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# Qualitative Analysis of a SIR Epidemic Model with a Nonlinear Relapse and Incidence Rate Stochastically Perturbed

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**Abstract:** In this paper, we analyze a stochastic SIR epidemic model in a constant population with a relapse and nonlinear perturbation. First, we illustrate show that the system has a unique global positive solution that belongs to a positively invariant set. Then, we obtain sufficient conditions for the extinction and persistence in the mean. Finally, numerical simulations are carried out to illustrate the theoretical results.

**Keywords:** Stochastic epidemic model, relapse, extinction, persistence

## 1 Introduction

The spread of infectious diseases, which has become one of the major concerns, threatens public health, as well as the economic and social developments of the community. Thus, it is necessary to control it. Based on the classical SIR epidemic model of Kermack and McKendrick which was established in 1927 [1], the spread of infectious diseases among population is mathematically described using compartmental models, such as SIS, SIR, SIRS, or SEIR where each letter refers to a compartment in which an individual can reside. Let $S(t)$ denotes the number of members of a population susceptible to the disease, $I(t)$ represents the number of infective members and $R(t)$ signifies the number of the members who have been removed from the possibility of infection with permanent or temporary immunity, at time $t$. A relapse may occur after recovery. According to Macquarie Dictionary, a relapse is "to fall or slip back into a former state or practice,...etc". According to the Australian Concise Oxford Dictionary, it is "deterioration in a patient's condition after a partial recovery" [21]. Incomplete treatment may lead to a relapse in case of catching some diseases, such as tuberculosis, see Martin [4]. Tudor [2] was the first to construct and study a compartmental epidemic model with relapse. In [16,20], the displacement of the recovered individuals to the infective class due to relapse is illustrated by a linear relapse rate $\eta R(t)$, where $\eta > 0$. If a relapse occurs due to a contact with an infected person, it is more reasonable to consider a bilinear relapse rate $\delta\beta R(t)I(t)$. That is, the SIR model with bilinear relapse takes the form:

$$\begin{cases} dS(t) = [\mu - \mu S(t) - \beta S(t)I(t)]dt, \\ dI(t) = [-(\mu+\alpha)I(t) + \beta S(t)I(t) + \delta\beta R(t)I(t)]dt, \\ dR(t) = [-\mu R(t) + \alpha I(t) - \delta\beta R(t)I(t)]dt, \end{cases} \tag{1.1}$$

where, $\mu$ is the rate at which new individuals enter the population as well as the natural death rate of the susceptible, infected and recovered individuals. $\alpha$ is the rate at which the infective individuals get recovered, $\beta$ is the infection coefficient, and $\delta \in [0,1]$ is the parameter that measures the intensity of the relapse. The incidence of a disease is the number of new cases per unit time. In our model, we adopt the bilinear incidence rate $\beta S(t)I(t)$, which is frequently used in many epidemic models. Another addition in the modeling of population dynamics of diseases is the introduction of stochasticity into

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epidemic models. Several scholars have investigated the effect of stochasticity on epidemic models [8,6,10]. For instance, to include stochastic demographic variability, Allen [8] studied a SDEs for simple SIS and SIR epidemic models with constant population size derived from a continuous time Markov chain model. In [6], the situation of white noise stochastic perturbations around the endemic equilibrium state was considered. The technique of parameter perturbation has been used by several researchers [9,10,17,13]. The case of color noise was investigated by Gray et al. as well as Settati and Lahrouz [12,15]. Both of them have conducted a detailed analysis on asymptotic behavior of an SIS epidemic model under a finite regimes-switching.

Let $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions that comprise all the random variables we meet in the following. In this paper, we assume that fluctuations in the environment will manifest themselves as fluctuations in the infection coefficient $\beta$. To incorporate the random nature of diseases transmission, we assume that the infection coefficient $\beta$ is a random variable governed by the equation $d\beta(t)=\widetilde{\beta}dt+\sigma dB(t)$, where $B(t)$ is a Brownian motion and $\sigma$ is the intensity of noise. Thus, if we still denote the mean value $\widetilde{\beta}$ by $\beta$, the corresponding stochastic system (1.1) can be described by the following stochastic differential equation:

$$\begin{aligned} dS(t) &= [\mu-\mu S(t)-\beta S(t)I(t)]dt-\sigma S(t)I(t)dB(t),\\ dI(t) &= [-(\mu+\alpha)I(t)+\beta S(t)I(t)+\delta\beta R(t)I(t)]dt\\ &\quad +(\sigma S(t)I(t)+\delta\sigma R(t)I(t))dB(t),\\ dR(t) &= [-\mu R(t)+\alpha I(t)-\delta\beta R(t)I(t)]dt\\ &\quad -\delta\sigma R(t)I(t)dB(t), \end{aligned} \tag{1}$$

subject to the initial conditions: $S(0)>0, I(0)>0, R(0)>0$ and $S(0)+I(0)+R(0)=1$. We mention here that a stochastic version of the deterministic model (1.1) is studied by El Myr et al [19], by perturbating the death rates in a population of varying size, which is not the case in our model. That is, the population remains constant in time. The other parts of the present paper is organized as follows: The next section addresses solving the problem of the model well posedness (1). The third section is devoted to obtaining sufficient conditions for the extinction of the disease described by (1). The fourth section sheds light on the disease persistence in the host population. The fifth section presents some numerical simulations to illustrate the theoretical findings. The last section involves conclusions and further considerations .

## 2 Existence and uniqueness of positive solution

Throughout the following sections, we denote .

$$\mathbb{R}^3_+=\{(x_1,x_2,x_3)|x_i>0,i=1,2,3\},$$

$$\Delta=\{x\in\mathbb{R}^3_+,x_1+x_2+x_3=1\}$$

In general, consider the $n-$dimensional stochastic differential equation

$$dX(t)=f(X(t),t)dt+g(X(t),t)dB(t),\ X(0)=x_0\in\mathbb{R}^n, \tag{2}$$

where $f:\mathbb{R}^n\to\mathbb{R}^n$, $g:\mathbb{R}^n\to\mathbb{R}^{n\times d}$ and $B(t)$ denotes a $d$-dimensional standard Brownian motion defined on the underlying probability space. If $A$ is a vector or matrix, its transpose is denoted by $A^T$. The $n\times n$ matrix

$$\Sigma(x)\overset{\Delta}{=}g(x)^Tg(x)$$

is called the diffusion matrix. For the convenience of a later presentation, we present the generator $\mathscr{L}$ associated with (1) as follows. For any twice continuously differentiable $\mathscr{V}(x)\in\mathscr{C}^2(\mathbb{R}^n)$

$$\mathscr{L}\mathscr{V}(x)=\nabla\mathscr{V}(x)f(x)+\frac{1}{2}trace\left(\Sigma(x)\nabla^2\mathscr{V}(x)\right),$$

where $\nabla\mathscr{V}$ and $\nabla^2\mathscr{V}$ denote the gradient, Hessian of $\mathscr{V}$ respectively. Next, we indicate that the solution of model (1) is global and positive using the Lyapunov analysis method.

**Theorem 1.***For any initial values $(S(0),I(0),R(0))\in\Delta$, there is a unique solution $(S(t),I(t),R(t))$ to system (1) on $t\geq 0$ and the solution will remain in $\Delta$ with probability 1.*

*Proof.*Since the coefficients of system (1) are locally Lipschitz continuous, for any initial value $(S(0),I(0),R(0))\in\Delta$, there is a unique local solution on $[0,\tau_e)$ where $\tau_e$ is the explosion time (see [5]). Let $p_0>0$ be sufficiently large such that $\min\{S(0),I(0),R(0)\}>\frac{1}{p_0}$. For each $p\geq p_0$, define the stopping time $\tau_p$ as follows:

$$\tau_p=\inf\left\{t\in[0,\tau_e)\ :\min\left\{S(t),I(t),R(t)\right\}\leq\frac{1}{p}\right\}, \tag{3}$$

we set $\inf\emptyset=\infty$. Obviously, $\tau_p$ is increasing as $p\to\infty$. Set $\tau_\infty=\lim_{p\to\infty}\tau_p$. Suppose that $P\{\tau_\infty\leq\infty\}>0$, then there is a pair of constants $T>0$ and $\varepsilon\in(0,1)$ such that $P\{\tau_\infty\leq T\}>\varepsilon$. Thus, there is an integer $p_1\geq p_0$ such that

$$P\{\tau_p\leq T\}\geq\varepsilon,\quad\forall p\geq p_1 \tag{4}$$

For $t\leq\tau_p$, we can see for each $k$

$$\frac{d(S+I+R)}{dt}=\mu\left(1-S(t)+I(t)+R(t)\right),$$

which implies that for all $t\geq 0$, we have

$$d(S(t)+I(t)+R(t))=1+e^{-\mu t}\left(S(0)+I(0)+R(0)-1\right).$$

Since, $S(0)+I(0)+R(0)=1$. Then

$$S(t)+I(t)+R(t)=1\qquad\text{for all } t\geq 0. \tag{5}$$

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Define the $C^2$ function $\mathtt{W}: \mathbb{R}^3_+ \to \mathbb{R}$ by

$$\mathtt{W}(S,I,R) = (S-1-lnS)+(I-1-lnI)+(R-1-lnR)$$

Hence, by Itô's formula, we obtain that for all $t \in [0,\tau_p)$

$$d\mathtt{W}(t) = L\mathtt{W}(t)dt + \sigma\Big[I(t)-S(t)-\delta R(t)+\delta I(t)\Big]dB(t), \tag{6}$$

where $L\mathtt{W}(t)$ is given by,

$$\begin{aligned} L\mathtt{W}(t) &= (1-\frac{1}{S(t)})\Big[\mu-\mu S(t)-\beta S(t)I(t)\Big]+\frac{1}{2}\sigma^2 I^2(t) \\ &\quad +(1-\frac{1}{I(t)})\Big[-(\mu+\alpha)I(t)+\beta S(t)I(t) \\ &\quad +\delta\beta R(t)I(t)\Big]+\frac{1}{2}\sigma^2\big(S(t)+\delta R(t)\big)^2 \\ &\quad +(1-\frac{1}{R(t)})\Big[-\mu R(t)+\alpha I(t)-\delta\beta R(t)I(t)\Big] \\ &\quad +\frac{1}{2}\sigma^2\delta^2 R^2(t) \\ &= 4\mu+\alpha+\beta I(t)+\delta\beta I(t)-\mu\big(S(t)+I(t)+R(t)\big) \\ &\quad -\frac{\mu}{S(t)}-\beta S(t)-\delta\beta R(t)-\alpha\frac{I(t)}{R(t)} \\ &\quad +\frac{1}{2}\sigma^2 I^2(t)+\frac{1}{2}\sigma^2(S(t)+\delta R(t))^2+\frac{1}{2}\sigma^2\delta^2 R^2(t) \\ &\le 4\mu+\alpha+\beta+\delta\beta+\frac{1}{2}\sigma^2+\frac{1}{2}\sigma^2(1+\delta)^2+\frac{1}{2}\sigma^2\delta^2 \\ &=: K, \end{aligned}$$

where $K$ is a positive constant independent of $S,I,R$ and $t$. Substituting the last inequality into (6) leads to

$$d\mathtt{W}(t) \le Kdt + \sigma\Big[I(t)-S(t)-\delta R(t)+\delta I(t)\Big]dB(t).$$

Integrating the above-mentioned inequality sides from 0 to $\tau_p \wedge T$ and taking expectation, we get

$$\begin{aligned} E\Big[\mathtt{W}\big(S(\tau_p\wedge T),I(\tau_p\wedge T),R(\tau_p\wedge T)\big)\Big] &\le \mathtt{W}\big(S(0),I(0),R(0)\big) \\ &\quad +E\Big[\int_0^{\tau_p\wedge T} Kdt\Big] \\ &\le \mathtt{W}\big(S(0),I(0),R(0)\big) \\ &\quad +KT. \end{aligned} \tag{7}$$

Let $\Omega_p = \{\tau_p \le T\}$ for $p \ge p_1$, we have by (4) $P(\Omega_p) \ge \varepsilon$. Thus, for every $\omega \in \Omega_p$, there is at least one of $S_k(\tau_p,\omega), I_k(\tau_p,\omega)$ and $R_k(\tau_p,\omega)$ equals $\frac{1}{p}$. Therefore, $\mathtt{W}\Big(S(\tau_p,\omega),I(\tau_p,\omega),R(\tau_p,\omega)\Big)$ is not less than $\frac{1}{p}-1-\ln\frac{1}{p}$. Hence,

$$\mathtt{W}\Big(S(\tau_p,\omega),I(\tau_p,\omega),R(\tau_p,\omega)\Big) \ge \frac{1}{p}-1+\ln p.$$

Then, it follows from (7) that

$$\begin{aligned} \mathtt{W}\big(S(0),I(0),R(0)\big)+KT &\ge E\Big[1_{\Omega_p}(\omega)\mathtt{W}\Big(S(\tau_p,\omega),I(\tau_p,\omega) \\ &\qquad ,R(\tau_p,\omega)\Big)\Big] \\ &\ge \varepsilon\Big(\frac{1}{p}-1+\ln p\Big) \end{aligned}$$

where $1_{\Omega_p}(\omega)$ is the indicator function of $\Omega_p$. Let $p\to\infty$ ,we have that

$$\infty > \mathtt{W}\big(S(0),I(0),R(0)\big) \ge +\infty.$$

This is a contradiction. Consequently, we must have $\tau_\infty = \infty$ as. Since, $\tau_\infty \le \tau_e$, we deduce that $\tau_e = \infty$ a.s. That is, the solution $S(t),I(t),R(t)$ will not explode in a finite time and remain positive for all $t \ge 0$. The invariance of the domain $\Delta$ follows from (5).

## 3 Extinction of the Disease

The following theorem provides sufficient conditions in which the disease will die out.

**Theorem 2.** *Let $(S(t),I(t),R(t))$ be the solution of system (1) with initial value $(S(0),I(0),R(0)) \in \Delta$. If one of the two following assumptions holds*

$$(C1)\ \frac{\beta^2}{2\sigma^2} < \mu+\alpha \tag{8}$$

$$(C2)\quad \frac{\sigma^2(\alpha(\delta+1)+\mu)}{\mu+\alpha} \le \beta \quad and$$

$$\mathcal{T}_e = \left[\frac{\frac{\beta[\alpha(\delta+1)+\mu]}{\mu+\alpha}}{(\mu+\alpha)+\frac{\sigma^2}{2}\left(\frac{\alpha(\delta+1)+\mu}{\mu+\alpha}\right)^2}\right] < 1, \tag{9}$$

*then the disease will die out exponentially a.s. That is,*

$$\limsup_{t\to\infty}\frac{\log I_t}{t} < 0, \quad a.s.. \tag{10}$$

*Moreover*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t R(s)ds = 0, \quad a.s..$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t S(s)ds = 1, \quad a.s..$$

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*Proof.* Applying Itô's formula to system (1), we obtain

$$d\log I(t) = \Big(\beta\big(S(t)+\delta R(t)\big)-(\mu+\alpha) - \frac{\sigma^2}{2}\big(S(t)+\delta R(t)\big)^2\Big)dt + \sigma\big(S(t)+\delta R(t)\big)dB(t).$$

Integrating the above equation from 0 to t and dividing by t results in

$$\begin{aligned}\frac{\log I(t)-\log I(0)}{t} &= \frac{\beta}{t}\int_0^t (S(s)+\delta R(s))ds-(\mu+\alpha) - \frac{\sigma^2}{2}\int_0^t \frac{(S(s)+\delta R(s))^2}{t}ds+\frac{M(t)}{t}\\ &\le \beta\Big(\int_0^t \frac{(S(s)+\delta R(s))}{t}ds\Big)-(\mu+\alpha) - \frac{\sigma^2}{2}\Big(\int_0^t \frac{(S(s)+\delta R(s))}{t}ds\Big)^2 + \frac{M(t)}{t}\\ &\le \frac{\beta^2}{2\sigma^2}-(\mu+\alpha) - \frac{\sigma^2}{2}\Big[\Big(\int_0^t \frac{(S(s)+\delta R(s))}{t}ds\Big)-\frac{\beta}{\sigma^2}\Big]^2 + \frac{M(t)}{t}\\ &\le \frac{\beta^2}{2\sigma^2}-(\mu+\alpha)+\frac{M(t)}{t}.\end{aligned}$$

Where $M(t) = \sigma\int_0^t \big(S(s)+\delta R(s)\big)dB(s)$ is a local martingale with $M(0)=0$. Moreover,

$$\limsup_{t\to\infty}\frac{\langle M,M\rangle_t}{t} \le \sigma^2(1+\delta)^2<\infty \quad a.s..$$

In virtue of Lemma 2, we get $\lim_{t\to+\infty}\frac{M(t)}{t}=0$ a.s..

$$\limsup_{t\to+\infty}\frac{\log I(t)}{t} \le \frac{\beta^2}{2\sigma^2}-(\mu+\alpha) \quad a.s.,$$

so, if condition (8) holds, the desired assertion (10) is verified. Now, we show that (9) is a sufficient condition for (10). First, we rewrite, using $S+I+R=1$, the third equation of the system (1) as

$$\begin{aligned}dR &= [-\mu R(t)+\alpha(1-S(t)-R(t))-\delta\beta R(t)I(t)]dt - \delta\sigma R(t)I(t)dB(t)\\ &= [\alpha-(\mu+\alpha)R(t)-\alpha S(t)-\delta\beta R(t)I(t)]dt - \delta\sigma R(t)I(t)dB(t).\end{aligned}$$

From the well-known variation of constant formula, we get

$$\begin{aligned}R(t) &= e^{-(\mu+\alpha)t}\Big[R(0)+\int_0^t e^{(\mu+\alpha)s}\big(\alpha-\alpha S(t) - \delta\beta R(s)I(s)\big)ds - \delta\sigma\int_0^t e^{(\mu+\alpha)s}R(s)I(s)dB(s)\Big]\\ &= \frac{\alpha}{\mu+\alpha}-\Big(\frac{\alpha}{\mu+\alpha}-R(0)\Big)e^{-(\mu+\alpha)t} - \alpha\int_0^t S(s)e^{-(\mu+\alpha)(t-s)}ds\\ &\quad -\delta\beta\int_0^t R(s)I(s)e^{-(\mu+\alpha)(t-s)}ds - \delta\sigma\int_0^t R(s)I(s)e^{-(\mu+\alpha)(t-s)}dB(s).\end{aligned}$$

Then

$$\begin{aligned}S(t)+\delta R(t) &= (1-R(t)-I(t)) + \delta\Big[\frac{\alpha}{\mu+\alpha}-\Big(\frac{\alpha}{\mu+\alpha}-R(0)\Big)e^{-(\mu+\alpha)t}\Big]\\ &\quad -\delta\Big(\alpha\int_0^t S(s)e^{-(\mu+\alpha)(t-s)}ds + \delta\beta\int_0^t R(s)I(s)e^{-(\mu+\alpha)(t-s)}ds + \delta\sigma\int_0^t R(s)I(s)e^{-(\mu+\alpha)(t-s)}dB(s)\Big)\\ &= \frac{\alpha(\delta+1)+\mu}{\mu+\alpha}-\psi_1(t)+\psi_2(t) - \delta^2\sigma\int_0^t R(s)I(s)e^{-(\mu+\alpha)(t-s)}dB(s), \quad (11)\end{aligned}$$

where

$$\psi_1(t) = R(t)+I(t)+\delta\alpha\int_0^t S(s)e^{-(\mu+\alpha)(t-s)}ds + \delta^2\beta\int_0^t R(s)I(s)e^{-(\mu+\alpha)(t-s)}ds,$$

$$\text{and} \quad \psi_2(t) = -\delta\Big(\frac{\alpha}{\mu+\alpha}-R(0)\Big)e^{-(\mu+\alpha)t}.$$

Since, $S,I,R\in(0,1)$, we have

$$\lim_{t\to 0}\frac{1}{t}\int_0^t \psi_2(s)ds = 0 \quad and \quad \frac{1}{t}\int_0^t \psi_1(s)ds \ge 0. \quad (12)$$

Next, from (11) we get

$$\begin{aligned}\frac{\log I(t)-\log I(0)}{t} &= \frac{\beta}{t}\int_0^t (S(s)+\delta R(s))ds-(\mu+\alpha)- \frac{\sigma^2}{2t}\int_0^t (S(s)+\delta R(s))^2 ds+\frac{M(t)}{t}\\ &\le \frac{\beta}{t}\int_0^t (S(s)+\delta R(s))ds-(\mu+\alpha) - \frac{\sigma^2}{2}\Big(\int_0^t \frac{(S(s)+\delta R(s))}{t}ds\Big)^2 + \frac{M(t)}{t},\end{aligned}$$

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where $M(t)=\sigma\int_0^t\big(S(s)+\delta R(s)\big)dB(s)$. By (11), we have

$$\frac{\log I(t)-\log I(0)}{t}\le\frac{\beta\Big[\alpha(\delta+1)+\mu\Big]}{\mu+\alpha}-(\mu+\alpha)-\frac{\sigma^2}{2}\left(\frac{\alpha(\delta+1)+\mu}{\mu+\alpha}\right)^2+\left(\frac{\sigma^2(\alpha(\delta+1)+\mu)}{\mu+\alpha}-\beta\right)\times\frac{1}{t}\int_0^t\psi_1(s)ds+\Gamma(t).$$

Where

$$\Gamma(t)=\left(-\sigma^2\left(\frac{\alpha(\delta+1)+\mu}{\mu+\alpha}-\frac{1}{t}\int_0^t\psi_1(s)ds\right)+\beta\right)\times\left(\frac{1}{t}\int_0^t\psi_2(s)ds-\Psi(t)\right)-\frac{\sigma^2}{2}\left(\frac{1}{t}\int_0^t\psi_2(s)ds-\Psi(t)\right)^2+\frac{M(t)}{t},$$

and

$$\Psi(t)=\frac{\delta^2\sigma}{t}\int_0^t\int_0^u R(s)I(s)e^{-(\mu+\alpha)(u-s)}dB(s)du.$$

In view of Lemma (1), we have $\lim_{t\to+\infty}\Psi(t)=0$ a.s., Lemma (2) with (12) lead to $\lim_{t\to+\infty}\Gamma(t)=0$ a.s.. Thus, if condition (9) holds, we get

$$\begin{aligned}\limsup_{t\to+\infty}\frac{\log I(t)}{t}&\le\frac{\beta\left[\alpha(\delta+1)+\mu\right]}{\mu+\alpha}-(\mu+\alpha)-\frac{\sigma^2}{2}\left(\frac{\alpha(\delta+1)+\mu}{\mu+\alpha}\right)^2\\&\le\left((\mu+\alpha)+\frac{\sigma^2}{2}\left(\frac{\alpha(\delta+1)+\mu}{\mu+\alpha}\right)^2\right)\times\left[\frac{\frac{\beta[\alpha(\delta+1)+\mu]}{\mu+\alpha}}{(\mu+\alpha)+\frac{\sigma^2}{2}\left(\frac{\alpha(\delta+1)+\mu}{\mu+\alpha}\right)^2}-1\right]\\&\le\left((\mu+\alpha)+\frac{\sigma^2}{2}\left(\frac{\alpha(\delta+1)+\mu}{\mu+\alpha}\right)^2\right)\times(\mathscr{T}_e-1)<0,\end{aligned}\tag{13}$$

which indicates that

$$\lim_{t\to+\infty}I(t)=0\ a.s..$$

Denote $\Omega_1=\{\omega\in\Omega:\lim_{t\to\infty}I(\omega,t)=0\}$. In view of (13), we have $\mathbb{P}(\Omega_1)=1$. It means, for any given $\varepsilon_1>0$, there exists a constant $T_1=T_1(\omega)$ such that $I(t)<\varepsilon_1$, a.s. for $t>T_1$. Integrating the third equation of (1) both sides from 0 to $t$ and dividing by $t$, we have

$$\begin{aligned}\frac{\mu}{t}\int_0^t R(s)ds&\le\frac{\alpha}{t}\int_0^t I(s)ds-\frac{\delta\beta}{t}\int_0^t R(s)I(s)ds-\frac{R(t)-R(0)}{t}-\frac{1}{t}\int_0^t\sigma\delta R(s)I(s)dB(s).\\&\le\frac{\alpha}{t}\int_0^t I(s)ds-\frac{R(t)-R(0)}{t}-\frac{1}{t}\int_0^t\sigma\delta R(s)I(s)dB(s).\\&\le\frac{\alpha}{t}\int_0^{T_1}I(s)ds+\frac{\alpha}{t}\int_{T_1}^t I(s)ds-\frac{R(t)-R(0)}{t}-\frac{1}{t}\int_0^t\sigma\delta R(s)I(s)dB(s).\\&\le\frac{\alpha T_1}{t}+\alpha\varepsilon_1-\frac{\alpha\varepsilon_1 T_1}{t}-\frac{R(t)-R(0)}{t}-\frac{1}{t}\int_0^t\sigma\delta R(s)I(s)dB(s).\end{aligned}$$

According to Lemma 2, we get $\lim_{t\to+\infty}\frac{1}{t}\int_0^t\sigma\delta R(s)I(s)dB(s)=0$, a.s.. Then, from the arbitrariness of $\varepsilon_1$, we get

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t R(s)ds\le 0,\quad a.s..$$

From the positivity of $R$, we get

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t R(s)ds=0,\quad a.s..$$

Let $\Omega_2=\{\omega\in\Omega_1:\lim_{t\to\infty}\frac{1}{t}\int_0^t R(\omega,s)ds=0\}\subset\Omega_1$. In view of (13), for any given $\varepsilon_2>0$, there exists a constant $T_2=T_2(\omega)\ge T_1$ such that $\frac{1}{t}\int_0^t R(s)ds<\varepsilon_2$, a.s. for $t>T_2$. For any $\omega\in\Omega_2$, we have

$$d(S(t)+I(t)+R(t))=\Big[\mu-\mu\big(S(t)+I(t)+R(t)\big)\Big]dt.$$

Integrating from 0 to $t$ and dividing by $t$ yield

$$\begin{aligned}\frac{1}{t}\int_0^t S(s)ds&=1-\frac{1}{t}\int_0^t I(s)ds-\frac{1}{t}\int_0^t R(s)ds-\frac{1}{\mu}\Big(\frac{N(t)-N(0)}{t}\Big).\\&=1-\frac{1}{t}\int_0^{T_2}\big(I(s)+R(s)\big)ds-\frac{1}{t}\int_{T_2}^t I(s)ds-\frac{1}{t}\int_{T_2}^t R(s)ds-\frac{1}{\mu}\Big(\frac{N(t)-N(0)}{t}\Big).\\&\ge 1-\varepsilon_1-\varepsilon_2+\varepsilon_1\frac{T_2}{t}-\frac{2T_2}{t}-\frac{1}{\mu}\Big(\frac{N(t)-N(0)}{t}\Big).\end{aligned}$$

From the fact that $\lim_{t\to\infty}\frac{1}{\mu t}\big(\frac{N(t)-N(0)}{t}\big)=0$ a.s., and by the arbitrariness of $\varepsilon_1$ and $\varepsilon_2$, we get

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t S(s)ds\ge 1,\quad a.s..$$

Since, $0 \le S \le 1$, we conclude

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S(s)ds = 1, \quad a.s..$$

Thus, the proof is complete.

## 4 Persistence of the Disease

**Theorem 3.** *For any initial values $\big(S(0), I(0), R(0)\big) \in \Delta$. If*

$$\mathscr{T}_p = \frac{\beta}{\mu+\alpha+\frac{1}{2}\sigma^2} > 1, \tag{14}$$

*then the solution of system (1) obeys*

$$(i)\liminf_{t\longrightarrow\infty} \frac{1}{t}\int_0^t S(u)du \ge \frac{\mu}{\mu+\beta} \quad a.s.,$$
$$(ii)\liminf_{t\longrightarrow\infty} \frac{1}{t}\int_0^t I(u)du \ge \frac{\mu}{\beta-\frac{1}{2}\sigma^2}\Big(1-\frac{1}{\mathscr{T}_p}\Big) \quad a.s.,$$
$$(iii)\liminf_{t\longrightarrow\infty} \frac{1}{t}\int_0^t R(u)du \ge \frac{\mu\alpha}{(\mu+\delta\beta)(\beta-\frac{1}{2}\sigma^2)}\Big(1-\frac{1}{\mathscr{T}_p}\Big) \quad a.s..$$

*Proof.*$(i)$ From the first equation of system (1) and while we have $\big(S(t), I(t), R(t)\big) \in \Delta$, for all $t \ge 0$, we get

$$dS(t) \ge (\mu - (\mu+\beta)S)dt - \sigma S(t)I(t)dB(t).$$

Integrating the above inequality from 0 to $t$ and dividing both sides by $t$ results in the following:

$$\frac{1}{t}\int_0^t S(u)du \ge \frac{1}{\mu+\beta}\Big[\mu - \frac{S(t)-S(0)}{t} - \frac{\sigma}{t}\int_0^t S(u)I(u)dB(u)\Big].$$

From the large number theorem for martingales and the fact that $0 \le S(t) \le 1$ for all $t \ge 0$, we have

$$\lim_{t\to\infty}\Big(\frac{S(t)-S(0)}{t} - \frac{\sigma}{t}\int_0^t S(u)I(u)dB(u) = 0\Big) \quad a.s..$$

Then

$$\liminf_{t\longrightarrow\infty} \frac{1}{t}\int_0^t S(u)du \ge \frac{\mu}{\mu+\beta}.$$

$(ii)$ Applying Itô's formula to system (1), we get

$$d\log I(t) = \Big(\beta\big(S(t)+\delta R(t)\big) - (\mu+\alpha) - \frac{\sigma^2}{2}\big(S(t)+\delta R(t)\big)^2\Big)dt + \sigma\big(S(t) + \delta R(t)\big)dB(t).$$

Using the fact that $\big(S(t), I(t), R(t)\big) \in \Delta$, we have $S(t) \le S(t)+\delta R(t) \le 1$. Then

$$\begin{aligned} d\log I(t) &\ge \Big(-\frac{1}{2}\sigma^2 S(t) + \beta S(t) - (\mu+\alpha)\Big)dt \\ &\quad + \sigma\big(S(t)+\delta R(t)\big)dB(t). \\ &\ge \Big(-(\mu+\alpha) + \beta - \frac{1}{2}\sigma^2 \\ &\quad -(\beta-\frac{1}{2}\sigma^2)\big(1-S(t)\big)\Big)dt \\ &\quad + \sigma(S(t)+\delta R(t))dB(t). \end{aligned} \tag{15}$$

On the other hand

$$\begin{aligned} dS(t) &= (\mu - \mu S(t) - \beta S(t)I(t))dt - \sigma S(t)I(t)dB(t) \\ &\ge \Big(\mu\big(1-S(t)\big) - \beta I(t)\Big)dt - \sigma S(t)I(t)dB(t). \end{aligned}$$

So,

$$-\big(1-S(t)\big)dt \ge -\frac{1}{\mu}dS(t) - \frac{\beta}{\mu}I(t)dt - \frac{\sigma}{\mu}S(t)I(t)dB(t). \tag{16}$$

Combining (14),(15),(16) and integrating from 0 to $t$ give

$$\begin{aligned} \log I(t) &\ge \Big(\beta - \big(\mu+\alpha+\frac{1}{2}\sigma^2\big)\Big)t \\ &\quad -\frac{\beta}{\mu}(\beta - \frac{1}{2}\sigma^2)\int_0^t I(u)du + \Upsilon(t). \end{aligned}$$

Where

$$\begin{aligned} \Upsilon(t) &= \log I(0) - \frac{\beta-\frac{1}{2}\sigma^2}{\mu}\big(S(t)-S(0)\big) + \int_0^t \sigma(S(u) \\ &\quad + \delta R(u))dB(u) - \frac{(\beta-\frac{1}{2}\sigma^2)\sigma}{\mu}\int_0^t S(u)I(u)dB(u). \end{aligned}$$

Moreover, based on the fact that $0 \le S(t) \le 1$ for all $t \ge 0$ and Lemma 2, we have $\lim_{t\to\infty}\frac{\Upsilon(t)}{t} = 0$. Applying Lemma 3, we get

$$\liminf_{t\longrightarrow\infty} \frac{1}{t}\int_0^t I(u)du \ge \frac{\mu}{\beta-\frac{1}{2}\sigma^2}\Big(1-\frac{1}{\mathscr{T}_p}\Big) \quad a.s..$$

$(iii)$ Integrating the third equation of the system (1), we have

$$\begin{aligned} R(t)-R(0) &= -\mu\int_0^t R(u)du + \alpha\int_0^t I(u)du \\ &\quad -\delta\beta\int_0^t R(u)I(u)du - \delta\sigma\int_0^t R(u)I(u)dB(u) \\ &\ge \alpha\int_0^t I(u)du - (\mu+\delta\beta)\int_0^t R(u)du \\ &\quad -\delta\sigma\int_0^t R(u)I(u)dB(u). \end{aligned}$$

Then

$$\int_0^t R(u)du \geq \frac{1}{\mu+\delta\beta}\Big( -\big(R(t)-R(0)\big) + \alpha\int_0^t I(u)du$$
$$-\delta\sigma\int_0^t R(u)I(u)dB(u)\Big).$$

According to Lemma (2) and the fact that $0 \leq R(t) \leq 1$ for all $t \geq 0$, we have

$$\lim_{t\to\infty}\left( -\frac{R(t)-R(0)}{t} - \frac{\delta\sigma}{t}\int_0^t R(u)I(u)dB(u) = 0\right) \quad a.s..$$

Then, using $(ii)$, we obtain

$$\liminf_{t\longrightarrow\infty}\frac{1}{t}\int_0^t R(u)du \geq \frac{\mu\alpha}{(\mu+\delta\beta)(\beta-\frac{1}{2}\sigma^2)}\left(1-\frac{1}{\mathscr{T}_p}\right) \quad a.s..$$

This finishes the proof of Theorem (3).

## Appendix

**Lemma 1.** *Let $g(t)$ be a continuous and bounded function on $[0,\infty)$ and $\theta > 1$, then*

$$\limsup_{t\to\infty}\frac{1}{\sqrt{t}\ln t}\left|\int_0^t g(s)dB(s)\right| \leq \theta \quad a.s.$$

*And for any constant $\xi > 0$,*

$$\limsup_{t\to\infty}\frac{1}{\sqrt{t}\ln t}\left|\int_0^t g(s)e^{-\xi(t-s)}dB(s)\right| \leq \theta \quad a.s.$$

For the proof of the lemma, we refer the reader to Liu et al [18]

**Lemma 2.** *[5] (Strong Law of Large Numbers).Let $M = \{M_t\}_{t\geq 0}$ be a real-value continuous local martingale vanishing at $t = 0$.Then*

$$\lim_{t\to\infty}\langle M,M\rangle_t = \infty \quad a.s. \Rightarrow \lim_{t\to\infty}\frac{M_t}{\langle M,M\rangle_t} = 0. \quad a.s.$$

*also we have*

$$\limsup_{t\to\infty}\frac{\langle M,M\rangle_t}{t} < \infty \quad a.s. \Rightarrow \lim_{t\to\infty}\frac{M_t}{t} = 0. \quad a.s.$$

**Lemma 3.** *Suppose $X \in \mathscr{C}(\mathbb{R}_+\times\Omega,\mathbb{R}_+)$ and $Y \in \mathscr{C}(\mathbb{R}_+\times\Omega,\mathbb{R})$. If there exist positive constants $\nu_0$ and $\nu$ such that for all $t \geq 0$:*

$$\log X(t) \geq \nu_0 t - \nu\int_0^t X(u)du + Y(t) \text{ and } \lim_{\infty}\frac{Y(t)}{t} = 0\, a.s..$$

*Then*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t X(u)du \geq \frac{\nu_0}{\nu} \quad a.s.$$

## 5 Discussion and Simulations

To illustrate the effectiveness of the results, we will perform some numerical simulations. Here, the numerical simulations are given by the Milstein scheme [7]. We consider the following discretization of equation (1)

$$\begin{cases} S_{k+1} = S_k + [\mu - \mu S_k - \beta S_k I_k]\Delta t - \sigma S_k I_k\sqrt{\Delta t}\tau_k \\ \quad -\frac{\sigma^2}{2}S_k I_k(\tau_k^2-1)\Delta t, \\ I_{k+1} = I_k + [-(\mu+\alpha)I_k + \beta S_k I_k + \delta\beta R_k I_k]\Delta t \\ \quad +\sigma S_k I_k\sqrt{\Delta t}\tau_k + \frac{\sigma^2}{2}S_k I_k(\tau_k^2-1)\Delta t \\ \quad +\sigma\delta R_k I_k\sqrt{\Delta t}\tau_k - \frac{\sigma^2}{2}\delta R_k I_k(\tau_k^2-1)\Delta t, \\ R_{k+1} = R_k + [-\mu R_k + \alpha I_k - \delta\beta R_k I_k]\Delta t \\ \quad -\sigma\delta R_k I_k\sqrt{\Delta t}\tau_k - \frac{\sigma^2}{2}\delta R_k I_k(\tau_k^2-1)\Delta t, \end{cases}$$

where $\tau_k$ $(k = 1,2,\ldots)$ are $N(0,1)$-distributed independent random variables.

**Example 51** *Let the parameters in model (1) take the following values: $\beta = 0.49$, $\mu = 0.05$, $\delta = 0.22$, $\alpha = 0.28$,, $\sigma = 0.6$, with initial values $S(0) = 0.7$, $I(0) = 0.2$, $R(0) = 0.1$. Therefore, it follows from Theorem (2) that the solution $(S(t), I(t), R(t))$ of model (1) obeys $\limsup_{t\to+\infty}\frac{\log I(t)}{t} \leq \left((\mu+\alpha)+\frac{\sigma^2}{2}\left(\frac{\alpha(\delta+1)+\mu}{\mu+\alpha}\right)^2\right)(\mathscr{T}_e - 1) = -0.00344 < 0$ , a.s.. That is, $I(t)$ will tend to zero exponentially with probability one, (see* Fig*1).*

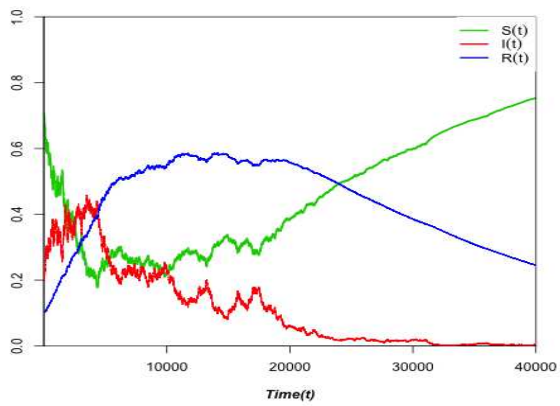


**Fig. 1:** The numerical simulation of one path of the solution $(S(t), I(t), R(t))$ of system (1) using the Milstein scheme with initial values $S(0) = 0.7$, $I(0) = 0.2$, $R(0) = 0.1$ and the parameter values $\beta = 0.49$, $\mu = 0.05$, $\delta = 0.22$, $\alpha = 0.28$,, $\sigma = 0.6$, and $\mathscr{T}_e = 0.99656 < 1$.

**Example 52** *Let the parameters in model (1) take the following values: $\beta = 0.52$, $\mu = 0.05$, $\delta = 0.5$,*

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$\alpha = 0.28,$, $\sigma = 0.6$, *with initial values* $S(0) = 0.7$, $I(0) = 0.2$, $R(0) = 0.1$, *so* $\mathcal{T}_p = 1.02 > 1$. *Therefore, it follows from Theorem* (3) *the disease will prevail in the population, we present the simulations to support our results (see* Fig2*).*

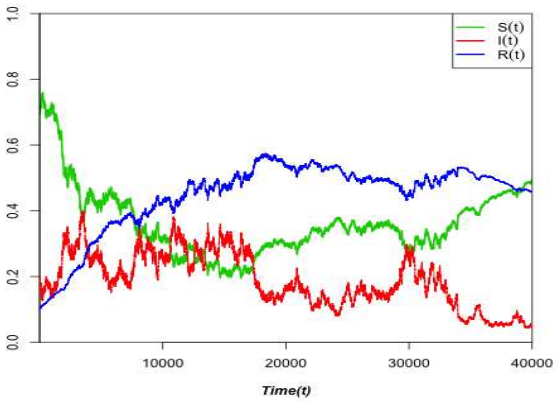


**Fig. 2:** The numerical simulation of one path of the solution $(S(t), I(t), R(t))$ of system (1) using the Milstein scheme with initial values $S(0) = 0.7$, $I(0) = 0.2$, $R(0) = 0.1$ and the parameters values $\beta = 0.52$, $\mu = 0.05$, $\delta = 0.5$, $\alpha = 0.28,$, $\sigma = 0.6$, and $\mathcal{T}_p = 1.02 > 1$.

## 6 Perspective

In this paper, we have considered a stochastic SIR epidemic model with a nonlinear relapse by perturbing the incidence rate. Applying theoretical analysis, we have found sufficient conditions for the extinction and the persistence of the disease. In addition, we have performed some numerical simulations to illustrate our findings. However, some interesting topics deserve further consideration. On the one hand, in Theorem (2), the condition $\frac{\sigma^2(\alpha(\delta+1)+\mu)}{\mu+\alpha} \leq \beta$ is necessary, what is the behavior of system (1) in the case of $\frac{\sigma^2(\alpha(\delta+1)+\mu)}{\mu+\alpha} > \beta$? On the other hand, white noise, in the present paper, is used to describe small-scale time environmental fluctuations, such as daily or weekly variations of meteorological factors. Nevertheless, climate conditions usually experience random switching between different environments, either dry or wet. It is worthwhile to introduce Markovian switching into all parameters of system (1). However, the present stage involves some technical obstacles. Our future work will address these cases.

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