Journal of Analysis & Number Theory
*An International Journal*

# Existence and Convergence of Best Proximity Points in $\mathscr{M}$ - Normal Cone Metric Spaces

*Sirajo Yahaya*[1,*], *Danlami Ibrahim Kubau*[2] *and Ibrahim Aliyu Fulatan*[2]

[1]Department of Mathematics and Statistics, American University of Nigeria, Yola , Adamawa State, Nigeria
[2]Department of Mathematics, Ahmadu Bello University, Zaria, Kaduna, Kaduna State, Nigeria

**Abstract:** In this paper, using the notion of Banach, Kannan and Charterjea contraction mappings, we introduce cyclic $\mathscr{M}\mathscr{T}$-contractions mapping and prove the existence and convergence of best proximity point in the context of complete $\mathscr{M}$ - normal cone metric spaces.

## 1 Introduction

The fact that fixed point theory gives a cohesive treatment and is a fundamental tool for solving equations of the form $Tx = x$ where $T$ is a self-mapping defined on a subset of some appropriate space and $x$ is a point in the space, leads to the importance of the area under consideration. But, almost all such results depend upon the existence of a fixed point for self-mappings.

The earliest techniques, largely devised by [1], involved the iteration of an integral operator to obtain solutions to such problems. In 1922, these techniques were given precise abstract formulation by [2] and [3] contraction mappings principle, a powerful tool in analysis for establishing existence and uniqueness of solution of problems in different kind.The fact that, fixed point theorem is an important tool for solving equation of the form $Tx = x$ where $T$ is a self-map defined on a subset of some suitable space, leads to the significance of this area. Various authors have generalized Banach-Cacciopoli contraction principle in different spaces.

[4] characterized metric completeness by generalized Banach Contraction principle. However, if $T$ is not a self-mapping, then it is probable that the equation $Tx = x$ has no solution. In this case, [5] introduced the concept of the best approximation theorems. Moreover, best proximity point theorem was also considered by different

authors in the context of metric space,which is a concept that analyzed the existence of an approximate solution that is optimal; that is, if $A$ and $B$ are non-empty subsets of considered space and $T : A \cup B \rightarrow A \cup B$ then we can find a point $x \in A \cup B$ such that $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf_{n \in \mathbb{N}} \{d(x, y) : x \in A \text{ and } y \in B\}$. [5], also proved the following interesting best proximity point theorem concerning cyclic contraction maps in metric spaces. However, [6], proved the above best proximity point theorem in the set up of multiplicative metric space.

On the other hand, [7] first introduced the notion of cone metric space, [8] introduced the notion of cone rectangular metric space, proved Banach contraction mapping principles and some fixed point theorems of such mappings on a complete cone metric space. [9] studied common fixed points for weak commutative mappings on a cone metric space. Moreover, [10] Introduced the concept of $M$- normal cone and proved fixed point results for several contraction mappings. [11] prove Common fixed Points for a class of Contraction Mapping in a Metric space. [12] studied some unique fixed point theorems in the set up of cone metric space. Their results carry fixed point results of [13] and [14] to cone metric spaces. For more details about fixed point see [[15],[16],[17],[18],[19],[20] e.t.c]. Best proximity point theorem for a subclass of generalised $\mathscr{M}\mathscr{T}$-cyclic contraction mappings was established in [21] in the context of metric space.

* Corresponding author e-mail: surajmt951@gmail.com

Let us recall that a function $\varphi : [0,\infty) \to [0,1)$ is said to be an $\mathcal{MT}$-function if $\limsup_{s \to t^+} \varphi(s) < 1$ for all $t \in [0,\infty)$. It is obvious that if $\varphi : [0,\infty) \to [0,1)$ is a non-decreasing function or a non-increasing function, then $\varphi$ is an $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-function is a rich class.

Thus, motivated by the results above, one of the main objectives of this paper is to consider a wider class of such introduced mappings which was considered by [10]. Also, some new existence theorems of such mappings have been studied and best proximity point theorem is established. This work improves many results in the literature. In particular, it extends the ideas of [10].

## 2 Preliminaries

**Definition 1.***Let E be a real Banach space and P a non empty subset of E. P is said to be a cone if and only if it satisfies the following conditions:*

*1.P is closed and $P \neq \{0\}$;*
*2.$a,b \in \Re$, $a,b \geqslant 0, x,y \in P \Rightarrow ax + by \in P$;*
*3.$P \cap (-P) = \{0\}$ Where $-P = \{-x : x \in P\}$.*

**Definition 2.***For any given cone $P \subseteq E$, we can define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ if $x \leq y$ but $x \neq y$ and $x \ll y$ to mean $y - x \in intP$, where $intP$ denote the interior of P. The cone P is called normal if there is a number K such that for all $x,y \in E$,*

$$0 \leq x \leq y \implies \|x\| \leq K\|y\|. \tag{1}$$

*The least positive integer K satisfying inequality (1) above is called normal constant of P. The cone P is called regular if every increasing sequence which is bounded above is convergent.*

**Definition 3.***Let X be a non-empty set. Suppose there is a cone P in E such that the mapping $d : X \times X \longrightarrow E$ satisfies*

*i$d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$ for all $x,y \in X$;*
*ii$d(x,y) = d(y,x)$ for all $x,y \in X$;*
*iii$d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y \in X$.*

*Then d is called a cone metric on X and $(X,d)$ is called a cone metric space.*

It is obvious that cone metric space generalizes metric space.

*Example 1.Let $E = R^2, P = (x,y) \in E : x,y \geq 0 \subset R^2, X = R$ and $d : X * X \to E$ such that $d(x,y) = (|x-y|, k|x-y|)$, where $k \geq 0$ is a constant. Then $(X,d)$ is a cone metric space.*

**Definition 4.***Let X be a cone metric space and $\{x_n\}$ be a sequence in X with $x \in X$, then,*

*i.for every $c \in E$ with $0 \ll c$, there exist a natural number $n_0$ such that for all $n \geq n_0, d(x_n,x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x.*
*ii.If for every $c \in E$ with $0 \ll c$ there exist an $n_0 \in N$ such that for all $n,m \geq n_0, d(x_n,x_m) \ll c$ the $\{x_n\}$ is called a Cauchy sequence in X. If a cauchy sequence in X converges to a point in X, then it is called a complete cone metric space.*

**Definition 5.***Two positive vectors x and y in a Banach space E are said to have reflective archimedian property if there exist an integer m such that $y < mx$ if and only if $y \leq m^\lambda x$, $0 < \lambda < 1$, where $\lambda$ is called a bi-conditional index.*

*The least positive integer m satisfying the above inequality is called the archimedean constant for P.*

A cone P in a Banach space E is called $\mathcal{M}$-normal if it possesses a reflective archimedian property.

**Definition 6.***Let X be a non empty set. A point $x \in X$ is said to be a fixed point of the self-mapping $T : X \longrightarrow X$ if $Tx = x$. For any other point $x^* \in X$ with $Tx^* = x^*$, if $x = x^*$ then T is said to have a unique fixed point in X.*

**Definition 7.***A mapping $T : X \to X$ is called contraction if there exist a positive real number $\alpha > 0$ such that $d(Tx,Ty) \leq \alpha d(x,y)$.*

**Definition 8.***Let A and B be non-empty subsets of a metric space $(X,d)$. Consider a map $T : A \cup B \longrightarrow A \cup B$. T is called a cyclic map if $T(A) \subset B$ and $T(B) \subset A$.*

**Definition 9.***Let A and B be non empty subset of a metric space $(X,d)$. A cyclic map $T : A \cup B \longrightarrow A \cup B$ is called a cyclic contraction map, if $\exists$ $k \in [0,1)$ such that $d(Tx,Ty) \leq kd(x,y) + (k-1)d(A,B)$ for all $x \in A$ and $y \in B$*

## 3 Results and Discussion

In this paper, we use the concept of cyclic contraction and prove for the existence of best proximity points instead of fixed points. First, we establish the following theorem that relate to cyclic $\mathcal{MT}$-contractions mappings.

**Theorem 1.***Let A and B be two non-empty closed subsets of a complete cone metric space X and P an $\mathcal{M}$ - normal cone. Suppose the mapping $T : A \cup B \to A \cup B$ satisfies the strict cyclic contraction condition*

$$d(Tx,Ty) < \alpha[d(Tx,x) + d(Ty,y) + kd(x,Ty)]$$
$$+ (1-\alpha)d(A,B),$$

*for any $x \in A$ and $y \in B$. Where $0 \leq \alpha < 1, 0 < k < 1$. Then there exists an iterative sequence $\{x_n\}_{n \in \mathbb{N}}$ in $A \cup B$ define by $x_{n+1} = Tx_n$, $n \in \mathbb{N}$, s.t*

$$\lim_{n \to \infty} d(x_n,x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n,x_{n+1}) = d(A,B).$$

*Proof.*First of all suppose $A \cap B \neq \phi$, then $x_{n+1} = x_n$.
So, Let $x_0 \in A$ be given. Define an iterative sequence
$\{x_n\}_{n \in \mathbb{N}}$ by $x_{n+1} = Tx_n$, $n \in \mathbb{N}$.
Then, $d(A,B) \leq d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ by definition.
If there exists $j \in \mathbb{N}$ such that $x_j = x_{j+1} \subset A \cap B$,
then by definition
$Tx_j = x_{j+1} = x_j$, $Sx_j = x_{j-1} = x_j$,
also $x_{j+2} = Tx_{j+1} = T(Tx_j) = Tx_j = x_j$,
$x_{j+2} = Sx_{j+1} = S(Sx_j) = Sx_j = x_j$.
So, $x_j = x_{j+1} = x_{j+2} = ...$ and therefore

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.$$

Then

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1})$$
$$= d(A,B) = 0.$$

Hence, it suffices to consider the case $x_j \neq x_{j+1}$
for all $n \in \mathbb{N}$, where $x_{n+1} \neq x_n$.
We shall show that the sequence $\{d(x_n, x_{n+1})\}$
is non-increasing.

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$< \alpha[d(Tx_n, x_n) + d(Tx_{n-1}, x_{n-1})$$
$$+ kd(x_n, x_n)] + (1-\alpha)d(A,B)$$
$$= \alpha[d(x_{n+1}, x_n) + d(x_n, x_{n-1})$$
$$+ kd(x_n, x_n)] + (1-\alpha)d(A,B)$$
$$< \alpha[d(x_{n+1}, x_n) + d(x_n, x_{n-1})]$$
$$+ (1-\alpha)d(A,B)$$
$$= \frac{\alpha}{1-\alpha}d(x_n, x_{n-1}) + d(A,B).$$

Also,

$$d(x_{n+2}, x_{n+1}) = d(Tx_{n+1}, Tx_n)$$
$$< \alpha[d(Tx_{n+1}, x_{n+1}) + d(Tx_n, x_n)$$
$$+ kd(x_{n+1}, Tx_n)] + (1-\alpha)d(A,B)$$
$$= \alpha[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)$$
$$+ kd(x_{n+1}, x_{n+1})] + (1-\alpha)d(A,B)$$
$$= \alpha[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)]$$
$$+ (1-\alpha)d(A,B)$$
$$< (\frac{\alpha}{1-\alpha})^2 d(x_{n+2}, x_{n+1})$$
$$+ (\frac{1}{1-\alpha})d(A,B)$$
$$< (\frac{\alpha}{1-\alpha})^2 d(x_n, x_{n-1}) + d(A,B).$$

Continuing in this manner, we get

$$d(x_{n+1}, x_n) < (\frac{\alpha}{1-\alpha})^n d(x_{n-1}, x_n) + d(A,B).$$

But

$$d(A,B) \leq d(x_{n+1}, x_n)$$

and since $\mathscr{P}$ is $\mathscr{M}-$ normal cone, we have

$$d(A,B) \leq d(x_{n+1}, x_n)$$
$$\leq \frac{\alpha^{n\lambda}}{(1-\alpha)^{\lambda}}d(x_{n-1}, x_n) + d(A,B).$$

Taking limit as $n \to \infty$ in the above inequality, we get

$$d(A,B) \leq \lim_{n \to \infty} d(x_{n+1}, x_n)$$
$$\leq \lim_{n \to \infty} \frac{\alpha^{n\lambda}}{(1-\alpha)^{\lambda}}d(x_1, x_0)$$
$$+ d(A,B).$$

This implies

$$d(A,B) \leq \lim_{n \to \infty} d(x_{n+1}, x_n) \leq d(A,B)$$

.
Hence

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = d(A,B)$$

Applying Theorems 1, we establish the following
best proximity point theorem.

**Theorem 2.***Let A and B be two non-empty closed subsets
of a complete cone metric space X and P an $\mathscr{M}$ - normal
cone. Suppose the mapping $T : A \cup B \to A \cup B$ is a strict
cyclic contraction. Define an iterative sequence by
$x_{n+1} = Tx_n$, $n \in \mathbb{N}$. Suppose further that $\{x_{n-1}\}_{n \in \mathbb{N}}$ has a
converging subsequence in A, then there exists $x \in A$ s.t*

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = d(A,B).$$

*Proof.*Let $\{x_{n_k-1}\}$ be a subsequence of $\{x_{n-1}\}$
converging to a point $x \in A$.
Now, observe that

$$d(A,B) \leq d(x, x_{n_k}) \leq d(x, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})$$

for all $n \in \mathbb{N}$. Since $\lim_{k \to \infty} d(x, x_{n_k-1}) = 0$ and
from Theorem 1, we know that
$\lim_{k \to \infty} d(x, x_{n_k-1}) = d(A,B)$
we have

$$d(A,B) \leq \lim_{n \to \infty} d(x, x_{n_k})$$
$$\leq \lim_{n \to \infty} d(x, x_{n_k-1}) + \lim_{n \to \infty} d(x_{n_k-1}, x_{n_k}).$$

By taking limit as $k \to \infty$

$$d(A,B) \leq \lim_{n \to \infty} d(x, x_{n_k}) \leq \lim_{n \to \infty} d(x_{n_k-1}, x_{n_k}) = d(A,B)$$

hence

$$\lim_{n \to \infty} d(x, x_{n_k}) = d(A,B).$$

Now

$$d(A,B) \leq d(x_{n_k+1}, Tx)$$
$$= d(x_{n_k}, Tx)$$
$$\leq \alpha[d(x_{n_k+1}, x_{n_k}) + d(Tx, x) + kd(x_{n_k}, Tx)]$$
$$+ (1 - \alpha)d(A,B)$$
$$= \alpha[d(x_{n_k}, x_{n_k}) + d(Tx, x) + kd(x_{n_k}, Tx)]$$
$$+ (1 - \alpha)d(A,B)$$
$$= \alpha[d(Tx, x) + kd(x_{n_k}, Tx)] + (1 - \alpha)d(A,B)$$
$$\leq \alpha kd(x_{n_k}, Tx) + (1 - \alpha)d(A,B)$$
$$\leq \alpha kd(x_{n_k}, x) + (1 - \alpha)d(x_{n_k}, x)$$
$$= (\alpha k - \alpha + 1)d(x_{n_k}, x).$$

hence

$$d(A,B) \leq d(x_{n_k+1}, Tx) \leq (\alpha k - \alpha + 1)d(x_{n_k+1}, x)$$

Taking limit as $k \to \infty$, we obtain

$$d(A,B) \leq \lim_{n \to \infty} d(x_{n_k+1}, Tx) \leq (\alpha k - \alpha + 1) \lim_{n \to \infty} d(x_{n_k+1}, x)$$

$$d(A,B) \leq d(x, Tx) \leq (\alpha k - \alpha + 1) \lim_{n \to \infty} d(A,B)$$

This implies $d(A,B) \leq d(x, Tx) \leq d(A,B)$
and it follows that $d(x, Tx) = d(A,B)$.

**Theorem 3.** *Let A and B be two non-empty closed subsets of a complete cone metric space X and P an $\mathcal{M}$ - normal cone. Suppose the mapping $T : A \cup B \to A \cup B$ satisfies the strict cyclic contraction condition*

$$d(Tx, Ty) < \alpha[d(Tx, y) + d(Ty, x) + kd(x, y)]$$
$$+ (1 - \alpha)d(A,B)$$

*for any $x \in A$ and $y \in B$. Where $0 < \alpha < 1$, $0 < k < 1$, with $m + k < m - \alpha m > 0$. Then there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $A \cup B$ defined by $x_{n+1} = Tx_n$, $n \in \mathbb{N}$, such that*

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = d(A,B).$$

*and $d(Tx, x) = d(A,B)$*

*Proof.* Suppose $k \neq 0$ and choose $x_0 \in A$. Let a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $A \cup B$ be defined as $x_1 = Tx_0 \subset A$, $x_2 = Tx_1 \subset B$, ..., $x_{n+1} = Tx_n$, $n \in \mathbb{N}$.
We want to show that the sequence $\{d(x_n, x_{n+1})\}$ is non-increasing.
Since, $T$ is strictly cyclic contraction, we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$< \alpha[d(Tx_{n-1}, x_{n-}) + d(Tx_n, x_n)$$
$$+ kd(x_{n-1}, x_n)] + (1 - \alpha)d(A,B)$$
$$= \alpha[d(x_n, x_{n-1}) + d(x_{n+1}, x_n)$$
$$+ kd(x_{n-1}, x_n)] + (1 - \alpha)d(A,B)$$
$$= [(\alpha + k)d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$
$$+ (1 - \alpha)d(A,B)$$
$$\leq \frac{\alpha k + 1}{1 - \alpha} d(x_n, x_{n-1})$$
$$\leq d(x_n, x_{n-1})$$

where $\frac{\alpha k + 1}{1 - \alpha} < 1$ Hence, $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence. Now by the completeness of $X$, $x_n \to x$ for some point $x \in A \cup B$.

$$d(Tx, x) \leq d(Tx, Tx_n) + d(Tx_n, x)$$
$$= d(Tx, Tx_n) + d(Tx_n, x)$$
$$< \alpha[d(Tx_n, x) + d(Tx, x_n) + kd(x_n, x)]$$
$$+ (1 - \alpha)d(A,B) + d(Tx_n, x)$$
$$= \alpha d(Tx_n, x) + \alpha d(Tx, x_n) + \alpha kd(x_n, x)$$
$$+ (1 - \alpha)d(A,B) + d(Tx_n, x)$$
$$= \alpha d(Tx_n, x) + \alpha d(Tx, x_n) + \alpha kd(x_n, x)$$
$$+ d(A,B) - \alpha d(A,B) + d(Tx_n, x)$$
$$\leq \alpha d(Tx_n, x) + \alpha d(Tx, x_n) + \alpha kd(x_n, x)$$
$$+ d(A,B) - \alpha d(A,B) + d(Tx_n, x)$$
$$\leq \alpha d(Tx_n, x) + \alpha d(Tx, x_n) + \alpha kd(x_n, x)$$
$$+ d(x_n, x) - \alpha d(A,B) + d(Tx_n, x)$$

Letting $n \to \infty$ and using the $\mathcal{M}$- normality of $P$, we see that

$$d(Tx, x) \leq \alpha d(Tx_n, x) + \alpha d(Tx, x_n) + \alpha kd(x, x)$$
$$+ d(x, x) - \alpha d(A,B) + d(Tx, x)$$
$$\leq \frac{1}{(1 - \alpha)^\lambda}[\alpha d(Tx_n, x) + \alpha kd(x_n, x)$$
$$+ d(x_{n+1}, x)] + \frac{1}{(1 - \alpha)^\lambda} d(A,B)$$

and since $\mathcal{P}$ is $M$- normal, we have

$$d(Tx, x) \leq \frac{1}{(1 - \alpha)^\lambda} d(A,B)$$
$$\leq \frac{1}{(1 - \alpha)^\lambda} d(A,B)$$
$$\leq d(A,B)$$

Hence, we have

$$d(A,B) \leq d(Tx, x) \leq d(A,B).$$

and this implies

$$d(Tx, x) = d(A,B).$$

Setting $A \cap B = \phi$ in Theorem 2, yields the following corollary.

**Corollary 1.** *([10]). Let X be a complete cone metric space and P an $\mathcal{M}$ - normal cone. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) < \alpha[d(Tx, x) + d(Ty, y) + kd(x, Ty)] \qquad (2)$$

*for any $x \in A$ and $y \in B$. Where $0 \leq \alpha < 1$, $0 < k < 1$. Then, $T$ has a unique fixed point in $X$.*

Setting $A \cap B = \phi$ in Theorem 3, yields the following corollary.

**Corollary 2.**(*[10]). Let X be a complete cone metric space and P an $\mathcal{M}$ - normal cone. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) < \alpha[d(Tx, y) + d(Ty, x) + kd(x, y)] \qquad (3)$$

*for any $x \in A$ and $y \in B$. Where $0 < \alpha < 1$, $0 < k < 1$, with $m + k < m - \alpha m > 0$. Then, T has a unique fixed point in X.*

## 4 Conclusion

[10] introduced the concept of m-normal cone metric spaces and proved the existence of fixed points for such contractions in the said spaces. This idea particularly extends the work of [7] from cone metric space to m-normal cone metric space. Even though the famous Banach, Kannan and Charterjea results can be deduced from [10] but best proximity point is not considered in m-normal cone metric spaces hence our results. Our result extend the work of [10] from fixed point to best proximity point. However, the introduced mapping can be used to obtained best proximity point in a bigger space.

## Acknowledgement

## References

[1] E. Picard, Sur I'application des methodes d'approximations successive a l'etude de certaines equations differentielles ordinaires, Journal of Mathematics, **9**, 217-271 (1893).

[2] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, Fundamental Mathematics **3**, 133 - 181 (1922).

[3] R. Cacciopoli, Un teorem generale sull'essistenza di elementiliaiti in una transformazione funzionale, Rendiconti Accademy Nazionale Lincei **13**, 498-502 (1931).

[4] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, proceedings of the American Mathematical Society **136**(5), 1861 - 1869 (2008).

[5] A. Eldreed and P. Veeramani, Existence and Convergence of Best Proximity Points, Mathematical Analysis Application **323**, 1001-1006 (2006).

[6] I. A. Fulatan, S. Yahaya and A. T. Imam, Best Proximity Points for Nonlinear Cyclic Mappings On Multiplicative Metric Spaces, Nigerian Journal of Scientific Research **15**(3), 587 - 590 (2016).

[7] H. Long and G. Zhang, Cone Metric Spaces and fixed point theorems of contractive mappings, Journal of Mathematical Analysis and Applications **332**, 1468-1476 (2007).

[8] A. Azam, M. Arshad and I. Beg, Banach Contraction Principle on Cone rectangular metric spaces, Applicable Analysis and Discrete Mathematics **3**, 236-241 (2009).

[9] P. Raja and S. M. Vaezpour, Some extensions of Banach?s contraction principle in complete cone metric spaces, Fixed Point Theory and Applications **11**(2008), Article ID 768294.

[10] M. S. Shagari, A. T. Imam and S. Yahaya, m-Normal Cone Metric Spaces, Journal of Analysis and Number Theory **6**(1), 21-26 (2018).

[11] S. Yahaya, Common fixed Points of Contraction Mapping in a Metric space, Pan American Mathematical Journal **30**(4), 1 - 10 (2020).

[12] A. Azam and N. Mehmood, Multivalued fixed point theorems in tvs-cone metric spaces, Fixed Point Theory and Applications **2013**, Article ID 184 (2013).

[13] S. Basha and N. Shahzad and R. Jeyaraj, Common best proximity points: global optimization of multi-objective functions, Applied Mathematics Letter **24**, 883-886 (2011).

[14] N. Mizoguchi and W. Takahashi, Fixed point theorems for multi-valued mappings on complete metric spaces, Journal of Mathematical Analysis and Applications **141**, 177-188, (1989).

[15] J. Merryfield, B. Rothschild and J.D. Stein, An application of Ramsey's theorem to the Banach contraction principle, proceedings of the American Mathematical Society **130**(4), 927 - 933 (2002).

[16] C. Byne, A unified tretmen of some iterative algorithms in signal processing and image restoration, Institute of Physics Publishing; Inverse Problems **20**, 103 - 120 (2004).

[17] H. Aydi , E. Karapinar and H. Lakzian, Fixed point results on a class of generalized metric spaces, Journal Mathematical Analysis Application **337**, 36 - 48 (2008).

[18] R. P. Agarwal, M. A. El-Gebily and D. ORegan, Generalized contractions in partially ordered metric spaces, Applied Analysis **87**, 109-116 (2008).

[19] W. S. Du, Some new results and generalizations in metric fixed point theory, Nonlinear Analysis: Theory, Methods and Applications **73**, 1439-1446 (2010).

[20] K. P. Sastry , G. A. Naidu , K. M. Sharma and I. L. Gayatri, Common fixed point for compatible mappings on 2-metric spaces, Global Journal mathematical science; Theory and practical **2**(3), 173-179 (2010).

[21] S. Niyom and P. Boriwan and N. Petrot, Existence of Best Proximity Points for a Class of Generalized Cyclic Contraction Mappings, Thai Journal of Mathematics **16**, 793 - 182 (2018).