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Fuzzy Differential Subordination Involving Generalized Noor-Salagean Operator

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Abstract: The well-known technique of subordination has recently been extended from the geometric function theory to the fuzzy set theory by several authors. In this paper, we use the notion of fuzzy differential subordination to introduce certain fuzzy classes using the generalized Noor-Salagean operator. Certain interesting results are established for these classes.

Keywords: Analytic functions, convex univalent functions, fuzzy best dominant, fuzzy differential subordination, fuzzy sets, operators.

1 Introduction and preliminaries

One of the most recent techniques of research in the one complex variable is the method of differential subordination, which has been introduced by Miller and Mocanu, see, for example, [1,2].
A new research direction in this area has been launched by combining the concept of differential subordination with the complex functions domain to the fuzzy set theory, which is named as fuzzy differential subordination. For some details, see [3,4,5].

In this paper, we shall use fuzzy subordination to obtain some interesting results in the context of geometric function theory for certain classes of analytic functions defined by generalized Noor-Salagean operator in the open unit disc.
The unit disc of the complex plain is denoted by $E = \{z :| z |< 1\}$ and the class of analytic functions in E by $\mathscr{A}[a,n]$ consisting of functions written in the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots. \tag{1.1}$$

Note that $\mathscr{A}[0,1] = \mathscr{A}$ is the normalized class of analytic functions in the unit disc E.
We recall here some definitions of basic concepts.

Definition 1.[6] *Let X be a nonempty set. A pair (A, F_A), where*

$$F_A : X \to [0,1] \quad and \quad A = \{x \in X : 0 < F_A(x) \le 1\}$$

is called the fuzzy subset. The function F_A is called membership function of the fuzzy subset (A, F_A).

For the applications of fuzzy sets, see [7,8] and the references therein.

Definition 2.[10] *Let $\psi : \mathbb{C}^3 \times \to \mathbb{C}$ and h be univalent in E with*
$\psi(a,0,0) = h(0) = a$. If p is analytic in E with $p(0) = a$ and satisfies the
(second order) fuzzy differential subordination

$$F_{\psi(\mathbb{C}^3\times E)}, \left(\psi(p(z), zp'(z), z^2 p''(z); z\right) \in F_{h(E)}, \quad z \in E \tag{1.2}$$

then p is called a fuzzy solution of the fuzzy subordination, or more simple a fuzzy dominant, if

$$F_{p(E)}(p(z)) \le F_{q(E)}(q(z)), \quad z \in E$$

for all p satisfying (1.2). A fuzzy dominant $\tilde{q}$ that satisfies

$$F_{\tilde{q}(E)}(\tilde{q}(z)) \le F_{q(E)}(q(z)), \quad z \in E$$

for all fuzzy dominant q of (1.2), is called the fuzzy dominant of (1.2).

Definition 3.[10] *Let (M, F_M) and (N, F_N) be two fuzzy subsets of X. We say that the fuzzy sets M and N are equal, if and only if,*

$$F_M(x) = F_N(x), \quad x \in X$$

and we denote it by $(M, F_M) = (N, F_N)$.
Also

$$(M, F_M) \subseteq (N, F_N),$$

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if and only if

$$F_M(x) \leq F_N(x), \quad x \in X.$$

Let $D \subseteq \mathbb{C}$ and let f, g be analytic functions. We denote by

$$F(D) = supp(F(D), F_{F(D)}) = \{f(z) : 0 < F_{F(D)}(f(z)) \leq 1, \quad z \in D\}$$

and

$$g(D) = supp(F(D), F_{g(D)}) = \{g(z) : 0 < F_{g(D)}(g(z)) \leq 1, \quad z \in D\}.$$

Definition 4.*[10] Let $D \subset \mathbb{C}$ and $z_0 \in D$ be a fixed point. Let the functions f, g be analytic in D. The function f is said to be fuzzy subordinate to g, written as*

$$f \prec_{\mathscr{F}} g \quad or \quad f(z) \prec_{\mathscr{F}} g(z),$$

if the following conditions are satisfied.
(i). $f(z_0) = g(z_0)$,
(ii). $F_{f(D)}(f(z)) \leq F_{g(D)}(g(z)), \quad z \in D.$

*Remark.*We say that $f \in \mathscr{A}$ is subordinate to $g \in \mathscr{A}$, if there exists a Schwarz function w analytic in E with $w(0) = 0$ and $|w(z)| < 1$, for $z \in E$, such that $f(z) = g(w(z)), \quad z \in E$ and we write $f \prec g$ or $f(z) \prec g(z), z \in E$.

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$. Then the Hadamard product (or convolution) of these power series is defined as the power series

$$(f \star g)(z) = f(z) \star g(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

For $m \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\lambda \geq 0$ and f given by $f(z) = \sum_{k=0}^{\infty} a_k z^k$, the generalized Salagean operator $S_\lambda^m : \mathscr{A} \to \mathscr{A}$ is defined as follows:

$$S_\lambda^m[f](z) = z + \sum_{k=2}^{\infty} \left(1 + \lambda(k-1)\right)^m a_m z^k, \quad z \in E. \quad (1.3)$$

We note that

$$S_\lambda^0[f] = f$$

$$S_\lambda^1[f](z) = S_\lambda[f](z) = (1-\lambda) f(z) + \lambda z f'(z),$$

and

$$S_\lambda^m[f](z) = S_\lambda^m f(z) = S_\lambda(S_\lambda^{m-1} f(z)), \quad m = 1, 2, 3, \ldots$$

For $\lambda = 1$, the operator $S_1^m = S$ is called Salagean operator, introduced in [9] and for S_λ^m, we refer to [10].

Also, for negative integral values of m and $\lambda > 0$, we have

$$S_\lambda^{-1}[f](z) = \frac{1}{\lambda} z^{1-(\frac{1}{\lambda})} \int_0^z t^{(\frac{1}{\lambda})-2} f(t) dt, z \in E$$

and, in general

$$S_\lambda^{-m}[f](z) = \frac{1}{\lambda} z^{1-(\frac{1}{\lambda})} \int_0^z t^{(\frac{1}{\lambda})-2} S_\lambda^{-m+1} f(t) dt, \quad z \in E$$
$$= \{S_\lambda^{-1}(\frac{z}{1-z}) \star S_\lambda^{-1}(\frac{z}{1-z}) \star \ldots.. \star S_\lambda^{-1}(\frac{z}{1-z})\} \star f(z). \quad (1.4)$$

Let $f_n(z) = \frac{z}{(1-z)^{n+1)}}$ and $f_n^{-1}(z)$ be defined as

$$f_n(z) \star f_n^{-1}(z) = \frac{z}{(1-z)^n}. \quad n \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots.\}.$$

The Noor integral operator $N_n : \mathscr{A} \to \mathscr{A}$ is defined [11] as

$$N_n f(z) = f_n^{-1}(z) \star f(z). \quad (1.5)$$

It can easily be observed that

$$N_0 f(z) = z f'(z) \quad \text{and} \quad N_1 f(z) = f(z).$$

From (5), we can derive the following recursive relation for the operator N_n :

$$z(N_n f(z))' = (n+1) N_n f(z) - n N_{n+1} f(z). \quad (1.6)$$

Many authors [10,12,13,14,15,16] have used Noor operator to introduce and investigate the properties of certain known and new classes of analytic functions. Also, see [17,18,19,20] for related classes of analytic functions and their variant forms.
We now define a new operator $L_{\lambda,n}^m : \mathscr{A} \to \mathscr{A}$, called Noor-Salagean operator, as follows.

Definition 5. *Let*

$$f \in \mathscr{A}, \quad \lambda \geq 0, \quad m \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots..\}, \quad n \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots.\}.$$

Then the operator $L_{\lambda,n}^m : \mathscr{A} \to \mathscr{A}$, is defined as:

$$L_{\lambda,n}^m f(z) = \left(S_\lambda^m \times N_n\right) f(z), \quad z \in E. \quad (1.7)$$

$f \in \mathscr{A}$ is said to belong to the class $\mathscr{R}_\lambda^m(h_\beta)$, if it satisfies the subordination fuzzy

$$(L_{\lambda,n}^m f(z))'(z) \prec_{\mathscr{F}} h_\beta(z) = \frac{1+\beta z}{1-\beta^2 z}, \quad z \in E.$$

The function $h_\beta(z) = \frac{1+\beta z}{1-\beta^2 z}$ is a special case of Janoswki function $\frac{1+Az}{1+Bz}$,
with $A = \beta$, $B = -\beta^2$, $\beta \in (0,1]$ and is convex univalent in E with

$$Re(h_\beta(z)) = Re\left(\frac{1+\beta z}{1-\beta^2 z}\right) > \frac{1-\beta}{1+\eta^2}, \quad z \in E$$

and $h_1(z) = \frac{1+z}{1-z}$.

Some special cases

(i).

$$L^1_{\lambda,n}f(z) = (1-\lambda)N_nf(z) + \lambda z\big(N_nf(z)\big)'$$

and

$$f \in R^1_{\lambda,n}, \quad \text{if} \quad \big(L^1_{\lambda,n}f(z)\big) \prec_{\mathscr{F}} h_\beta(z)\big), \quad z \in E.$$

That is,

$$(1-\lambda)(N_nf(z))') + \lambda z\big(N_nf(z)\big)'' \prec_{\mathscr{F}} h_\beta(z), \quad z \in E.$$

(ii). $L^0_{\lambda,n}f(z) = N_nf(z)$ and $f \in R^0_{\lambda,n}(h_\beta(z))$ implies

$$\big(zN_nf(z)\big)' \prec_{\mathscr{F}} h_\beta(z), \quad z \in E.$$

(iii).

$$L^{-1}_{\lambda,n}f(z) = \frac{1}{\lambda}z^{1-(\frac{1}{\lambda})}\int_0^z t^{(\frac{1}{\lambda})-2}N_nf(t)dt, \lambda > 0, \quad z \in E.$$

and for $m = 2$, we have

$$L^{-2}_{\lambda,n}f(z) = \frac{1}{\lambda}z^{1-(\frac{1}{\lambda})}\int_0^z t^{(\frac{1}{\lambda})-2}L^{-1}_{\lambda,n}f(t)dt, \quad z \in E.$$

In general, by convolution properties, see [21],

$$\begin{aligned} &L^{-m}_{\lambda,n}f(z) \\ &= \frac{1}{\lambda}z^{1-(\frac{1}{\lambda})}\int_0^z t^{(\frac{1}{\lambda})-2}L^{-m+1}_{\lambda,n}f(t)dt, \quad (m \in \mathbb{N}, \quad z \in E) \\ &= \{L^{-1}_{\lambda,n}(\frac{z}{1-z}) \star L^{-1}_{\lambda,n}(\frac{z}{1-z}) \star \ldots.. \star L^{-1}_{\lambda,n}(\frac{z}{1-z})\} \star f(z). \end{aligned}$$

Thus

$$f \in R^{-m}_{\lambda,n}(h_\beta(z)), \qquad \text{if and only if,}$$

$$\big(L^{-m}_{\lambda,n}f(z)\big)' \prec_{\mathscr{F}} h_\beta(z), \quad \beta(0,1].$$

Definition 6. *Let $\delta \in (0,a]$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$. Then a function $f \in \mathscr{A}$ is said to belong to the class $\mathscr{F}R^m_{\lambda,n}(\delta)$, if it satisfies the following inequality*

$$F_{(L^m_{\lambda,n})'(E)}(L^m_{\lambda,n})'(E) > \delta, \quad z \in E.$$

2 Preliminary Results

We shall need the following set of lemmas to prove our main results.

Lemma 1.[1,2] *For $a,b,c \in \mathbb{C}$ $(c \neq 0,-1,-2,\ldots..)$, the Gauss hypergeometric function ${}_2F_1$ is defined as*

$${}_2F_1(a,b;c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots$$

This series is absolutely convergent for $z \in E$ and therefore it represents an analytic function in E.
The following identities are well known.

(i). $\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,b;c;z), \quad \big(Re(c) > Re(b) > 0\big).$

(ii). ${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z)$

(iii). ${}_2F_1(a,b;c;z) = (1-z)^{-a}\ {}_2F_1(a,c-b;c;\frac{z}{1-z})$

(iv).

$$\begin{aligned}(a+1) + az\ {}_2F_1(1,a+1;a+z;z) \\ = (a+1)\ {}_2F_1(1,a;a+1;z)\end{aligned}$$

Lemma 2.[2,22] *Let the function h be analytic and convex (univalent) in E with $h(0) = 1$. Suppose also that the function ϕ is analytic in E with $\phi(0) = 1$. If*

$$\phi(z) + \frac{z\phi'(z)}{\gamma} \prec h(z), \quad \big(Re(\gamma) \geq 0, \quad \gamma \neq 0, \quad z \in E\big),$$

then

$$\phi(z) \prec \psi(z) = \frac{\gamma}{z^\gamma}\int_0^z t^{\gamma-1}h(t)dt \prec h(z), \quad z \in E,$$

and ψ is the best dominant.
If the function ϕ also satisfies the inequality

$$Re\big(\phi(z)\big) > \rho, \quad \big(0 \leq \rho < 1\big), \quad z \in E\big),$$

then

$$Re\big[\phi(z)\big] \geq (2\rho-1) + \frac{z(1-\rho)}{1+|z|}.$$

For the following two Lemmas, we refer to [23].

Lemma 3. *Let h be a convex function with $h(0) = 1$ and $\gamma \in \mathbb{C}$, $Re(\gamma) \geq 0, \gamma \neq 0$. If p is analytic in E with*

$$p(0) = 1, \quad \psi : \mathbb{C}^2 \times E \to \mathbb{C},$$

$\psi(p(z), zp'(z), z = p(z) + \frac{zp'(z)}{\gamma}$ *is analytic function in E and*

$$F_{\psi(\mathbb{C}^2\times E)}\big(p(z) + \frac{1}{\gamma}zp'(z)\big) \leq F_{h(E)}(h(z),$$

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that is

$$p(z)+\frac{1}{\gamma}zp'(z)\prec_{\mathscr{F}} h(z), \quad z\in E,$$

then

$$F_{p(E)}(p(z))\leq F_{g(E)}(g(z))\leq F_{h(E)}(h(z)),$$

that is

$$p(z)\prec_{\mathscr{F}} g(z)\prec_{\mathscr{F}} h(z), \quad z\in E,$$

where

$$g(z)=\frac{\gamma}{z^{\gamma}}\int_0^z t^{\gamma-1}h(t)dt, \quad z\in E,$$

and g is convex and fuzzy dominant.

Lemma 4. *Let g be a convex function in E and let*

$$h(z)=g(z)+n\alpha g'(z), \quad z\in E, \quad \alpha>0,$$

and n is a positive integer.
If

$$p(z)=g(0)+p_nz^n+p_{n+1}z^{n+1}+\ldots, z\in E$$

is holomorphic in E and

$$F_{[p+\alpha zp'](E)}\big(p(z)+\alpha zp'(z)\big)\leq F_{h(E)}(h(z)),$$

that is

$$p(z)+\alpha zp'(z)\prec_{\mathscr{F}} h(z), \quad z\in E,$$

then

$$F_{p(E)}(p(z))\leq F_{g(E)}(g(z)), \quad z\in E.$$

That is

$$p(z)\prec_{\mathscr{F}} g(z), \quad z\in E.$$

3 Main Results

Theorem 1. *The class $\mathscr{F}R^m_{\lambda,n}(\delta)$ is a convex set*

Proof. Let $f_1,f_2\in\mathscr{F}R^m_{\lambda,n}(\delta)$ and let $f=\alpha_1f_1+\alpha_2f_2=1$, where α_1,α_2 are nonnegative with $\alpha1+\alpha_2=1$. To prove this result, it is sufficient to establish that f is also in $\mathscr{F}R^m_{\lambda,n}(\delta)$ for $z\in E$.
Now

$$f'(z)=\alpha_1f_1'(z)+\alpha_2f_2'(z)$$

and

$$\big(L^m_{\lambda,n}f(z)\big)'=\alpha_1\big(L^m_{\lambda,n}f_1(z)\big)'+\alpha_2\big(L^m_{\lambda,n}f_2(z)\big)'.$$

Since $f_1,f_2\in\mathscr{F}R^m_{\lambda,n}(\delta)$, we have

$$\delta<F\big(L^m_{\lambda,n}f_1\big)'(E)\leq 1 \quad \text{and} \quad \delta<F\big(L^m_{\lambda,n}f_2\big)'(E)\leq 1,$$

which implies

$$\delta=(\alpha_1+\alpha_2)\delta<F\big(L^m_{\lambda,n}\big)'<1. \tag{3.8}$$

That is

$$F_{(L^m_{\lambda,n})'(E)}\big(L^m_{\lambda,n}f(z)\big)'=F_{\alpha_1(L^m_{\lambda,n})'(E)}\big(L^m_{\lambda,n}f_1(z)\big)' +F_{\alpha_2(L^m_{\lambda,n})'(E)}\big(L^m_{\lambda,n}f_2(z)\big)' \tag{3.9}$$

From definition (6), (3.8) and (3.9), we obtain the required result.
If we take $f_1,f_2\in R^m_{\lambda,n}(h_\beta(z))$, then $\delta=\frac{1-\beta}{1+\beta^2}$ and in this case,
$f\in\mathscr{F}R^m_{\lambda,n}\big(\frac{1\beta}{1+\beta^2}\big), \quad \beta\in(0,1]$.
Also $f_1,f_2\in R^m_{\lambda,n}(h_1(z))$ implies

$$f\in\mathscr{F}R_{\lambda,n}(\delta_1), \quad \delta_1=0.$$

This completes the proof.

Theorem 2. *Let*

$$f\in R^1_{\lambda,n}h_\beta(z).$$

Then $f\in R^1_{\lambda,n+1}h_\beta(z), \quad n\in\mathbb{N}$.
That is,

$$R^1_{\lambda,n}h_\beta(z)\subseteq_{\mathscr{F}} R^1_{\lambda,n+1}h_\beta(z).$$

Proof. Let

$$f\in R^1_{\lambda,n}h_\beta(z).$$

Then $\big(L^1_{\lambda,n}f(z)\big)'\prec_{\mathscr{F}}(h_\beta(z))$.
That is,

$$\{(1-\lambda)\big(N_nf(z)\big)'\big)+\lambda z\big(N_nf(z)\big)''\big)\prec_{\mathscr{F}}(h_\beta(z)).$$

Let $p(z)=\big(L^1_{\lambda,}f(z)\big)'$. It is easy to show that p is analytic in E with $p(0)=1$. Simple computation together with the use of (1.6) leads us to

$$\begin{aligned}p(z)+\frac{zp'(z)}{n}&=(1-\lambda)\big(N_nf(z)\big)'+\lambda z\big(N_nf(z)\big)''\\&=\big(L^1_{\lambda,n}f(z)\big)'\prec_{\mathscr{F}}(h_\beta(z)).\end{aligned}$$

Using Lemma 3, it follows that $F_{p(E)}p(z)\leq F_{g(E)}(g(z))$, that is,

$$p(z)\prec_{\mathscr{F}} g(z)\prec_{\mathscr{F}}(h_\beta(z).$$

where

$$\begin{aligned}g(z)&=\frac{n}{z^n}\int_0^z t^{n-1}\frac{1+\beta t}{1-\beta^2t}dt\\&=-\frac{1}{\beta}\\&+(1+\frac{1}{\beta})(\frac{1}{1-\beta^2})\quad {}_2F_1\big(1,1,n+1,\frac{-\beta^2}{1-\beta^2}\big),\end{aligned} \tag{3.10}$$

by using identities in Lemma 2 with a change of variables. Thus $L^1_{\lambda,n+1}f(z)\prec_{\mathscr{F}} g(z), \quad g(z)$ is given by (3.10) is fuzzy best dominant.

Putting $\beta=1$, we have

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Corollary 1.

$$R^1_{\lambda,n}(h_1) \subseteq_{\mathscr{F}} R^1_{\lambda,n+1}(\alpha_1),$$

$\alpha_1 =_2 F_1(1,1;n+1;\frac{1}{2})-1.$ *This result is best possible.* $h_1(z)=\frac{1+z}{1-z}$. *This implies* $Re(h_1(z))>0$. $g(z))$ *given by* (3.10) *is the best dominant and*

$$\alpha_1 = \inf_{z\in E}\big(Re(g(z)) = g(-1).$$

Now the result follows from Theorem 2.

Corollary 2. *Let* g *be convex,* $g(0)=1$ *and let* $h(z)=g(z)+zg'(z)$.
Let

$$f\in R^1_{\lambda,0}\Big(\frac{1+(2\delta-1)z}{1-z}\Big),\quad \delta=\frac{1-\beta}{1+\beta^2}.$$

Then, form Theorem 2, we have

$$f\in R^1_{\lambda,0}\Big(\frac{1+(2\delta^\star-1)z}{1-z}\Big),\quad z\in E,$$

where

$$\begin{aligned}\delta^\star &= g(1)\\ &= (2\delta-1)+2(1-\delta)\log 2,\quad \delta=\frac{1-\beta}{1+\beta^2}\end{aligned}$$

and

$$\begin{aligned}g(z) &= \frac{1}{z}\int_0^z \frac{1+(2\delta-1)t}{1+t}dt\\ &= (2\delta-1)+\frac{2(1-\delta)}{z}\int_0^z\frac{1}{1+t}dt\\ &= (2\delta-1)+\frac{2(1-\delta)}{z}\log(1+z). \qquad (3.11)\end{aligned}$$

Proof. Since

$$\begin{aligned}\big(L^1_{\lambda,0}f(z)\big)' &= \big(S^1\lambda\star N_0f(z)\big)'\\ &= \big(S^1\lambda(zf(z))'\big)'\\ &= \big[(1-\lambda)f'(z)+\lambda(f'(z))'\big]\\ &= f'(z)+\lambda f''(z).\end{aligned}$$

So, we have

$$F_{(L^1_{\lambda,0}(f)'(E)}\big(L^1_{\lambda,0}f(z)\big)' \le F_{h_\delta(E)}(h_\delta(z)),$$

$$\text{where}\quad h_\delta(z)=\frac{1+(2\delta-1)z}{1+z},$$

implies that

$$F_{(L^1_{(\lambda,0}(f)'(E)}\big(L^1_{\lambda,0}f(z)\big)' \le F_{g(E)}(g(z)),\quad z\in E,$$

where g is given by (3.11). That is,

$$\frac{1}{z}\big(L^1_{\lambda,0}(f(z)\big) \prec_{\mathscr{F}} g(z),\quad z\in E.$$

Corollary 3. *Let* $L^1_{\lambda,1}f(z)=zp(z)$, *where*

$$\begin{aligned}L^1_{\lambda,1}f(z) &= \big(S^1_\lambda\star N_1\big)f(z)\\ &= \big(S^1_\lambda\star(N_1f(z)\big)\\ &= S_\lambda f(z)\\ &= (1-\lambda)f(z)+\lambda zf'(z).\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{z}\big(L^1_{\lambda,1}f(z)\big) = p(z) &= (1-\lambda)\frac{f(z)}{z}+\lambda f'(z)\\ &= 1+\sum_{n=2}^{\infty}\{(1-\lambda)+n\lambda\}a_nz^{n-1},\end{aligned}$$

so $p(z)$ *is analytic in* E *and* $p(0)=1$. *Now*

$$\begin{aligned}\big(L^1_{\lambda,1}f(z)\big)' &= \big(zp(z)\big)'\\ &= p(z)+zp'(z) \prec_{\mathscr{F}} \frac{1-(2\delta-1)z}{1-z}.\end{aligned}$$

Using Lemma 2 with $\gamma=1$, *we obtain from Theorem 2,* $p\prec_{\mathscr{F}} g$, *where* g *is given by (3.11).*

By using convolution properties given in [21], following result can easily proved.

Theorem 3. *Let*
$f_i\in R^m_{\lambda,n}\big(h_\beta(z)\big),\quad i=1,2,\quad \beta_i\in(0,1],\lambda>0.$
Let

$$L^m_{\lambda,n}=L^m_{\lambda,n}(f_1\star f2)(z),\quad z\in E.$$

Then $f\in\mathscr{A}$ *belongs to the class* $R^m_{\lambda,n}\big(\frac{1-2(\sigma-1)z}{1-z}\big)$,
where

$$\begin{aligned}\sigma &= (2\sigma_3-1)+(1-\sigma_3)(\lambda+2(1-\lambda)\ln 2\\ \sigma_3 &= 1-2(1-\sigma_1)(1-\sigma_2)\end{aligned}$$

and

$$\sigma_i=-\frac{1}{\beta_i}+(1-\frac{1}{\beta_i})(1+\beta_i)^{-1}\quad {}_2F_1\big(1,1;1+\frac{1}{\lambda};\frac{\beta^2 i}{1-\beta_i^2}\big)$$

Theorem 4. *Let* g *be a convex function in* E *and let*

$$h_\beta(z)=g(z)+\frac{1}{b+2}zg'(z),\quad z\in E,\quad b>-2.$$

If $f\in R^m_{\lambda,n}(h_\beta(z))$ *and*

$$G(z)=N_bf(z)=\frac{b+2}{z^{b+1}}\int_0^z t^bf(t)dt,\quad z\in E, \qquad (3.12)$$

then

$$F_{\big(L^m_{\lambda,n}f\big)'(E)}\big(L^m_{\lambda,n}f(z)\big)' \le F_{h_\beta(E)}h_\beta(z),$$

that is $\big(L^m_{\lambda,n}f(z)\big)' \prec_{\mathscr{F}} h_\beta(z)$ *implies that*

$$F_{\big(L^m_{\lambda,n}G\big)'(E)}\big(L^m_{\lambda,n}g(z)\big)' \le F_{g(E)}g(z),$$

that is

$$\big(L^m_{\lambda,n}G(z)\big)' \prec_{\mathscr{F}} g(z),\quad z\in E.$$

This result is sharp

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Proof. Let $p(z) = \left(L^m_{\lambda,n}G(z)\right)'$. Then p is analytic with $p(0) = 1$.
Differentiating (3.12), we have

$$(b+1)G(z) + zG'(z) = (b+2)f(z). \quad (3.13)$$

We apply operator $L^m_{\lambda,n}$ on both sides of (3.13) and differentiate to obtain

$$p(z) + \frac{1}{b+2}zp'(z) = \left(L^m_{\lambda,n}f(z)\right)'.$$

That is

$$\begin{aligned} &p(z) + \frac{1}{b+2}zp'(z) \\ &= \left(L^m_{\lambda,n}G(z)\right)' + \frac{1}{b+2}z\left(L^m_{\lambda,n}G(z)\right)'' \\ &= \left(L^m_{\lambda,n}f(z)\right)' \prec_{\mathscr{F}} h_\beta(z), \quad z \in E. \end{aligned} \quad (3.14)$$

This implies that

$$\begin{aligned} &F_{(L^m_{\lambda,n}f)'(E)}\{\left(L^m_{\lambda,n}G\right)' + \frac{1}{b+2}z\left(L^m_{\lambda,n}G\right)''\}(z) \\ &leq\ F_{h_\beta(E)}(h_\beta(z)) \\ &\leq\ F_{h_\beta(E)}\left(g + \frac{1}{b+2}zg'\right)(z). \end{aligned}$$

Now applying Lemma 4, we obtain

$$F_{p(E)}p(z) \leq F_{G(E)}g(z), \quad z \in E.$$

where g is the fuzzy best dominant.
This proves $G \in R^m_{\lambda,n}(g)$.

Corollary 4. *Let, in Theorem 4,*

$$b = -1, \quad h_\beta(z) = \frac{1-(2\delta-1)z}{1-z}, \quad \delta = \frac{1-\beta}{1+\beta^2}.$$

Then, for $m = 1, \quad n = 1$

$$\left(L^1_{\lambda,1}G(z)\right)' \prec_{\mathscr{F}} g(z), = \{(2\delta -) + \frac{2(1-\delta)}{z}\log(1+z)\}.$$

When $\beta = 1$, *then* $\delta = 0$ *and in this case*

$$g(z) = \frac{2}{z}\log(1+z) - 1.$$

Corollary 5. *Let* $b = 0$ *in (3.12). Then*

$$N_0\left(R^m_{\lambda,1}G(z)\right)'\left(\frac{1-(2\delta-1)z}{1-z}\right) \subseteq_{\mathscr{F}} R^m_{\lambda,n}\left(\frac{1-(2\delta^\star-1)z}{1-z}\right),$$
$$\delta\frac{1-\beta}{1+\beta^2}$$

and

$$\delta^\star = 2\delta - 1 + 4(1-\delta)\int_0^z \frac{t}{1+t}dt.$$

Proof follows directly from Theorem 4 and
That is

$$\begin{aligned} &F_{p(E)}p(z) \leq F_{g(E)}g(z) \leq F_{h_\delta(E)}(h_\delta(z)), \\ &\qquad h_\delta(z) = \frac{1-2\delta-1)z}{1-z}. \end{aligned}$$

That is

$$F_{(L^m_{\lambda,n})'(E)}\left(L^m_{\lambda,n}G(z)\right)' \leq F_{g_0(E)}g_0(z) \leq F_{h_\delta(E)}\left(h_\delta(z)\right),$$

where

$$\begin{aligned} g_0(z) &= \frac{2}{z}\int_0^z \{t - \frac{1+(2\delta-1)t}{1+t}\}dt \\ &= 2\delta - 1 + 4(1-\delta)\int_0^z \frac{t}{1+t}dt \end{aligned}$$

and

$$\delta^\star = g_0(1).$$

4 Conclusion

In this paper, we have introduced and investigated various properties of fuzzy subordination connected with Noor-Salagean operators in geometric function theory. It is an interesting problems to discuss the applications of the fuzzy set and system in the management sciences, modelling, optimization statistics and probability theory. See [26,27,28,29] and the therein for more details.

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References

[1] S. S. Miller and P. T. Mocanu, Differential subordinattion and univalent functions, *Michig. Math. J.,* **28**, 157-171(1981),

[2] S. S. Miller and P. T. Mocanu, *Differential Subordination: Theory and Applications,* Monograh Text Books, Vol. 225, Marcel Dekker, New York, 2000.

[3] A. Alb Lupas and G. Oros, On special fuzzy differential subordinations using Salagean and Ruscheweyh operators, *Appl. Math. Comput.,* **261**, 119-127(2015).

[4] G. I. Oros and Gh. Oros, Fuzzy differential subordination, *Acta Universtatics Aplulensis,* **30**, 55-64(2012).

[5] A. Alb Lupas, A note on special fuzzy differential subordinations using generalized Salagean and Rscheweyh derivative,*J. Comput. Anal. Appl.,* **15(8)** 1476-1483(2013).

[6] I. A. Zadeh, Fuzzy sets, *Inform. Control,* **8**, 335-353(1965).

[7] B. Padhy, P. W. Samanta, S. K. Paikray and U. K. Mishra, An EOQ model for items having fuzzy amelioration and deterioration, *Appl. Math. Inform. Sci.,* **16(2)**, 353-360(2022).

[8] Q. A. Tantawy, and F. M. Selim, Soft fuzzy syntopogenous spaces, *Appl. Math. Inform. Sci.,* **9(1)**, 95-102(2015).

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[9] G. S. Salagean, *Subclasses of univalent functions, Complex Analysis,* in Fifth Romanian-Finnish Seminar, Part I,(Bucharest, 1981), Lecture Notes in Math. Vol. 1o13, Springer, (19830.
[10] A. Alb Lupas, On special fuzzy differential sobordination using convolution product of Salagean operator and Ruscheweyh derivative, *J. Comput. Anal Appl.,* **15(8)**, 1484-1489(2013).
[11] K. I. Noor and M. A. Noor, On integral operators, *J. Math. Anal. Appl.,* **238(1)**, 341-352(1999).
[12] K. I. Noor and M. A. Noor, Some applications of fuzzy differential subordinations, *J. Advan. Math. Stud.,* **15(3)**, xx-xx(2022).
[13] G. I. Oros and G. Oros, The notion of subordination in fuzzy sets theory, *General Math.,* **19**, 97-103(2011).
[14] J. L. Liu and K. I. Noor, On subordinationsfor certain analytic functions associaated with Noor integral operator, *Appl. Math. Comput.,* **137**, 1453- 1460(2007).
[15] K. I. Noor, On new classes of integral operators, *J. Natur. Geometry,* **16**, 71-80(1999).
[16] K. I. Noor and M. A. Noor, On certain classes of analytic functions defined by Noor integral operaor, *J. Math. Anal. Appl.,***281(1)**, 244-250(2003).
[17] K. I. Noor, On generalized close-to-convexity, *Appl. Math. Inform. Sci.,* **11(5)**, 1383-1388(2017).
[18] K. I. Noor and M. A. Noor, Linear combinations of generalized q-starlike functions, *Appl. Math. Inform. Sci.,* **11(3)**, 745-748(2017).
[19] K. I. Noor, M. A. Noor and H. M. Mohamed,*Appl. Math. Inform. Sci.,* **15(4)**, 437-443(2021).
[20] Sheza M. El-Deeb and Bassant M. El-Matary, Coefficients bounds of multivalent function connected with q-analogue of Salagean operator, *Appl. Math. Inform. Sci.,* **14(6)**, 745-748(2020).
[21] P. Sharma and R. K. Maurya, Certain ssubordination results on the convoltion of analytic functions, *J. Math. Appl.,* **37**, 107-114(2014).
[22] D. T. Hallenbeck and St. Ruscheweyh, Subordination by convex functions, *Proc. Amer. Math. Soc.,* **52**, 191-195(1975).
[23] W. G. Atshaw and K. O. Hussain, Differential superordination,*Theory Appl. Math. Computer Sci.,* **7(1)**, 27-38(2017).
[24] F. Al-Oboudi, On univalent functions defined by a generalized Salagean operator, *Int. J. Math. Sci.,* **27**, 1429-1436(2004).
[25] G. I. Oros and G. Oros, Dominants and best dominants in fuzzy differential subordinations, *Stud. Univ. Babes-Bdlyai. Math.,* **57**, 239-248(2012).
[26] F. A. Farahat, M. A. El Sayed and M. A. Elsisy, Interactive approach for solving multi-level multiobjective quadratic fractional programming problems with fuzzy parameters in the constraints,*J. Stat. Appl. Probability,* **11(1)**, 265-276(2022).
[27] R. A. Saifi and I. M. Hatamleh, Electronic management requirements and their role in improving job performance, *J. Stat. Appl. Probability,* **11(1)**, 353-376(2022).
[28] M. Shakil, M. Ahsanullah and B. M. G. Kibria, Some characterizations and applications of a size-biased weighted distribution useful in lifetime modelling, *J. Stat. Appl. Probability,* **9(1)**, 127-137(2020).
[29] O. E. Emam, M. A. A. Fattah and S. M. Azzam, A fuzzy approach for solving a three-level linear programming problem with neutrosophic parameters in the objective functions and constraints, *J. Stat. Appl. Probability,* **10(3)**, 677-686(2021).

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