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Fixed Points of Non-Self Mappings in Partial Metric Spaces

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Abstract: A number of theorems on contractive mappings for common fixed points in partial metric spaces have been proved and many of them apply to self maps. In this paper, we extend a common fixed point theorem on a partial metric space by Karapinar *et al.* so that it can apply to a non-self mapping in a metrically convex partial metric space under specified conditions.

Keywords: Partial metric spaces, common fixed points, metrically convex sets, non-self mapping

1 Introduction and Preliminaries

A number of researchers have developed theorems which stipulate the conditions for fixed points to exist for partial metric spaces, and in some cases, designed algorithms to locate them.

The purpose of this study is to discuss the existence of common fixed points of a pair of non-self mappings in the context of partial metric spaces under certain conditions.

At this stage we introduce a few preliminaries that will be useful in developing the theorem proved in this study.

Definition 1.[4] *A partial metric space is a pair (X,p), where X is a non-empty set and the mapping $p : X \times X \to \mathbb{R}$ is a function such that for all $x,y,z \in X$*

(P0): $0 \leq p(x,x) \leq p(x,y)$,
(P1): $p(x,x) = p(x,y) = p(y,y) \Leftrightarrow x = y$,
(P2): $p(x,y) = p(y,x)$, *and*
(P3): $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The function p in the partial metric space is called the *partial metric*.

Note that, owing to (P0) and (P1), we have

$$p(x,y) = 0 \Leftrightarrow x = y. \tag{1}$$

An example of a partial metric space (X,p) is where $X = \mathbb{R}^+$, the set of non-negative real numbers, and the mapping $p : X \times X \to \mathbb{R}^+$ is defined as $p(x,y) = \max\{x,y\}$ for all $x,y \in \mathbb{R}^+$.

Each partial metric p on X generates a T_0 topology τ_p on X with a base being the family of open balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 2. [4] *Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then*

(i) $\{x_n\}$ *converges to* $x \in X$ *if and only if* $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.
(ii) $\{x_n\}$ *is called a Cauchy sequence if and only if there exists (and is finite)* $\lim_{n,m\to+\infty} p(x_n,x_m)$.

Definition 3. *A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that*

$$p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m).$$

Lemma 1. [4] *Let (X,p) be a partial metric space. Then (X,p^s) is a metric space defined for all $x,y \in X$, where*

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y).$$

In this study, we term p^s as the metric derived from the partial metric p.

We take note of the following lemma.

Lemma 2. [1]

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(i) A sequence $\{x_n\}$ is a Cauchy sequence in the partial metric space (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).

(ii) A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete.

Lemma 3. *Let (X,p) be a partial metric space and let p^s be the metric derived from the partial metric p as defined in Lemma 1. Then for all $a,b,c \in X$,*

$$p^s(a,b) \le p^s(a,c) \Rightarrow p(a,b) \le 2p(a,c).$$

Proof.

$$\begin{aligned} & p^s(a,b) \le p^s(a,c) \\ \Rightarrow 2p(a,b) - p(a,a) - p(b,b) & \\ & \le 2p(a,c) - p(a,a) - p(c,c) \\ \Rightarrow 2p(a,b) & \le 2p(a,c) - p(c,c) + p(b,b) \\ & \le 2p(a,c) + p(b,b) \\ & \le 2p(a,c) + p(a,b) \end{aligned}$$

because $p(b,b) \le p(a,b)$ from Definition 1, (P0).

$$\Rightarrow p(a,b) \le 2p(a,c).$$

The metrically convex metric space is defined as follows:

Definition 4. [2] *A metric space (X,d) is said to be metrically convex if for all x,y in X with $x \ne y$, there exists a point z in X, $(x \ne z \ne y)$ such that*

$$d(x,y) = d(x,z) + d(z,y).$$

Remark 1. When (X,d) is a metrically convex metric space, we define

$$seg[x,y] = \{z : d(x,y) = d(x,z) + d(z,y)\}.$$

We get the following lemma from Assad and Kirk [2].

Lemma 4. *Let C be a closed subset of the complete and convex metric space X. If $x \in C$ and $y \notin C$, then there exists a point $z \in \partial C$ (the boundary of C) such that*

$$d(x,z) + d(z,y) = d(x,y). \tag{2}$$

Using Remark 1, we can rephrase Lemma 4 as follows

Lemma 5. *Let C be a closed subset of the complete and convex metric space X. If $x \in C$ and $y \notin C$, then there exists a point $z \in \partial C$ (the boundary of C) such that $z \in \mathrm{seg}[x,y]$.*

We introduce the metrically convex partial metric space.

Definition 5. *A partial metric space (X,p) is said to be metrically convex if the corresponding metric space (X,p^s) is metrically convex in the sense of Lemma 3, where $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ for all $x,y \in X$.*

As an example, the partial metric space $(\mathbb{R}^+,p)$ where $p(x,y) = \max\{x,y\}$ for all $x,y \in \mathbb{R}^+$ is metrically convex. This is because the metric space (X,p^s) is metrically convex, where $p^s(x,y) = |x-y|$, which is the metric derived from the partial metric p.

Lemma 6. *Let (X,p) be a metrically convex partial metric space. Let $x,y \in X$. If $z \in \mathrm{seg}[x,y]$ then:*

(i) $p(x,y) - p(x,z) = p(z,y) - p(z,z)$,

(ii) $p(x,y) \ge p(x,z)$.

Proof. Applying (2) to Definition 5, if $z \in \mathrm{seg}[x,y]$, then we have:

$$\begin{aligned} & p^s(x,y) = p^s(x,z) + p^s(z,y) \Rightarrow \\ & 2p(x,y) - p(x,x) - p(y,y) = 2p(x,z) - p(x,x) - p(z,z) \\ & \qquad + 2p(z,y) - p(z,z) - p(y,y) \\ & \Rightarrow p(x,y) = p(x,z) + p(z,y) - p(z,z) \\ & \Rightarrow p(x,y) - p(x,z) = p(z,y) - p(z,z). \end{aligned}$$

As $p(z,y) - p(z,z) \ge 0$, from (P0) of Definition 1 we have

$$p(x,y) \ge p(x,z).$$

This proves Lemma 6.

Lemma 7. *Let C be a non-empty subset of a metrically convex partial metric space (X,p) which is closed in (X,p^s). If $x \in C$ and $y \in X \backslash C$, then there exists a point $z \in \partial C$ $\big($the boundary of C with respect to $(X,p^s)\big)$ such that*

$$p(x,y) + p(z,z) = p(x,z) + p(z,y).$$

Proof. From Definition 5, if the partial metric space (X,p) is metrically convex, then (X,p^s) is metrically convex. From Lemma 5, this means that if $x \in C$ and $y \in X \backslash C$ then there exists z in ∂C, (the boundary of C), such that $p^s(x,y) = p^s(x,z) + p^s(z,y)$. Using Lemma 1, this means

$$\begin{aligned} & p^s(x,y) = p^s(x,z) + p^s(z,y) \\ \Rightarrow & 2p(x,y) - p(x,x) - p(y,y) = 2p(x,z) - p(x,x) - p(z,z) \\ & \qquad + 2p(z,y) - p(z,z) - p(y,y) \\ & \qquad \Rightarrow p(x,y) = p(x,z) + p(z,y) - p(z,z) \\ & \qquad \Rightarrow p(x,y) + p(z,z) = p(x,z) + p(z,y). \end{aligned}$$

We now prove the following lemma, which is modified from Theorem 1 of Assad and Kirk [2], and is necessary for our work.

Lemma 8. *Consider a sequence $\{w_n\}_{n \in \mathbb{N}} \in \mathbb{R}^+$ such that, for all $n \ge 2$, we have*

$$w_n \le k \max\{w_{n-2}, w_{n-1}\}, k \in (0,1], \tag{3}$$

then

$$w_n \le k^{n/2} k^{-1/2} \max\{w_0, w_1\}. \tag{4}$$

Proof. We prove the lemma by the induction. First we show that Lemma 8 holds for $n=2$.

We note that $k \in (0,1]$ implies $k \le k^{1/2}$. Hence, if $n = 2$, then (3) leads to

$$\begin{aligned} w_2 &\le k\max\{w_0,w_1\} \\ &\le k^{1/2}\max\{w_0,w_1\} \\ &= k^{2/2}k^{-1/2}\max\{w_0,w_1\}. \end{aligned} \tag{5}$$

We then show that the lemma holds for $n=3$. If $n=3$, then (3) leads to
$w_3 \le k\max\{w_1,w_2\}$. If $w_1 \ge w_2$, then we get

$$\begin{aligned} & w_3 \le k\max\{w_1,w_2\} \\ \Rightarrow\ & w_3 \le kw_1 \\ & \le k\max\{w_0,w_1\} \\ & = k^{3/2}\cdot k^{-1/2}\max\{w_0,w_1\}. \end{aligned}$$

If however $w_1 < w_2$, we get

$$\begin{aligned} & w_3 \le k\max\{w_1,w_2\} \\ \Rightarrow\ & w_3 \le kw_2 \\ \Rightarrow\ & w_3 \le k\times k^{2/2}k^{-1/2}\max\{w_0,w_1\},\ \text{from (5)} \\ & \le k^{3/2}\max\{w_0,w_1\} \\ & \le k^{3/2}\cdot k^{-1/2}\max\{w_0,w_1\},\ \text{because } k^{-1/2} \ge 1. \end{aligned}$$

We now show that, if Lemma 8 holds for $1 \le n \le j$ where $j \ge 2$, then it must be hold for $j+1$. Hence we have from (3)

$$w_{j+1} \le k\max\{w_{j-1},w_j\}. \tag{6}$$

We consider two cases.
Case (i): Suppose $w_{j-1} \ge w_j$. Then (6) becomes

$$\begin{aligned} w_{j+1} &\le kw_{j-1} \\ &\le k\cdot k^{(j-1)/2}k^{-1/2}\max\{w_0,w_1\}\ \text{from (4)} \\ &= k^{(j+1)/2}k^{-1/2}\max\{w_0,w_1\}. \end{aligned} \tag{7}$$

Case (ii): Suppose $w_{j-1} < w_j$. Then (6) becomes

$$\begin{aligned} w_{j+2} &\le kw_j \\ &\le k\cdot k^{j/2}k^{-1/2}\max\{w_0,w_1\}\ \text{from (4)} \\ &= k^{(j+2)/2}k^{-1/2}\max\{w_0,w_1\}. \end{aligned} \tag{8}$$

We note that for $j \ge 2$ and $k \in (0,1]$ we have $k^{(j+1)/2} \ge k^{(j+2)/2}$. Hence (7) and (8) imply that

$$w_{j+1} \le k^{(j+1)/2}k^{-1/2}\max\{w_0,w_1\}.$$

A mapping $S: C \to X$ is called a *self mapping* if $C = X$. Otherwise it is called a *non-self mapping*.

Given a mapping $S: C \to X$, where $C \subseteq X$, $x \in C$ is said to be a *fixed point* of S in X if $Sx = x$.

Consider two mappings $S,T: C \to X$ where $C \subseteq X$. A point $x \in X$ is called a *coincidence point* of S and T in X if there is a $w \in X$ such that $w = Sx = Tx$. The point w in this case is called a *point of coincidence*. If in addition $w = x$, then x is called the *common fixed point* of S and T.

We now introduce the following theorem by Karapinar et al. [3] which states the conditions under which two self maps have a common fixed point in a partial metric space.

Theorem 1. *Let $S,T: X \to X$ be mappings on a complete partial metric space (X,p).*
(i) Let there be $r \in [0,1)$ such that

$$\max\{p(Sx,TSx),p(Tx,STx)\} \le r\min\{p(x,Sx),p(x,Tx)\}$$

for all $x \in X$ and let
(ii) $\alpha(y) = \inf\{p(x,y)+\min\{p(x,Sx),p(x,Tx)\}\} > 0$ for all $x \in X$ where y is not a common fixed point of S and T.
Then S,T have a unique common fixed point $z = S(z) = T(z)$, with
$p(z,z) = 0$.

The aim of this paper is to extend the above theorem to apply to non-self mappings.

2 Main Results

We start with stating the theorem which we intend to prove.

Theorem 2. *Let (X,p) be a complete metrically convex partial metric space and C be a non-empty closed subset of X, the closure being with respect to (X,p^s). Let ∂C, the boundary of C with respect to (X,p^s), be non-empty. Let $S,T: C \to X$ be continuous mappings. Let there be $r \in \mathbb{R}$ such that $0 \le 2r < 1$ and whenever $Sy,Ty \in C$, the following conditions hold:*
(a)
$\max\{p(Sx,TSy),p(Tx,STy)\} \le r\min\{p(x,Sy),p(x,Ty)\}$,
(b) $x \in \partial C \Rightarrow Sx$ and $Tx \in C$,
(c) SC and TC are closed in (X,p^s).
Then there is a unique point $z \in C$ such that $z = Tz = Sz$ and $p(z,z) = 0$.

Proof. We generate a sequence $\{x_n\}$ in the following way. We commence with an arbitrary $x_0 \in \partial C$. From assumption (b), it means we can choose $x_1 = Sx_0 \in C$. We determine Tx_1. If $Tx_1 \in C$, then we set $x_2 = Tx_1$. If however $Tx_1 \notin C$, we choose $x_2 \in \partial C$ such that $x_2 \in \text{seg}[Sx_0,Tx_1]$. We then find Sx_2.

We compute inductively the other elements of the sequence $\{x_n\}$. If $Sx_{2n} \in C$, then $x_{2n+1} = Sx_{2n}$. Otherwise, if $Sx_{2n} \notin C$ for $n \ge 1$, then we choose $x_{2n+1} \in \partial C$ such that $x_{2n+1} \in \text{seg}[Tx_{2n-1},Sx_{2n}]$.

From assumption (b), $x_{2n+1} \in \partial C$ implies $x_{2n+2} = Tx_{2n+1} \in C$. We then find Tx_{2n+1}.

Similarly, if $Tx_{2n+1} \in C$, then $x_{2n+2} = Tx_{2n+1}$. Otherwise, if $Tx_{2n+1} \notin C$, then we choose $x_{2n+2} \in \partial C$ such that $x_{n+2} \in \text{seg}[Sx_{2n},Tx_{2n+1}]$. We then find Sx_{2n+1}.

We now partition the set $\{x_n, n \in \mathbb{N}\}$ into $P = \{x_{2n+1} : x_{2n+1} = Sx_{2n}\} \cup \{x_{2n+2} : x_{2n+2} = Tx_{2n+1}\}$ and $Q = \{x_{2n+1} : x_{2n+1} \neq Sx_{2n}\} \cup \{x_{2n+2} : x_{2n+2} \neq Tx_{2n+1}\}$.

We note that if $x_{2n} \in P$, then $x_{2n} = Tx_{2n-1}$ for $n \geq 1$. Similarly if $x_{2n+1} \in P$, then $x_{2n+1} = Sx_{2n}$. We also note that if $x_{2n} \in Q$, then $x_{2n} \in \partial C$. Similarly if $x_{2n+1} \in Q$, then $x_{2n+1} \in \partial C$.

Remark 2. We show that if $x_{2n} \in Q$, then $x_{2n-1} \in P$.

Suppose $x_{2n-1} \in Q$. This means $x_{2n-1} \in \partial C$, which, from assumption (b), implies $x_{2n} = Tx_{2n-1} \in C$. Hence $x_{2n} \in P$ which is a contradiction. Using a similar argument, $x_{2n+1} \in Q \Rightarrow x_{2n} \in P$.

We consider the following cases:

Case 1. Let $(x_m, x_{m+1}) \in P \times P$.

Consider when m is even, that is $m = 2n$ for some $n \in \mathbb{N}, n \geq 1$. In this case $x_{2n} = Tx_{2n-1}$, $x_{2n+1} = Sx_{2n}$. We have from the assumption:

$$\begin{aligned} p(x_{2n}, x_{2n+1}) &= p(x_{2n}, Sx_{2n}) \\ &= p(Tx_{2n-1}, STx_{2n-1}) \\ &\leq \max\{p(Sx_{2n-1}, TSx_{2n-1}), p(Tx_{2n-1}, STx_{2n-1})\} \\ &\leq r\min\{p(x_{2n-1}, Sx_{2n-1}), p(x_{2n-1}, Tx_{2n-1})\} \\ &\leq rp(x_{2n-1}, Tx_{2n-1}) \\ &= rp(x_{2n-1}, x_{2n}). \end{aligned}$$

We get a similar result when m is odd, that is

$$p(x_{2n+1}, x_{2n+2}) \leq rp(x_{2n}, x_{2n+1}).$$

Hence, in all cases where $(x_m, x_{m+1}) \in P \times P$, we have

$$p(x_m, x_{m+1}) \leq rp(x_{m-1}, x_m). \tag{9}$$

Case 2. Suppose $(x_m, x_{m+1}) \in P \times Q$. Consider when m is even, implying $m = 2n$ for some $n \in \mathbb{N}, n \geq 1$. Then, from the construction of proof, we have $x_m = x_{2n} = Tx_{2n-1}$ and $x_{m+1} = x_{2n+1} \in \operatorname{seg}\{Tx_{2n-1}, Sx_{2n}\}$.

Hence, from Lemma 6 (ii), we have

$$\begin{aligned} p(x_{2n}, x_{2n+1}) &\leq p(x_{2n}, Sx_{2n}) \\ &= p(Tx_{2n-1}, STx_{2n-1}) \\ &\leq rp(x_{2n-1}, x_{2n}). \end{aligned} \tag{10}$$

Here we use the argument shown in Case 1.

We get a similar result when m is odd, that is

$$p(x_{2n+1}, x_{2n+2}) \leq rp(x_{2n}, x_{2n+1}).$$

Hence, in all cases when $(x_m, x_{m+1}) \in P \times Q$, we have

$$p(x_m, x_{m+1}) \leq rp(x_{m-1}, x_m). \tag{11}$$

Case 3. Let $(x_m, x_{m+1}) \in Q \times P$, which implies $x_{m-1} \in P$.

Let us consider first when m is odd, implying $m = 2n + 1$ for some $n \in \mathbb{N}$. From the construction of sequence, we have $x_{2n+1} \in \operatorname{seg}\{x_{2n}, Sx_{2n}\}$ and $x_{2n+2} = Tx_{2n+1}$ and $x_{2n} = Tx_{2n-1}$, (see Remark 2).

Using (P3) of Definition 1, we have

$$\begin{aligned} p(x_{2n+1}, x_{2n+2}) &= p(x_{2n+1}, Tx_{2n+1}) \\ &\leq p(x_{2n+1}, Sx_{2n}) + p(Sx_{2n}, Tx_{2n+1}) \\ &\quad - p(Sx_{2n}, Sx_{2n}) \\ &\leq p(x_{2n+1}, Sx_{2n}) + p(Sx_{2n}, Tx_{2n+1}) \\ &\leq p(x_{2n}, Sx_{2n}) + p(Sx_{2n}, Tx_{2n+1}), \\ &\qquad \text{by Lemma 6 (ii)} \\ &= p(Tx_{2n-1}, Sx_{2n}) + p(Sx_{2n}, Tx_{2n+1}), \\ &\qquad \text{as } x_{2n} = Tx_{2n-1} \end{aligned}$$

This implies

$$\begin{aligned} &p(x_{2n+1}, x_{2n+2}) \\ &\quad \leq 2\max\{p(Tx_{2n-1}, Sx_{2n}), p(Sx_{2n}, Tx_{2n+1})\}. \end{aligned} \tag{12}$$

If $p(Tx_{2n-1}, Sx_{2n}) \geq p(Sx_{2n}, Tx_{2n+1})$, then (12) becomes

$$\begin{aligned} p(x_{2n+1}, x_{2n+2}) &\leq 2p(Tx_{2n-1}, Sx_{2n}) \\ &= 2p(Tx_{2n-1}, STx_{2n-1}), \\ &\qquad \text{as } x_{2n} = Tx_{2n-1} \\ \Rightarrow p(x_{2n+1}, x_{2n+2}) &\leq 2rp(x_{2n-1}, x_{2n}), \end{aligned} \tag{13}$$

using the argument in Case 2, (see (10)).

If $p(Tx_{2n-1}, Sx_{2n}) < p(Sx_{2n}, Tx_{2n+1})$, then (12) becomes

$$\begin{aligned} p(x_{2n+1}, x_{2n+2})) &\leq 2p(Sx_{2n}, Tx_{2n+1}) \\ &= 2p(STx_{2n-1}, Tx_{2n+1}), \\ &\qquad \text{because } Tx_{2n-1} = x_{2n} \\ &= 2p(Tx_{2n+1}, STx_{2n-1}) \\ &\leq 2\max\{p(Sx_{2n+1}, TSx_{2n-1}), p(Tx_{2n+1}, STx_{2n-1})\} \\ &\leq 2r[\min\{p(x_{2n+1}, Sx_{2n-1}), p(x_{2n+1}, Tx_{2n-1})\} \\ &\leq 2rp(x_{2n+1}, Tx_{2n-1}) \\ &= 2rp(Tx_{2n-1}, x_{2n+1}) \\ \Rightarrow p(x_{2n+1}, x_{2n+2}) &\leq 2rp(x_{2n}, x_{2n+1}), \\ &\qquad \text{as } Tx_{2n-1} = x_{2n}. \end{aligned} \tag{14}$$

By (13) and (14), we have

$$p(x_{2n+1}, x_{2n+2}) \leq 2r\max\{p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1})\}.$$

A similar argument is valid when m is even. Thus in general if $(x_m, x_{m+1}) \in Q \times P$, we have

$$p(x_m, x_{m+1}) \leq 2r\max\{p(x_{m-2}, x_{m-1}), p(x_{m-1}, x_m).\} \tag{15}$$

By Remark 2, the case where $(x_m, x_{m+1}) \in Q \times Q$ is not possible.

Hence, considering all three cases, and their corresponding results as shown in (9), (11) and (15), we have

$$p(x_m, x_{m+1}) \leq 2r\max\{p(x_{m-2}, x_{m-1}), p(x_{m-1}, x_m)\}.$$

We use Lemma 8 with $w_i = p(x_i, x_{i+1})$. For $m \in \mathbb{N}$, we get $p(x_m, x_{m+1}) \leq (2r)^{m/2}\delta$ where

$$\delta = (2r)^{-1/2} \max\{p(x_0, x_1), p(x_1, x_2)\}.$$

Let $m > n$. Then, from (P3) of Definition 1 inductively, we have

$$\begin{aligned} p(x_n, x_m) &\leq \sum_{i=n}^{m-1} p(x_i, x_{i+1}) - \sum_{i=n}^{m-1} p(x_{i+1}, x_{i+1}) \\ &\leq \sum_{i=n}^{m-1} p(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{+\infty} p(x_i, x_{i+1}) \\ &\leq \delta \sum_{i=n}^{+\infty} (2r)^{i/2} \\ &= \delta (2r)^{n/2} \frac{1}{1-2r}. \end{aligned}$$

Taking $n, m \to +\infty$, we get

$$\lim_{n \to +\infty} p(x_n, x_m) = 0 < +\infty.$$

which makes $\{x_n\} \subset C$ a Cauchy sequence in (X, p).

Now, as the partial metric space (X, p) is complete, so is the corresponding metric space (X, p^s), (see Lemma 2). We have C as a closed subset of the complete metric space (X, p^s). Therefore, C is complete in (X, p^s) and hence it is also complete in (X, p).

Hence, there is $z \in C$ such that

$$z = \lim_{n \to +\infty} x_n.$$

We consider the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ in P. Then we have
$x_{2n_k} = Tx_{2n_k - 1}$. As T is a continuous mapping, we have

$$\begin{aligned} & x_{2n_k} = Tx_{2n_k - 1} \\ \Rightarrow & \lim_{n_k \to +\infty} x_{2n_k} = \lim_{n_k \to +\infty} Tx_{2n_k - 1} \\ & \Rightarrow z = Tz. \end{aligned}$$

Similarly, as S is a continuous mapping, we have

$$\begin{aligned} & x_{2n_k+1} = Sx_{2n_k} \\ \Rightarrow & \lim_{n_k \to +\infty} x_{2n_k+1} = \lim_{n_k \to +\infty} Sx_{2n_k} \\ & \Rightarrow z = Sz. \end{aligned}$$

Thus, z is a common fixed point of S and T.

We will show that z is unique. Suppose z' is also a common fixed point of S and T. Then

$$\begin{aligned} p(z', z) &= p(Sz', Tz) \\ &= p(Sz', TSz) \\ &\leq \max\{p(Sz', TSz), p(Tz', STz)\} \\ &\leq r \min\{p(z', Sz), p(z', Tz)\} \\ &\leq r p(z', Sz) \\ &= r p(z', z) \\ \Rightarrow p(z', z) &= 0 \\ &\Rightarrow z' = z, \text{ by } (1). \end{aligned}$$

Therefore, z is a unique fixed point of S and T. On replacing z' by z in above argument, we get $p(z, z) = 0$.

We have completed the proof of Theorem 2.

Remark 2. If we set $x = y$ in Theorem 2 and consider $C = X$ so that only Case 1 applies, then we get Theorem 1 found in Karapinar *et al.* [3].

If we set $T = I$, where I is the identity mapping, then we get the following corollary, which is an extension of Matthews Contraction Principle [4] to non-self mappings.

Corollary 1. *Let (X, p) be a complete metrically convex partial metric space and C be a non-empty closed subset of X, the closure being with respect to (X, p^s). Let ∂C, the boundary of C with respect to (X, p^s), be non-empty. Let $S : C \to X$ be a continuous mapping. Let there be $r \in \mathbb{R}$ such that $0 \leq 2r < 1$ and for all $x, y \in C$ the following conditions hold:*
(a) $p(Sx, Sy) \leq r p(x, y)$,
(b) $x \in \partial C \Rightarrow Sx \in C$ and
(c) SC is closed.
Then there is a unique point $z \in C$ such that $z = Sz$ with $p(z, z) = 0$.

If we set $S = T$, then we get the following corollary:

Corollary 2. *Let (X, p) be a complete metrically convex partial metric space and C be a non-empty closed subset of X, the closure being with respect to (X, p^s). Let ∂C, the boundary of C with respect to (X, p^s), be non-empty. Let $S : C \to X$ be a continuous mapping. Let there be $r \in \mathbb{R}$ such that $0 \leq 2r < 1$ and whenever $Sx, Sy \in C$ the following conditions hold:*
(a) $p(Sx, S^2y) \leq r p(x, Sy)$,
(b) $Sx \in \partial C \Rightarrow x \in C$, and
(c) SC is closed.
Then there is a unique point $z \in C$ such that $z = Sz$ and $p(z, z) = 0$.

Example 1. Consider the partial metric space (X, p), where $X = \mathbb{R}^+$ and $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. The metric derived from this partial metric is $p^s(x, y) = |x - y|$.

Let $C = [0, 5] \cup [10, 30]$. Define the mappings $S, T : \mathbb{R}^+ \to \mathbb{R}^+$ as $Sx = 0.4x$ and $Tx = 0.45x$ for all $x \in C$.

We note that $STx = TSx = 0.18x$.

We also note that for some $x \in C$, we have $Sx \notin C$ or $Tx \notin C$. When $x = 20$, we have $Sx = 8 \notin C$ and $Tx = 9 \notin C$, which make both S and T to be non-self mappings.

We observe that δC with respect to (X, p^s) is $\{0, 5, 10, 30\}$. Thus $x \in \delta C$ implies $Sx \in \{0, 2, 4, 12\} \subset C$ and $Tx \in \{0, 2.25, 4.5, 13.5\} \subset C$.

We have $SC = [0, 2] \cup [4, 12]$ and we also have $TC = [0, 2.25] \cup [4.5, 13, 5]$, both sets being closed in (X, p^s).

The condition that Sy, $Ty \in C$ implies that $y \in [0, 5] \cup [25, 30]$.

As $p(x, y) = \max\{x, y\}$, we have

$$\begin{aligned}\max\{&p(Sx, TSy), p(Tx, STy)\}\\ &= \max\{p(0.4x, 0.18y), p(0.45x, 0.18y)\}\\ &= \max\left\{\max\{0.4x, 0.18y\}, \max\{0.45x, 0.18y\}\right\}\\ &= \max\{0.45x, 0.18y\}. \qquad (16)\end{aligned}$$

Similarly

$$\begin{aligned}\min\{p(x, Sy), p(x, Ty)\} &= \min\{p(x, 0.4y), p(x, 0.45y)\}\\ &= \min\{\max\{x, 0.4y\}, \max\{x, 0.45y\}\}\\ &= \max\{x, 0.4y\}. \qquad (17)\end{aligned}$$

We use $h = 0.48 < 1/2$.

We consider the following cases.

Case (i). Let $x, y \in [0, 5]$ or $x, y \in [25, 30]$ with $y < 2.5x$. This means $0.18y < 0.45x$ and $0.4y < x$. Hence from (16) we have

$$\begin{aligned}\max\{p(Sx, TSy), p(Tx, STy)\} &= \max\{0.45x, 0.18y\}\\ &= 0.45x.\end{aligned} \qquad (18)$$

Similarly, from (17), we have

$$\min\{p(x, Sy), p(x, Ty)\} = \max\{x, 0.4y\} = x. \qquad (19)$$

Comparing (18) with (19), and recalling that $h = 0.48 < 1/2$, we have

$$\max\{p(Sx, TSy), p(Tx, STy)\} < h \min\{p(x, Sy), p(x, Ty)\}. \qquad (20)$$

Case (ii). Let $x, y \in [0, 5]$ with $y \geq 2.5x$. This implies $0.18y \geq 0.45x$ and $0.4y \geq x$.

From (16) we have

$$\begin{aligned}\max\{p(Sx, TSy), p(Tx, STy)\} &= \max\{0.45x, 0.18y\}\\ &= 0.18y.\end{aligned} \qquad (21)$$

From (17) we have

$$\min\{p(x, Sy), p(x, Ty)\} = \max\{x, 0.4y\} = 0.4y. \qquad (22)$$

Comparing (21) with (22), and recalling that $h = 0.48 < 1/2$, we have in this case

$$\max\{p(Sx, TSy), p(Tx, STy)\} < h \min\{p(x, Sy), p(x, Ty)\}. \qquad (23)$$

The situation where $x, y \in [25, 30]$ with $y \geq 2.5x$ is not possible.

Case (iii). We now consider $x \in [0, 5], y \in [25, 30]$. In this case, we have

$$0.45x \leq 0.45 \times 5 = 2.25 < 4.5 = 0.18 \times 25 \leq 0.18y. \qquad (24)$$

The equation (24) implies $x < 0.4y$.

From (16) we have

$$\begin{aligned}\max\{p(Sx, TSy), p(Tx, STy)\} &= \max\{0.45x, 0.18y\}\\ &= 0.18y.\end{aligned} \qquad (25)$$

From (17) we have

$$\min\{p(x, Sy), p(x, Ty)\} = \max\{x, 0.4y\} = 0.4y. \qquad (26)$$

Comparing (25) with (26), and recalling that $h = 0.48 < 1/2$, we have in this case

$$\max\{p(Sx, TSy), p(Tx, STy)\} < h \min\{p(x, Sy), p(x, Ty)\}. \qquad (27)$$

Case (iv). Finally, we consider when $y \in [0, 5]$ and $x \in [25, 30]$. In this case we have

$$0.18y < 0.18 \times 5 = 0.9 < 11.25 = 0.45 \times 25 \leq 0.45x. \qquad (28)$$

The equation (28) implies $0.4y < x$.

From (16) we have

$$\begin{aligned}\max\{p(Sx, TSy), p(Tx, STy)\} &= \max\{0.45x, 0.18y\}\\ &= 0.45x.\end{aligned} \qquad (29)$$

From (17), we have

$$\min\{p(x, Sy), p(x, Ty)\} = \max\{x, 0.4y\} = x. \qquad (30)$$

Comparing (29) with (30), and recalling that $h = 0.48 < 1/2$, we have in this case

$$\max\{p(Sx, TSy), p(Tx, STy)\} < h \min\{p(x, Sy), p(x, Ty)\}. \qquad (31)$$

Thus in all possible cases we have

$$\max\{p(Sx, TSy), p(Tx, STy)\} < h \min\{p(x, Sy), p(x, Ty)\}.$$

We note that, because $S(0) = T(0) = 0$, the point $z = 0$ is a common fixed point of the mappings S and T. We also note that $p(z, z) = 0$.

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