\alpha\pi}+1]\cos^\alpha\varphi\sin^\alpha\varphi.\end{aligned}$$

$$\begin{aligned}\tilde{\Gamma}^r_{\varphi r} &= \frac{\partial^\alpha}{\partial \varphi^\alpha}(\frac{\partial x}{\partial r})^\alpha\frac{\partial r^\alpha}{\partial x^\alpha} + \frac{\partial^\alpha}{\partial \varphi^\alpha}(\frac{\partial y}{\partial r})^\alpha\frac{\partial r^\alpha}{\partial y^\alpha} \\ &= \frac{1}{(1-\alpha)!}(-1)^\alpha\alpha^\alpha\cos^\alpha\varphi\sin^\alpha\varphi + \frac{1}{(1-\alpha)!}\alpha^\alpha\sin^\alpha\varphi\cos^\alpha\varphi \\ &= \frac{1}{(1-\alpha)!}\alpha^\alpha[(-1)^\alpha+1]\cos^\alpha\varphi\sin^\alpha\varphi \\ &= \frac{1}{(1-\alpha)!}\alpha^\alpha[e^{i\alpha\pi}+1]\cos^\alpha\varphi\sin^\alpha\varphi.\end{aligned}$$

$$\begin{aligned}\tilde{\Gamma}^r_{\varphi\varphi} &= \frac{\partial^\alpha}{\partial \varphi^\alpha}(\frac{\partial x}{\partial \varphi})^\alpha\frac{\partial r^\alpha}{\partial x^\alpha} + \frac{\partial^\alpha}{\partial \varphi^\alpha}(\frac{\partial y}{\partial \varphi})^\alpha\frac{\partial r^\alpha}{\partial y^\alpha} \\ &= \frac{1}{(1-\alpha)!}(-1)^\alpha\alpha^\alpha r^\alpha\cos^{2\alpha}\varphi + \frac{1}{(1-\alpha)!}(-1)^\alpha r^\alpha\sin^{2\alpha}\varphi \\ &= \frac{1}{(1-\alpha)!}(-1)^\alpha r^\alpha(\alpha^\alpha\cos^{2\alpha}\varphi+\sin^{2\alpha}\varphi) \\ &= \frac{1}{(1-\alpha)!}e^{i\alpha\pi}r^\alpha(\alpha^\alpha\cos^{2\alpha}\varphi+\sin^{2\alpha}\varphi).\end{aligned}$$

$$\begin{aligned}\tilde{\Gamma}^\varphi_{rr} &= \frac{\partial^\alpha}{\partial r^\alpha}(\frac{\partial x}{\partial r})^\alpha\frac{\partial \varphi^\alpha}{\partial x^\alpha} + \frac{\partial^r}{\partial r^\alpha}(\frac{\partial y}{\partial r})^\alpha\frac{\partial \varphi^\alpha}{\partial y^\alpha} \\ &= \frac{1}{(1-\alpha)!}(\cos^{\alpha-\alpha^2}\varphi)(0)((-1)^\alpha r^{-\alpha}\sin^\alpha\varphi) \\ &\quad + \frac{1}{(1-\alpha)!}(\sin^{\alpha-\alpha^2})(0)(r^{-\alpha}\cos^\alpha\varphi) \\ &= 0.\end{aligned}$$

$$\begin{aligned}\tilde{\Gamma}^\varphi_{r\varphi} &= \frac{\partial^\alpha}{\partial r^\alpha}(\frac{\partial x}{\partial \varphi})^\alpha\frac{\partial \varphi^\alpha}{\partial x^\alpha} + \frac{\partial^\alpha}{\partial r^\alpha}(\frac{\partial y}{\partial \varphi})^\alpha\frac{\partial \varphi^\alpha}{\partial y^\alpha} \\ &= \frac{1}{(1-\alpha)!}(-1)^{2\alpha}r^{-\alpha}\sin^{2\alpha}\varphi \\ &\quad + \frac{1}{(1-\alpha)!}\alpha^\alpha r^{-\alpha}\cos^{2\alpha}\varphi \\ &= \frac{1}{(1-\alpha)!}r^{-\alpha}[(-1)^{2\alpha}\sin^{2\alpha}\varphi+\alpha^\alpha\cos^{2\alpha}\varphi] \\ &= \frac{1}{(1-\alpha)!}r^{-\alpha}[e^{2i\alpha\pi}\sin^{2\alpha}\varphi+\alpha^\alpha\cos^{2\alpha}\varphi] = \tilde{\Gamma}^\varphi_{\varphi r}.\end{aligned}$$

$$\begin{aligned}\tilde{\Gamma}^\varphi_{\varphi\varphi} &= \frac{\partial^\alpha}{\partial \varphi^\alpha}(\frac{\partial x}{\partial \varphi})^\alpha\frac{\partial \varphi^\alpha}{\partial x^\alpha} + \frac{\partial^\alpha}{\partial \varphi^\alpha}(\frac{\partial y}{\partial \varphi})^\alpha\frac{\partial \varphi^\alpha}{\partial y^\alpha} \\ &= \frac{1}{(1-\alpha)!}(-1)^{2\alpha}\alpha^\alpha\sin^\alpha\varphi\cos^\alpha\varphi \\ &\quad + \frac{1}{(1-\alpha)!}(-1)^\alpha\alpha^\alpha\cos^\alpha\varphi\sin^\alpha\varphi \\ &= \frac{1}{(1-\alpha)!}\alpha^\alpha((-1)^{2\alpha}+(-1)^\alpha)\cos^\alpha\varphi\sin^\alpha\varphi \\ &= \frac{1}{(1-\alpha)!}\alpha^\alpha(e^{2i\alpha\pi}+e^{i\alpha\pi})\cos^\alpha\varphi\sin^\alpha\varphi.\end{aligned}$$

Hence we have that the fractional covariant derivative (12) in polar coordinates have the following appearance

$$\bigtriangledown_r^\alpha\partial_r^\alpha = \tilde{\Gamma}^r_{rr}\partial_r^\alpha + \tilde{\Gamma}^\varphi_{rr}\partial_\varphi^\alpha = 0. \tag{13}$$

$$\begin{aligned}
\bigtriangledown_r^\alpha \partial_\varphi^\alpha &= \tilde{\Gamma}_{r\varphi}^r \partial_r^\alpha + \tilde{\Gamma}_{r\varphi}^\varphi \partial_\varphi^\alpha \\
&= \frac{1}{(1-\alpha)!} r^{\alpha-\alpha^2}[(-1)^\alpha + 1]\cos^\alpha \varphi \sin^\alpha \varphi \partial_r^\alpha \\
&\quad + \frac{1}{(1-\alpha)!} r^{-\alpha}[(-1)^{2\alpha}\sin^{2\alpha}\varphi + \alpha^\alpha \cos^{2\alpha}]\partial_\varphi^\alpha \\
&= \frac{1}{(1-\alpha)!} r^{\alpha-\alpha^2}[e^{i\alpha\pi}+1]\cos^\alpha \varphi \sin^\alpha \varphi \partial_r^\alpha \\
&\quad + \frac{1}{(1-\alpha)!} r^{-\alpha}[e^{2i\alpha\pi}\sin^{2\alpha}\varphi + \alpha^\alpha \cos^{2\alpha}]\partial_\varphi^\alpha .
\end{aligned}$$

$$\begin{aligned}
\bigtriangledown_\varphi^\alpha \partial_r^\alpha &= \tilde{\Gamma}_{\varphi r}^r \partial_r^\alpha + \tilde{\Gamma}_{\varphi r}^\varphi \partial_\varphi^\alpha \\
&= \frac{1}{(1-\alpha)!} \alpha^\alpha [(-1)^\alpha + 1]\cos^\alpha \varphi \sin^\alpha \varphi \partial_r^\alpha \\
&\quad + \frac{1}{(1-\alpha)!} r^{-\alpha}[(-1)^{2\alpha}\sin^{2\alpha}\varphi + \alpha^\alpha \cos^{2\alpha}]\partial_\varphi^\alpha \\
&= \frac{1}{(1-\alpha)!} \alpha^\alpha [e^{i\alpha\pi}+1]\cos^\alpha \varphi \sin^\alpha \varphi \partial_r^\alpha \\
&\quad + \frac{1}{(1-\alpha)!} r^{-\alpha}[e^{2i\alpha\pi}\sin^{2\alpha}\varphi + \alpha^\alpha \cos^{2\alpha}]\partial_\varphi^\alpha
\end{aligned}$$

$$\begin{aligned}
\bigtriangledown_\varphi^\alpha \partial_\varphi^\alpha &= \tilde{\Gamma}_{\varphi\varphi}^r \partial_r^\alpha + \tilde{\Gamma}_{\varphi\varphi}^\varphi \partial_\varphi^\alpha \\
&= \frac{1}{(1-\alpha)!} (-1)^\alpha r^\alpha (\alpha^\alpha \cos^{2\alpha}\varphi + \sin^{2\alpha}\varphi)\partial_r^\alpha \\
&+ \frac{1}{(1-\alpha)!} \alpha^\alpha ((-1)^{2\alpha}\sin^\alpha \varphi \cos^\alpha \varphi \qquad (14)\\
&\quad + + (-1)^\alpha \cos^\alpha \varphi \sin^\alpha \varphi)\partial_\varphi^\alpha \\
&= \frac{1}{(1-\alpha)!} e^{i\alpha\pi} r^\alpha (\alpha^\alpha \cos^{2\alpha}\varphi + \sin^{2\alpha}\varphi)\partial_r^\alpha \\
&+ \frac{1}{(1-\alpha)!} \alpha^\alpha (e^{2i\alpha\pi}\sin^\alpha \varphi \cos^\alpha \varphi \qquad (15)\\
&\quad + e^{i\alpha\pi}\cos^\alpha \varphi \sin^\alpha \varphi)\partial_\varphi^\alpha .
\end{aligned}$$

Notice that, as expected, when $\alpha = 1$, one recovers the classical formula.

Proposition 4. *Consider $A_m = \tilde{\Gamma}_{ik}^j$, then the transformation law for A_m is given by*

$$A_{\grave{m}} = \frac{\partial x_i^\alpha}{\partial x_{\grave{i}}^\alpha} A_m + \frac{\partial^\alpha}{\partial x_{\grave{i}}^\alpha}\left(\frac{\partial x_r^\partial}{\partial x_{\grave{k}}^\alpha}\right)\frac{\partial x_{\grave{j}}^\alpha}{\partial x_r^\alpha}.$$

Proof:

$$\begin{aligned}
A_{\grave{m}} &= \tilde{\Gamma}_{\grave{i}\grave{k}}^{\grave{j}} \\
&= \frac{\partial x_{\grave{j}}^\alpha}{\partial x_j^\alpha}\frac{\partial x_k^\alpha}{\partial x_{\grave{k}}^\alpha}\frac{\partial x_i^\alpha}{\partial x_{\grave{i}}^\alpha}\tilde{\Gamma}_{ik}^j + \frac{\partial^\alpha}{\partial x_{\grave{i}}^\alpha}\left(\frac{\partial x_r^\alpha}{\partial x_{\grave{k}}^\alpha}\right)\frac{\partial x_{\grave{j}}^\alpha}{\partial x_r^\alpha} \\
&= \frac{\partial x_{\grave{j}}^\alpha}{\partial x_j^\alpha}\frac{\partial x_k^\alpha}{\partial x_{\grave{j}}^\alpha}\frac{\partial x_i^\alpha}{\partial x_{\grave{i}}^\alpha}\tilde{\Gamma}_{ik}^j + \frac{\partial^\alpha}{\partial x_{\grave{i}}^\alpha}\left(\frac{\partial x_r^\alpha}{\partial x_{\grave{j}}^\alpha}\right)\frac{\partial x_{\grave{j}}^\alpha}{\partial x_r^\alpha}.
\end{aligned}$$

Since $\frac{\partial x_{\grave{j}}}{\partial x_j}\frac{\partial x_k}{\partial x_{\grave{j}}} = \delta_j^k$, then

$$\begin{aligned}
A_{\grave{m}} &= \tilde{\Gamma}_{\grave{i}\grave{k}}^{\grave{j}} \\
&= (\delta_j^k)^\alpha \frac{\partial x_i^\alpha}{\partial x_{\grave{i}}^\alpha}\tilde{\Gamma}_{ik}^j + \frac{\partial^\alpha}{\partial x_{\grave{i}}^\alpha}\left(\frac{\partial x_r^\alpha}{\partial x_{\grave{j}}^\alpha}\right)\frac{\partial x_{\grave{j}}^\alpha}{\partial x_r^\alpha} \\
&= \frac{\partial x_i^\alpha}{\partial x_{\grave{i}}^\alpha} A_M + \frac{\partial^\alpha}{\partial x_{\grave{i}}^\alpha}\left(\frac{\partial x_r^\alpha}{\partial x_{\grave{j}}^\alpha}\right)\frac{\partial x_{\grave{j}}^\alpha}{\partial x_r^\alpha}.
\end{aligned}$$

Let $\tilde{T}_{\grave{i}\grave{k}}^{\grave{j}} = {}^1\tilde{\Gamma}_{\grave{i}\grave{k}}^{\grave{j}} - {}^2\tilde{\Gamma}_{\grave{i}\grave{k}}^{\grave{j}}$ then

$$\begin{aligned}
\tilde{T}_{\grave{i}\grave{k}}^{\grave{j}} &= \frac{\partial x_k^\alpha}{\partial x_{\grave{k}}^\alpha}\frac{\partial x_i^\alpha}{\partial x_{\grave{i}}^\alpha}\frac{\partial x_{\grave{j}}^\alpha}{\partial x_j^\alpha}{}^1\tilde{\Gamma}_{ik}^j + \frac{\partial^\alpha}{\partial x_{\grave{i}}^\alpha}\left(\frac{\partial x_r^\alpha}{\partial x_{\grave{k}}^\alpha}\right)\frac{\partial x_{\grave{j}}^\alpha}{\partial x_r^\alpha} \\
&- \frac{\partial x_k^\alpha}{\partial x_{\grave{k}}^\alpha}\frac{\partial x_i^\alpha}{\partial x_{\grave{i}}^\alpha}\frac{\partial x_{\grave{j}}^\alpha}{\partial x_j^\alpha}{}^2\tilde{\Gamma}_{ik}^j - \frac{\partial^\alpha}{\partial x_{\grave{i}}^\alpha}\left(\frac{\partial x_r^\alpha}{\partial x_{\grave{k}}^\alpha}\right)\frac{\partial x_{\grave{j}}^\alpha}{\partial x_r^\alpha} \\
&= \frac{\partial x_k^\alpha}{\partial x_{\grave{k}}^\alpha}\frac{\partial x_i^\alpha}{\partial x_{\grave{i}}^\alpha}\frac{\partial x_{\grave{j}}^\alpha}{\partial x_j^\alpha}({}^1\tilde{\Gamma}_{ik}^j - {}^2\tilde{\Gamma}_{ik}^j) \\
&= \frac{\partial x_k^\alpha}{\partial x_{\grave{k}}^\alpha}\frac{\partial x_i^\alpha}{\partial x_{\grave{i}}^\alpha}\frac{\partial x_{\grave{j}}^\alpha}{\partial x_j^\alpha}(\tilde{T}_{ik}^j).
\end{aligned}$$

3 The Torsion Tensor of a Fractional Connection

Definition 2. *On a differential manifold M with a fractional connection $\bigtriangledown^\alpha$, the expression*

$$\bigtriangledown_{X^\alpha}^\alpha Y^\alpha - \bigtriangledown_{Y^\alpha}^\alpha X^\alpha - [X^\alpha, Y^\alpha] = \tilde{T}(X^\alpha, Y^\alpha)$$

called the torsion (or the torsion tensor) of the fractional connection $\bigtriangledown^\alpha$.

One can easily see the following properties

$$\begin{aligned}
\tilde{T}(X^\alpha, Y^\alpha) &= -\tilde{T}(Y^\alpha, X^\alpha), \\
\tilde{T}(X^\alpha + Y^\alpha, Z^\alpha) &= \tilde{T}(X^\alpha, Z^\alpha) + \tilde{T}(Y^\alpha, Z^\alpha) \\
\tilde{T}(fX^\alpha, Y^\alpha) &= \bigtriangledown_{fX^\alpha}^\alpha Y^\alpha - \bigtriangledown_{Y^\alpha}^\alpha fX^\alpha - [fX^\alpha, Y^\alpha] \\
&= f\bigtriangledown_{X^\alpha}^\alpha Y^\alpha - (Y^\alpha f)X^\alpha \\
&\quad - f\bigtriangledown_{Y^\alpha}^\alpha X^\alpha + (Y^\alpha f)X^\alpha - f[X^\alpha, Y^\alpha] \\
&= f(\bigtriangledown_{X^\alpha}^\alpha Y^\alpha - \bigtriangledown_{Y^\alpha}^\alpha X^\alpha - [X^\alpha, Y^\alpha]) \\
&= f\tilde{T}(X^\alpha, Y^\alpha).
\end{aligned}$$

Remark. If $T^\alpha = 0$, then $\bigtriangledown^\alpha$ is symmetric.

Similarly, for fractional vector fields X^α and Y^α we define the difference tensor

$$\tilde{B}(X^\alpha, Y^\alpha) = \bigtriangledown_{1X^\alpha}^\alpha Y^\alpha - \bigtriangledown_{2X^\alpha}^\alpha Y^\alpha.$$

Proposition 5. *1. The linearity of $\tilde{B}$ is trivial from properties of the fractional connection.*

2.

$$\begin{aligned}
\tilde{B}(X^\alpha, fY^\alpha) &= \bigtriangledown_{1X^\alpha}^\alpha fY^\alpha - \bigtriangledown_{2X^\alpha}^\alpha fY^\alpha \\
&= (X^\alpha f)Y^\alpha + f\bigtriangledown_{1X^\alpha}^\alpha Y^\alpha - (X^\alpha f)Y^\alpha - f\bigtriangledown_{2X^\alpha}^\alpha Y^\alpha \\
&= f(\bigtriangledown_{1X^\alpha}^\alpha Y^\alpha - \bigtriangledown_{2X^\alpha}^\alpha Y^\alpha) \\
&= f\tilde{B}(X^\alpha, Y^\alpha).
\end{aligned}$$

3. Let $\tilde{B}(X^\alpha, Y^\alpha) = \tilde{S_1}(X^\alpha, Y^\alpha) + \tilde{S_2}(X^\alpha, Y^\alpha)$ where

$$\tilde{S_1}(X^\alpha, Y^\alpha) = \frac{1}{2}[\tilde{B}(X^\alpha, Y^\alpha) + \tilde{B}(Y^\alpha, X^\alpha)],$$

Table 1: $c = \cos\varphi, s = \sin\varphi$

$\alpha =$	.1	.2	.3
$\tilde{\Gamma}^r_{rr}$	0	0	0
$\tilde{\Gamma}^r_{r\varphi}$	$1.0398\, r^{.09}[i+1]c^{.1}s^{.1}$	$2.1474\, r^{.16}c^{.2}s^{.2}$	$1.1005r^{.21}[i+1]c^{.3}s^{.3}$
$\tilde{\Gamma}^r_{\varphi r}$	$.8259[i+1]c^{.1}s^{.1}$	$1.5564c^{.2}s^{.2}$	$.7668[i+1]c^{.2}s^{.2}$
$\tilde{\Gamma}^r_{\varphi\varphi}$	$1.0398\, ir^{.1}(.7943c^{.2}+s^{.2})$	$1.0737\, r^{.2}(.7248c^{.4}+s^{.4})$	$1.1005r^{.3}(.6931c^{.6}+s^{.6})$
$\tilde{\Gamma}^\varphi_{rr}$	0	0	0
$\tilde{\Gamma}^\varphi_{r\varphi} = \tilde{\Gamma}^\varphi_{\varphi r}$	$1.0398r^{-.1}[s^{.2}+.7943c^{.2}]$	$1.0737r^{-.2}[s^{.4}+.7248c^{.4}]$	$1.1005r^{-.3}[s^{.6}+.6968c^{.6}]$
$\tilde{\Gamma}^\varphi_{\varphi\varphi}$	$.8259(1+i)c^{.1}s^{.1}$	$1.5564c^{.2}s^{.2}$	$.7668(i+1)c^{.3}s^{.3}$

Table 2: $c = \cos\varphi, s = \sin\varphi$

$\alpha =$	.4	.5	.6
$\tilde{\Gamma}^r_{rr}$	0	0	0
$\tilde{\Gamma}^r_{r\varphi}$	$2.2384\, c^{.4}s^{.4}$	$1.1284\, r^{.25}[i+1]c^{.5}s^{.6}$	$2.2542r^{.24}c^{.6}s^{.6}$
$\tilde{\Gamma}^r_{\varphi r}$	$1.5514c^{.4}s^{.4}$	$.7979[i+1]c^{.5}s^{.5}$	$1.6591c^{.6}s^{.6}$
$\tilde{\Gamma}^r_{\varphi\varphi}$	$1.1192\, r^{.4}(.6931c^{.8}+s^{.8})$	$1.1284r^{.5}(.7071c+s)$	$1.1271r^{.6}(.736c^{1.2}+s^{1.2})$
$\tilde{\Gamma}^\varphi_{rr}$	0	0	0
$\tilde{\Gamma}^\varphi_{r\varphi} = \tilde{\Gamma}^\varphi_{\varphi r}$	$1.1192r^{-.4}[s^{.8}+.6931c^{.8}]$	$1.1284r^{-.5}[-s+.7071c]$	$1.1271r^{-.6}[s^{1.2}+.736c^{1.2}]$
$\tilde{\Gamma}^\varphi_{\varphi\varphi}$	$1.5514c^{.4}s^{.4}$	$.7979[-1+i]c^{.5}s^{.5}$	$1.6591c^{.6}s^{.6}$

Table 3: $c = \cos\varphi, s = \sin\varphi$.

$\alpha =$	.7	.8	.9	1
$\tilde{\Gamma}^r_{rr}$	0	0	0	0
$\tilde{\Gamma}^r_{r\varphi}$	$1.1142\, r^{.21}[i+1]c^{.7}s^{.7}$	$2.1782\, r^{.16}c^{.8}s^{.8}$	$1.0511r^{.09}[i+1]c^{.9}s^{.9}$	0
$\tilde{\Gamma}^r_{\varphi r}$	$.8681[i+1]c^{.7}s^{.7}$	$1.8219c^{.8}s^{.8}$	$.95598[i+1]c^{.9}s^{.9}$	0
$\tilde{\Gamma}^r_{\varphi\varphi}$	$1.1142\, ir^{.7}(.7791c^{.8}+s^{.8})$	$1.0891r^{.8}(.8365c^{1.6}+s^{1.6})$	$1.0511ir^{.9}(.9095c^{1.8}+s^{1.8})$	$-r$
$\tilde{\Gamma}^\varphi_{rr}$	0	0	0	0
$\tilde{\Gamma}^\varphi_{r\varphi} = \tilde{\Gamma}^\varphi_{\varphi r}$	$1.1142r^{-.7}[s^{1.4}+.7791c^{1.4}]$	$1.0891r^{-.8}[-s^{1.6}+.8365c^{1.6}]$	$1.0511r^{-.9}[s^{1.8}+.9095c^{1.8}]$	$\frac{1}{r}$
$\tilde{\Gamma}^\varphi_{\varphi\varphi}$	$.8681[i+1]c^{.7}s^{.7}$	$1.8221c^{.8}s^{.8}$	$.95598[i+1]c^{.9}s^{.9}$	0

and

$$\tilde{S_2}(X^\alpha, Y^\alpha) = \frac{1}{2}[\tilde{B}(X^\alpha, Y^\alpha) - \tilde{B}(Y^\alpha, X^\alpha)].$$

Actually, we can express $\tilde{S_2}(X^\alpha, Y^\alpha)$ *in terms of the torsion tensors* ${}^1\tilde{T}$ *and* ${}^2\tilde{T}$ *of* $\bigtriangledown^\alpha_1$ *and* $\bigtriangledown^\alpha_2$*, respectively as follows*

$$\begin{aligned}
&2\tilde{S_2}(X^\alpha, Y^\alpha) \\
&= \frac{1}{2}\tilde{B}(X^\alpha, Y^\alpha) - \tilde{B}(Y^\alpha, X^\alpha) \\
&= \bigtriangledown^\alpha_{1X^\alpha}Y^\alpha - \bigtriangledown^\alpha_{2X^\alpha}Y^\alpha - \bigtriangledown^\alpha_{1Y^\alpha}X^\alpha + \bigtriangledown^\alpha_{2Y^\alpha}X^\alpha \\
&= {}^1\tilde{T}(X^\alpha, Y^\alpha) + [X^\alpha, Y^\alpha] -{}^2\tilde{T}(X^\alpha, Y^\alpha) - [X^\alpha, Y^\alpha] \\
&= {}^1\tilde{T}(X^\alpha, Y^\alpha) -{}^2\tilde{T}(X^\alpha, Y^\alpha).
\end{aligned}$$

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