Sohag Journal of Mathematics
An International Journal

http://dx.doi.org/10.18576/sjm/050103

Some Properties of Univalent Analytic Functions Defined by Multiplier Transformation

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Received: 25 Jul. 2016, Revised: 19 Jul. 2017, Accepted: 21 Jul. 2017
Published online: 1 Jan. 2018

Abstract: In this paper, we study certain subclasses of analytic functions involving multiplier transformations in the open unit disc. We derive few subordination results for the functions in these subclasses and discuss the applications of subordination results with the help of complex functions. We obtain coefficient estimates, radii of starlikeness, convexity and close-to-convexity, extreme points, and integral means inequalities, growth and distortion theorems for these classes.

Keywords: subordination theorem, distortion inequalities, extreme points, integral means inequalities, radii of starlikeness, convexity and close-to-convexity.

1 Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic, normalized in the open unit disc $\mathscr{U} = \{z : |z| < 1\}$. We also denote by $\mathscr{C}$ the class of functions $f \in \mathscr{A}$ that are convex in $\mathscr{U}$.

Consider the function $\phi(z)$ defined by

$$\phi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k \tag{2}$$

where $\mu_k \geq 0$ for $k \in \mathbb{N} \setminus \{1\}$.

Definition 1. *Let $g(z)$ be analytic and univalent in $\mathscr{U}$. If $f(z)$ is analytic in $\mathscr{U}$, $f(0) = g(0)$ and $f(\mathscr{U}) \subset g(\mathscr{U})$, then we say that f is subordinate to g, and write $f \prec g$ or $f(z) \prec g(z)$. We also say that g is super ordinate to f in $\mathscr{U}$.*

Definition 2. *An infinite sequence $\{d_k\}_{k=1}^{\infty}$ of complex numbers will be called a subordinating factor sequence if for every univalent function f in $\mathscr{C}$, the class of convex functions in $\mathscr{U}$, we have*

$$\sum_{k=1}^{\infty} d_k a_k z^k \prec f(z), \quad (z \in \mathscr{U}; a_1 = 1).$$

Next we give a characterizing result for subordinating factor sequence in the form of a lemma due to Wilf [15].

Lemma 1. *The sequence $\{d_k\}_{k=1}^{\infty}$ is a subordinating factor sequence, if and only, if*

$$\mathscr{R}e\left\{1 + 2\sum_{k=1}^{\infty} d_k z^k\right\} > 0, \quad (z \in \mathscr{U}).$$

The hadamard product(or convolution) of two functions $f(z)$ given by (1) and

$$g(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

$(\nu)_k$ is the Pochhammer symbol defined by

$$(\nu)_k = \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \begin{cases} 1 & : k = 0 \\ \nu(\nu+1)\cdots(\nu+k-1) & : k \in \mathbb{N}. \end{cases} \tag{3}$$

For the complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ where $(\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots, s\};\ j = 1, 2, \ldots, s)$, we consider

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the generalized hypergeometric function ${}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ by

$$ {}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_q)_k}{(\beta_1)_k\ldots(\beta_s)_k(1)_k}z^k, \quad (4)$$

where $q\le s+1;\ q,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\};\ z\in\mathscr{U}$ and $(x)_n$ is the Pochhammer symbol defined by (3).
Let

$$\begin{aligned} h_{q,s}(\alpha_1,\beta_1;z)&=z.{}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)\\ &=z+\sum_{k=2}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_q)_k}{(\beta_1)_k\ldots(\beta_s)_k(1)_k}z^{k+1}, \end{aligned}\quad (5)$$

and using the hadamard product, the following family of linear operators $\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f:\mathscr{U}\to\mathscr{U}$, defined by

$$\mathscr{I}^{0,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)=f(z)*h_{q,s}(\alpha_1,\beta_1;z);$$

$$\mathscr{I}^{1,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)=(1-\lambda)f(z)*h_{q,s}(\alpha_1,\beta_1;z)+\frac{\lambda}{(1+l)z^{l-1}}\left(z^lf(z)*h_{q,s}(\alpha_1,\beta_1;z)\right);$$

and

$$\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)=\mathscr{I}^{1,l}_{q,s,\lambda}(\alpha_1,\beta_1)\left(\mathscr{I}^{m-1,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)\right). \quad (6)$$

If $f(z)\in\mathscr{A}$ then, from (5) and (6), we see that

$$\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)=z+\sum_{k=2}^{\infty}\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m\frac{(\alpha_1)_{k-1}\ldots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\ldots(\beta_s)_{k-1}(1)_{k-1}}a_kz^k, \quad (7)$$

where $m\in\mathbb{N}_0,\ l\ge 0,\ \lambda\ge 0$. We denote the subclass of $\mathscr{A}$ by $\mathscr{M}^l_m(\beta,\delta,\lambda,k)$ consisting of functions $f(z)$ satisfying

$$\mathscr{R}e\left\{1-\frac{2}{c}+\frac{2}{c}\frac{(\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z))'}{\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)}\right\}>\delta \quad (8)$$

where $c\ne 0,\ \mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\ \beta_1)f(z)*\phi(z)\ne 0,\ 0\le\delta<1$.
A function $f(z)\in\mathscr{A}$ is said to be in the class of β-uniformly starlike functions of order δ denoted by $SD(\beta,\delta)$, if

$$\mathscr{R}e\left\{\frac{zf'(z)}{f(z)}\right\}>\beta\left|\frac{zf'(z)}{f(z)}-1\right|+\delta \quad (9)$$

for some $\beta\ge 0, 0\le\delta<1$ and $z\in\mathscr{U}$.
Similarly, $f(z)\in\mathscr{A}$ is in the class of β-uniformly convex functions of order δ denoted by $KD(\beta,\delta)$, if

$$\mathscr{R}e\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\beta\left|\frac{zf''(z)}{f'(z)}\right|+\delta \quad (10)$$

for some $\beta\ge 0,\ 0\le\delta<1$ and $z\in\mathscr{U}$.
We introduce the class $KD^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$ as a subclass of $\mathscr{A}$ consisting of functions f which satisfy

$$\begin{aligned}&\mathscr{R}e\left\{1-\frac{2}{c}+\frac{2}{c}\frac{z(\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z))'}{\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)}\right\}\\ &\quad>\beta\left|\frac{2}{c}\frac{z(\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z))'}{\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)}-1\right|+\delta\end{aligned} \quad (11)$$

where $c\ne 0,\ \mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\ \beta_1)f(z)*\phi(z)\ne 0,\ \beta\ge 0$ and $0\le\delta<1$. From (2) and (7), we have

$$\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)=z+\sum_{k=2}^{\infty}\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m\frac{(\alpha_1)_{k-1}\ldots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\ldots(\beta_s)_{k-1}(1)_{k-1}}\mu_ka_kz^k$$

For convenience, we can write $\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)$ as follows:

$$\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)=z+\sum_{k=2}^{\infty}\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m\Gamma_{k-1}[\alpha_1,\beta_1]\mu_ka_kz^k \quad (12)$$

where$\Gamma_{k-1}[\alpha_1,\beta_1]=\frac{(\alpha_1)_{k-1}\ldots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\ldots(\beta_s)_{k-1}(1)_{k-1}}$

*Remark.*We notice that for the parametric values

$$c=2,\ \alpha_1=\beta_1,\ s=q,\ \phi(z)=\frac{z}{1-z}$$

obtain the class $SD(\beta,\delta)$ and

$$c=2,\ \alpha_1=\beta_1,\ s=q,\ \phi(z)=\frac{z}{(1-z)^2}$$

obtain the class $KD(\beta,\delta)$.

2 Coefficient inequalities for the class $KD^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$

We prove some coefficient inequalities for the class $KD^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$.

Theorem 1.*Let* $l\ge 0,\ \lambda\ge 0,\ q\le s+1,\ q,\ s\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ \beta\ge 0,\ 0\le\delta<1,\ z\in\mathbb{U},\ c\notin[0,2),\ \mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)\ne 0$. *If* $f(z)\in\mathscr{A}$ *satisfies the inequality given below*

$$\sum_{k=2}^{\infty}\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_k)|a_k|\le 2-\frac{2}{c}-\delta \quad (13)$$

where

$$\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_k) = \left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]\right| \left[\frac{2}{c}(1+\beta)(k-1)+2-\frac{2}{c}-\delta\right]\mu_k$$

then $f(z)\in KD^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$.

Proof. Suppose that condition (13) holds for $l\geq 0,\ \lambda\geq 0,\ q\leq s+1,\ q,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ \beta\geq 0,\ 0\leq\delta<1,\ c\neq 0,\ \mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\ \beta_1)f(z)*\phi(z)\neq 0$. From (11), we have

$$\beta\left|\frac{2}{c}\frac{z(\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z))'}{\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)}-1\right| -\mathscr{R}e\left\{1-\frac{2}{c}+\frac{2}{c}\frac{z(\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z))'}{\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)}\right\}\leq-\delta.$$

Thus

$$\beta\left|\frac{2}{c}\frac{z(\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z))'}{\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)}-1\right| -\mathscr{R}e\left\{\frac{2}{c}\frac{z(\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z))'}{\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)}-1\right\}\leq 2-\frac{2}{c}-\delta.$$

Notice that

$$\beta\left|\frac{2}{c}\frac{z(\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z))'}{\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)}-1\right| -\mathscr{R}e\left\{\frac{2}{c}\frac{z(\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z))'}{\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)}-1\right\}$$
$$\leq(\beta+1)\left|\frac{2}{c}\frac{z(\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z))'}{\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)}-1\right|$$
$$=(\beta+1)\left|\frac{2}{c}\frac{z+\sum\limits_{k=2}^{\infty}k\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m\Gamma_{k-1}[\alpha_1,\beta_1]\mu_k a_k z^k}{z+\sum\limits_{k=2}^{\infty}\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m\Gamma_{k-1}[\alpha_1,\beta_1]\mu_k a_k z^k}-1\right|$$
$$\leq(\beta+1)\frac{2}{c}\frac{\sum\limits_{k=2}^{\infty}(k-1)\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m\Gamma_{k-1}[\alpha_1,\beta_1]\right|\mu_k|a_k|}{1-\sum\limits_{k=2}^{\infty}\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m\Gamma_{k-1}[\alpha_1,\beta_1]\right|\mu_k|a_k|}.$$

The above inequality is bounded above by $(2-\frac{2}{c}-\delta)$ if

$$\sum_{k=2}^{\infty}\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m\Gamma_{k-1}[\alpha_1,\beta_1]\right| \left[\frac{2}{c}(1+\beta)(k-1)+2-\frac{2}{c}-\delta\right]\mu_k|a_k| \leq 2-\frac{2}{c}-\delta.$$

Hence the proof is complete. ■

Next, we give the coefficient inequalities for functions $f(z)$ which belong to the class $KD^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$.

Theorem 2.. *Let* $l\geq 0,\ \lambda\geq 0,\ q\leq s+1,\ q,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ \beta\geq 0,\ 0\leq\delta<1,\ z\in\mathbb{U},\ c\notin[0,2),\ \mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)\neq 0$. *If* $f(z)\in KD^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$, *then*

$$|a_k|\leq \frac{c(1-\delta)}{(k-1)(1-\beta)\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m\Gamma_{k-1}[\alpha_1,\beta_1]\right|\mu_k}\prod_{j=1}^{k-2}\left(1+\frac{c(1-\delta)}{j(1-\beta)}\right) \tag{14}$$

$(k\in\mathbb{N}\setminus\{1\})$. *The result is sharp.*

Proof. We note that

$$\mathscr{R}e\left\{1-\frac{2}{c}+\frac{2}{c}\frac{z(\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z))'}{\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)}\right\}>\frac{\delta-\beta}{1-\beta}$$

which follows from the relation (11) and the fact that $\mathscr{R}e(z)\leq|z|$ for any complex number z. We define the function $p(z)$ by

$$p(z):=\frac{(1-\beta)\left\{1-\frac{2}{c}+\frac{2}{c}\frac{z(\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z))'}{\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)}\right\}-(\delta-\beta)}{1-\beta}, \quad =1+p_1z+p_2z^2+\cdots.$$

where $z\in\mathscr{U}$. Then $p(z)$ is analytic in $\mathscr{U}$ with $p(0)=1$ and $\mathscr{R}e(p(z))>0,(z\in\mathscr{U})$. We thus have

$$1-\frac{2}{c}+\frac{2}{c}\frac{z(\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z))'}{\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)}=1+\left(\frac{1-\delta}{1-\beta}\right)\sum_{k=1}^{\infty}p_kz^k \tag{15}$$

which implies

$$2[z(\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z))'-\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z)] =c\left(\frac{1-\delta}{1-\beta}\right)(\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z))\sum_{k=1}^{\infty}p_kz^k.$$

Using (5) and comparing the coefficients of z^k on both sides of above equation, we obtain

$$2(k-1)\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m\Gamma_{k-1}[\alpha_1,\beta_1]\mu_ka_k=c\left(\frac{1-\delta}{1-\beta}\right)\times \left(p_{k-1}+p_{k-2}\left[\frac{1+l+\lambda}{1+l}\right]^m\Gamma_1[\alpha_1,\beta_1]\mu_2a_2+\cdots+p_1\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m\Gamma_{k-2}[\alpha_1,\beta_1]\mu_{k-1}a_{k-1}\right)$$

Applying the coefficient estimates $|p_k|\leq 2$ using the Carath*é*odory's Lemma[see ref.[5]], we obtain

$$|a_k|\leq\frac{c(1-\delta)}{(1-\beta)(k-1)\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m\Gamma_{k-1}[\alpha_1,\beta_1]|\mu_k\right|}\times \left(1+\left|\left[\frac{1+l+\lambda}{1+l}\right]^m\Gamma_1[\alpha_1,\beta_1]\right|\mu_2|a_2|+\cdots+\left|\left[\frac{1+l+\lambda(k-2)}{1+l}\right]^m\Gamma_{k-2}[\alpha_1,\beta_1]\mu_{k-1}|a_{k-1}|\right|\right).$$

We prove the result (14) by using the principle of mathematical induction.

For $k=2$,

$$|a_2| \le \frac{c(1-\delta)}{(1-\beta)\left|\left[\frac{1+l+\lambda}{1+l}\right]^m \Gamma_1[\alpha_1,\beta_1]\right|\mu_2}$$

.

For $k = 3$
$$|a_3| \le \frac{c(1-\delta)}{2(1-\beta)\left|\left[\frac{1+l+2\lambda}{1+l}\right]^m \Gamma_2[\alpha_1,\beta_1]\right|\mu_2}\left(1+\frac{c(1-\delta)}{(1-\beta)}\right).$$

Assume that (14) holds for $k \le n$, then

$$\begin{aligned}
|a_{n+1}| \le & \frac{c(1-\delta)}{n(1-\beta)\left|\left[\frac{1+l+n\lambda}{1+l}\right]^m \Gamma_n[\alpha_1,\beta_1]\right|\mu_{n+1}}\times \\
& [1+\left|\left[\frac{1+l+\lambda}{1+l}\right]^m \Gamma_1[\alpha_1,\beta_1]\right|\mu_2|a_2|+ \\
& \left|\left[\frac{1+l+2\lambda}{1+l}\right]^m \Gamma_2[\alpha_1,\beta_1]\right|\mu_3|a_3|+\cdots+ \\
& \left|\left[\frac{1+l+\lambda(n-1)}{1+l}\right]^m \Gamma_{n-1}[\alpha_1,\beta_1]\right|\mu_n|a_n|] \\
\le & \frac{c(1-\delta)}{n(1-\beta)\left|\left[\frac{1+l+n\lambda}{1+l}\right]^m \Gamma_n[\alpha_1,\beta_1]\right|\mu_{n+1}}\times \\
& \left[\prod_{j=1}^{n-2}\left(1+\frac{c(1-\delta)}{j(1-\beta)}\right)+\frac{c(1-\beta)}{(n-1)(1-\beta)}\prod_{j=1}^{n-2}\left(1+\frac{c(1-\beta)}{j(1-k)}\right)\right] \\
= & \frac{c(1-\delta)}{n(1-\beta)\left|\left[\frac{1+l+n\lambda}{1+l}\right]^m \Gamma_n[\alpha_1,\beta_1]\right|\mu_{n+1}}\prod_{j=1}^{n-1}\left(1+\frac{c(1-\delta)}{j(1-\gamma)}\right).
\end{aligned} \tag{16}$$

Thus the result (14) is true for $k=n+1$. Finally the result is sharp for the function $f(z)$ given by

$$f(z) = z+\frac{c(1-\delta)}{(k-1)(1-\beta)\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]|\mu_k\right|}\prod_{j=1}^{k-2}\left(1+\frac{c(1-\delta)}{j(1-\beta)}\right). \tag{17}$$

∎

3 Distortion inequalities for the class $KD^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$

Theorem 3.*Let $f(z) \in KD^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$ and $l \ge 0$, $\lambda \ge 0$, $q \le s+1$, $q,s \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, $\beta \ge 0$, $0 \le \delta < 1$, $z \in \mathbb{U}$, $c \notin [0,2)$, $\mathscr{I}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1)f(z)*\phi(z) \neq 0$. Then*

$$||f(z)|-r| \le \frac{c(1-\delta)}{(1-\beta)}r^2\sum_{k=2}^{\infty}\frac{1}{(k-1)\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]\right|\mu_k}\prod_{j=1}^{k-2}\left(1+\frac{c(1-\delta)}{j(1-\beta)}\right)\quad (|z|=r<1) \tag{18}$$

and

$$||f'(z)|-1| \le \frac{c(1-\delta)}{(1-\beta)}r\sum_{k=2}^{\infty}\frac{k}{(k-1)\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]|\mu_k\right|}\prod_{j=1}^{k-2}\left(1+\frac{c(1-\delta)}{j(1-\beta)}\right)\quad (|z|=r<1). \tag{19}$$

Proof. Let $f(z) \in \mathscr{A}$ be of the form (1). Then by using theorem 2, we have

$$\begin{aligned}
|f(z)| &\le |z|+\sum_{k=2}^{\infty}|a_k||z|^k \\
&\le r+r^2\sum_{k=2}^{\infty}\frac{c(1-\delta)}{(k-1)(1-\beta)\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]|\mu_k\right|}\prod_{j=1}^{k-2}\left(1+\frac{c(1-\delta)}{j(1-\beta)}\right)\quad (|z|=r<1)
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
|f(z)| &\ge |z|-\sum_{k=2}^{\infty}|a_k||z|^k \\
&\ge r-r^2\sum_{k=2}^{\infty}\frac{c(1-\delta)}{(k-1)(1-\beta)\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]|\mu_k\right|}\prod_{j=1}^{k-2}\left(1+\frac{c(1-\delta)}{j(1-\beta)}\right)\quad (|z|=r<1).
\end{aligned} \tag{21}$$

Combining (20) and (21) we obtain the result (18). Proof of result (19) follows similarly. ∎

4 Radii of starlikeness, convexity and close-to-convexity for the class $KD^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$

Theorem 4.*Let the function $f(z) \in KD^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$. Then f is starlike of* *order* ξ, $0 \le \delta < 1$, *in the disc* $|z| \le r = r_1(q,s,\lambda,m,l,\alpha_1,\beta_1,\xi,\delta,\beta,c)$, *where*

$$|z| < r_1 = \inf_{k\ge2}\left\{\frac{(1-\delta)[(k-1)(1-\beta)\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]\right|\mu_k]}{(k-2-\xi)[c(1-\delta)]\prod_{j=1}^{k-2}\left(1+\frac{c(1-\delta)}{j(1-\beta)}\right)}\right\}^{\frac{1}{k-1}}. \tag{22}$$

Proof. For $0 \le \xi < 1$, it is sufficient to show that

$$\left|z\frac{f'(z)}{f(z)} - 1\right| \le 1-\xi. \tag{23}$$

For $|z| \le r_1$, we have

$$\left|z\frac{f'(z)}{f(z)} - 1\right| \le \left|-\frac{-\sum_{k=2}^{\infty}(k-1)a_n z^{k-1}}{1-\sum_{k=2}^{\infty}a_n z^{k-1}}\right| \le 1-\xi. \tag{24}$$

Hence, (24) holds true if

$$\sum_{k=2}^{\infty}\frac{(k-2-\xi)}{(1-\xi)}|a_k|z^{k-1} \le 1. \tag{25}$$

With the aid of (14) and (25) is true if

$$|z|^{k-1} \le \left\{\frac{(1-\xi)[(k-1)(1-\beta)\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]\right|\mu_k]}{(k-2-\xi)[c(1-\delta)]\prod_{j=1}^{k-2}\left(1+\frac{c(1-\delta)}{j(1-\beta)}\right)}\right\}.$$

Therefore,

$$|z| < \inf_{k\ge2}\left\{\frac{(1-\xi)[(k-1)(1-\beta)\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]\right|\mu_k]}{(k-2-\xi)[c(1-\delta)]\prod_{j=1}^{k-2}\left(1+\frac{c(1-\delta)}{j(1-\beta)}\right)}\right\}^{\frac{1}{k-1}}. \tag{26}$$

Setting $|z| = r_1(q,s,\lambda,m,l,\alpha_1,\beta_1,\xi,\delta,\beta,c)$ in (26), we get the radius of starlikeness,which completes the proof of the theorem 4. ■

Theorem 5. *Let the function* $f(z) \in KD^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$. *Then* f *is convex of order* ξ, $0 \le \xi < 1$, *in the disc* $|z| \le r = r_2(q,s,\lambda,m,l,\alpha_1,\beta_1,\xi,\delta,\beta,c)$, *where*

$$|z| < r_2 = \inf_{k\ge2}\left\{\frac{(1-\xi)(k-1)(1-\beta)\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]\right|\mu_k}{[k(k+\xi-2)]c(1-\delta)\prod_{j=1}^{k-2}\left(1+\frac{c(1-\delta)}{j(1-\beta)}\right)}\right\}^{\frac{1}{k-1}}. \tag{27}$$

Proof. For $0 \le \xi < 1$, it is sufficient to show that

$$\left|z\frac{f''(z)}{f'(z)}\right| \le 1-\xi. \tag{28}$$

For $|z| \le r_2$, we have

$$\left|z\frac{f''(z)}{f'(z)}\right| \le \left|-\frac{\sum_{k=2}^{\infty}k(k-1)a_k z^{k-1}}{1-\sum_{k=2}^{\infty}ka_k z^{k-1}}\right| \le 1-\xi. \tag{29}$$

Hence, (29) holds true if

$$\sum_{k=2}^{\infty}\frac{[k(k+\xi-2)}{(1-\xi)}|a_k|z^{k-1} \le 1. \tag{30}$$

With the aid of (13) and (30) is true if

$$|z|^{k-1} \le \left\{\frac{(1-\xi)(k-1)(1-\beta)\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]\right|\mu_k}{[k(k+\xi-2]c(1-\delta)\prod_{j=1}^{k-2}\left(1+\frac{c(1-\delta)}{j(1-\beta)}\right)}\right\}.$$

Therefore,

$$|z| < r_2 = \inf_{k\ge2}\left\{\frac{(1-\xi)(k-1)(1-\beta)\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]\right|\mu_k}{[k(k+\xi-2)]c(1-\delta)\prod_{j=1}^{k-2}\left(1+\frac{c(1-\delta)}{j(1-\beta)}\right)}\right\}^{\frac{1}{k-1}}. \tag{31}$$

Setting $|z| = r_2(q,s,\lambda,m,l,\alpha_1,\beta_1,\xi,\delta,\beta,c)$ in (31), we get the radius of the convexity, which completes the proof of the theorem 5. ■

Theorem 6. *Let the function* $f(z) \in KD^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$. *Then* f *is close-to-convex of order* δ, $0 \le \xi < 1$, *in the disc* $|z| \le r = r_3(q,s,\lambda,m,l,\alpha_1,\beta_1,\xi,\delta,\beta,c)$, *where*

$$|z| < r_3 = \inf_{k\ge2}\left\{\frac{(1-\xi)(k-1)(1-\beta)\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]\right|\mu_k}{kc(1-\delta)\prod_{j=1}^{k-2}\left(1+\frac{c(1-\delta)}{j(1-\beta)}\right)}\right\}^{\frac{1}{k-1}}. \tag{32}$$

Proof. For $0 \le \xi < 1$, it is sufficient to show that

$$|\{f'(z)\} - 1| \le 1-\xi. \tag{33}$$

For $|z| \le r_3$, we have

$$|f'(z) - 1| \le \left|\sum_{k=2}^{\infty}ka_k z^{k-1}\right| \le 1-\xi. \tag{34}$$

Hence, (34) holds true if

$$\sum_{n=2}^{\infty}\frac{k}{(1-\xi)}|a_k|z^{k-1} \le 1. \tag{35}$$

With the aid of (13) and (35) is true if

$$|z|^{k-1} \leq \left\{ \frac{(1-\xi)(k-1)(1-\beta)\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]\right|\mu_k}{kc(1-\delta)\prod_{j=1}^{k-2}\left(1+\frac{c(1-\delta)}{j(1-\beta)}\right)} \right\}.$$

Therefore,

$$|z| < r_3 = \inf_{k\geq 2} \left\{ \frac{(1-\xi)(k-1)(1-\beta)\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]\right|\mu_k}{kc(1-\delta)\prod_{j=1}^{k-2}\left(1+\frac{c(1-\delta)}{j(1-\beta)}\right)} \right\}^{\frac{1}{k-1}}. \quad (36)$$

Setting $|z| = r_3(q,s,\lambda,m,l,\alpha_1,\beta_1,\xi,\delta,\beta,c)$ in (36), we get the radius of close-to-convex, which completes the proof of the theorem 6. ∎

5 Extreme points of the class $KD^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$

Let $\widetilde{KD}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$ be the subclass of $KD^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$ consisting of all functions $f(z) \in \mathscr{A}$ and satisfying condition (13). We determine the extreme points of the class $\widetilde{KD}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$.

Theorem 7.*Let* $f_1(z) := z$ *and*

$$f_k(z) := z + \frac{2-\frac{2}{c}-\delta}{\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]\right|\left[\frac{2}{c}(1+\beta)(k-1)+2-\frac{2}{c}-\beta\right]\mu_k\right|} z^k,$$

$k \in \mathbb{N}\setminus\{1\}$. *Then* $f(z) \in \widetilde{KD}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,b,c)$, *if and only, if* $f(z)$ *can be expressed in the form*

$$f(z) = \sum_{k=1}^{\infty}\lambda_k f_k(z) \quad \text{where} \quad \lambda_k > 0, \quad \sum_{k=1}^{\infty}\lambda_k = 1.$$

Proof. Suppose that

$$f(z) = \sum_{k=1}^{\infty}\lambda_k f_k(z) = z + \sum_{k=2}^{\infty} \frac{\lambda_k(2-\frac{2}{c}-\delta)}{\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]\mu_k\left[\frac{2}{c}(1+\beta)(k-1)+2-\frac{2}{c}-\delta\right]} z^k.$$

Then

$$\sum_{k=2}^{\infty}\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]\left[\frac{2}{c}(1+\beta)(k-1)+2-\frac{2}{c}-\delta\right]\times$$

$$\mu_k\lambda_k \frac{(2-\frac{2}{c}-\delta)}{\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]\right|\mu_k\left[\frac{2}{c}(1+\beta)(k-1)+2-\frac{2}{c}-\delta\right]}$$

$$= \left(2-\frac{2}{c}-\delta\right)\sum_{k=2}^{\infty}\lambda_k$$

$$= \left(2-\frac{2}{c}-\delta\right)(1-\lambda_1) \leq 2-\frac{2}{c}-\delta.$$

Thus by definition of the function class $\widetilde{KD}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$, we have

$$f(z) \in \widetilde{KD}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c).$$

Conversely, assume that

$$f(z) \in \widetilde{KD}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c).$$

Then in view of equation (13), we set

$$\lambda_k = \frac{\left|\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^m \Gamma_{k-1}[\alpha_1,\beta_1]\right|\left[\frac{2}{c}(1+\alpha)(k-1)+2-\frac{2}{c}-\delta\right]\mu_k\right|}{2-\frac{2}{c}-\beta}|a_k|$$

where $k \in \mathbb{N}\setminus\{1\}$ and $\lambda_1 = 1-\sum_{k=2}^{\infty}\lambda_k$ which together imply that

$$f(z) = \sum_{k=1}^{\infty}\lambda_k f_k(z)$$

. ∎

6 Subordination theorem and integral mean inequalities

Here, we give a sharp subordination result associated with the class $\widetilde{KD}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$. Some applications of the main result that gives interesting results of analytic functions are also investigated.

Theorem 8.*Let* $f(z) \in \widetilde{KD}^{m,l}_{q,s,\lambda}(\alpha_1,\beta_1,\phi,\delta,\beta,c)$ *and* $\{\mu_k\}_{k=2}^{\infty}$ *be a non-decreasing sequence, then*

$$\frac{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)}{2\{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)+2-\frac{2}{c}-\delta\}}(f*g)(z) \prec g(z) \quad (37)$$

for every function g in $\mathscr{C}$, *the class of convex functions and*

$$\mathscr{R}e\ f(z) > -\frac{\{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)+2-\frac{2}{c}-\delta\}}{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)}. \quad (38)$$

The constant factor $\frac{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)}{2\{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)+2-\frac{2}{c}-\delta\}}$ *cannot be replaced by larger one.*

Proof. Let $f(z) \in \widetilde{KD}^{m,l}_{q,s,\lambda}(\alpha_1, \beta_1, \phi, \delta, \beta, c)$ and

$$g(z) = z + \sum_{k=2}^{\infty} d_k z^k$$

be any function in the class $\mathscr{C}$. Then we have

$$\frac{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)}{2\{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)+2-\frac{2}{c}-\delta\}}(f*g)(z)$$
$$= \frac{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)}{2\{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)+2-\frac{2}{c}-\delta\}} \times \left(z + \sum_{k=2}^{\infty} a_k d_k z^k\right).$$

Thus by definition (1), the subordination result (37) will hold true if the sequence

$$\left\{\frac{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)a_k}{2\{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)+2-\frac{2}{c}-\delta\}}\right\}_{k=1}^{\infty} \tag{39}$$

is a subordinating factor sequence with $a_1 = 1$. In view of Lemma (1), this is equivalent to the following inequality: For $z \in \mathscr{U}$

$$\mathscr{R}e\left\{1+\sum_{k=1}^{\infty}\frac{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)}{\{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)+2-\frac{2}{c}-\}}a_k z^k\right\} > 0. \tag{40}$$

In view of (13) and $|z| = r \quad (0 < r < 1)$, we obtain

$$\mathscr{R}e\left\{1+\sum_{k=1}^{\infty}\frac{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)}{\{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)+2-\frac{2}{c}-\delta\}}a_k z^k\right\}$$
$$= \mathscr{R}e\left\{1+\frac{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)}{\{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)+2-\frac{2}{c}-\delta\}}z + \frac{1}{\{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)+2-\frac{2}{c}-\}}\times \sum_{k=2}^{\infty}\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)a_k z^k\right\}$$
$$> 1-\frac{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)}{\{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)+2-\frac{2}{c}-\delta\}}r - \frac{1}{\{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)+2-\frac{2}{c}-\delta\}}\times \sum_{k=2}^{\infty}\left(2-\frac{2}{c}-\delta\right)|a_k|r$$
$$> 0, \quad |z| = r.$$

This evidently establishes the inequality (40), and consequently the subordination relation (37) is proved. The assertion (38) is proved using (37) by choosing $g(z)$ as

$$g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k. \tag{41}$$

The sharpness of the multiplying factor in (37) can be proved by considering the function

$$f_1(z) = z - \frac{2-\frac{2}{c}-\delta}{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)}z^2 \tag{42}$$

which belongs to $\widetilde{KD}^{m,l}_{q,s,\lambda}(\alpha_1, \beta_1, \phi, \delta, \beta, c)$. Using (37), we conclude that

$$\frac{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)}{2\{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)+2-\frac{2}{c}-\delta\}}\ f_1(z) \prec \frac{z}{1-z}.$$

From $f_1(z)$ defined by (42) and the fact that $\frac{z}{1-z}$ maps the unit disc onto the domain $\mathscr{R}e\ w > -\frac{1}{2}$, we infer that

$$\inf_{z\in\Delta}\left\{\mathscr{R}e\left(\frac{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)}{2\{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)+2-\frac{2}{c}-\beta\}}f_1(z)\right)\right\} = -\frac{1}{2}.$$

Thus the proof is complete. ■

*Remark.*We observe that we can apply Theorem (7) to obtain some known results by other authors. Taking $c = 2,\ \alpha_1 = \beta_1,\ s = q$ we obtain the results due to R. K. Raina [14]. Similarly all the results on bounds for $|a_k|, k = 2, 3, \cdots$ and the Corollaries in [10] also follow by accordingly specializing the parameters as stated in Remark (6).

Next we state the following Littlewood's subordination theorem which we will use in our investigation to obtain the integral mean inequality.

Lemma 2.*If $f(z)$ and $g(z)$ are analytic in Δ with $f(z) \prec g(z)$, then*

$$\int_0^{2\pi}|f(re^{i\theta})|^p d\theta \le \int_0^{2\pi}|g(re^{i\theta})|^p d\theta$$

where $0 < p < \infty, z = re^{i\theta}$ and $0 \le r < 1$. Strict inequality holds for $0 < r < 1$ unless f is constant or $w(z) = \alpha z, |\alpha| = 1$.

Applying Lemma (2) for functions $f(z)$ in the class $\widetilde{KD}^{m,l}_{q,s,\lambda}(\alpha_1, \beta_1, \phi, \delta, \beta, c)$ we arrive at the following result.

Theorem 9.*Let $p > 0$. If $f(z) \in KD'(\phi, \alpha, s, b, c)$ is given by (1) and $\{\mu_k\}_{k=2}^{\infty}$ is a non-decreasing sequence, then for $z = re^{i\theta} \quad (0 < r < 1)$*

$$\int_0^{2\pi}|f(re^{i\theta})|^p d\theta \le \int_0^{2\pi}|f_1(re^{i\theta})|^p d\theta$$

where

$$f_1(z) = z - \frac{2-\frac{2}{c}-\delta}{\psi(\beta,\delta,l,m,\lambda,\alpha_1,\beta_1,c,\mu_2)}z^2.$$

We conclude this section with the following remark.

*Remark.*Utilizing Theorem 9 we can deduce integral mean inequalities for the class $SD(\alpha,\beta)$ and $KD(\alpha,\beta)$ by specializing the parameters as stated in Remark (6).

7 Perspective

By using Hadamard product we defined family of linear operators and using this operator we defined the subclass of $\mathscr{A}$ as certain subclass of analytic functions involving multiplier transformations in the open unit disc. We have derived few subordination results for the functions in these subclasses and discussed the applications of subordination results with the help of complex functions. We have obtained coefficient estimates, radii of starlikeness, convexity and close-to-convexity, extreme points, and integral means, growth and distortion theorems for these classes.

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