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# Some New Constructions of Minimal Circular Partially Neighbor Balanced Designs

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**Abstract:** Neighbor balanced designs (NBDs) are used to balance the neighbor effects in the experiments where performance of a treatment is influenced with the treatments applied in its neighboring units. Among these designs, experimenters prefer minimal NBDs but minimal circular NBDs can be generated only for $v$ (number of treatments) odd. For v even, minimal partially neighbor balanced designs (PNBDs) may be considered as better alternates to the minimal NBDs. In this article, minimal circular PNBDs are constructed in which $3v/2$ unordered pairs of distinct treatments do not appear as neighbors.

**Keywords:** Neighbor effects; Neighbor designs; GNDs; Robust to neighbor effects.

## 1 Introduction

There are several experiments where response of a treatment is influenced by the treatment(s) applied in its neighboring units. Effect of a treatment is known as its direct effect while its neighboring units effect is called neighbor effect. Neighbor effect often becomes the major source of bias, which can be balance with the use of neighbor balanced designs (NBDs).

(i) A design where each pair of distinct treatments appears $\lambda'$ times as neighbors is called NBD. If $\lambda' = 1$ then it is called minimal NBD.

(ii) If each pair of distinct treatments appears as neighbors at most once, design is called partially NBD (PNBD).

(iii) A block formed in a cycle such that its first and last units are considered as adjacent neighbors is called a circular block. In circular blocks, each unit has one left-neighbor and one right-neighbor.

[1] used neighbor designs in virus research. The bias due to neighbor effects can be minimized with the use of NBDs, see [2]. [3] presented NBDs using border plots. [4] observed through simulations that NBDs or PNBDs are minimize that bias. [5], [6], [7] and [8] are some more references for CNBDs. [9] constructed generalized neighbor designs (GNDs) by relaxing constancy condition of $\lambda'$ with further condition that $\lambda'_i \neq 0$. The GNDs are more economical if $\lambda'_i$ takes only two values $\lambda'_1 = 1$ and $\lambda'_2 = 2$. [10], [11], [12] constructed some classes of circular GNDs. [13], [14] and [15] developed some infinite series to obtain the minimal circular GNDs. [16] presented list of CGNDs for blocks of sizes three.

[17] suggested that PNBDs should be used if minimal NBDs cannot be generated. [18] presented minimal circular PNBDs (MCPNBDs) for some specific cases. [19] developed a series of CPNBDs for $v = n$. [20] presented two new series of non-binary CPNBDs. In this article, for complete solution of MCPNBDs-I, some generators are developed through method of cyclic shifts $(Rule\,I)$. Here, $v$ is number of treatments and MCPNBDs-I are MCPNBDs in which $3v/2$ pairs of different treatments do not appear as neighbors. Catalogues of these designs are also compiled in blocks of two different and three different sizes which are presented as Appendix A and Appendix B respectively.

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## 2 Method of cyclic shifts

Method of cyclic shifts $(Rule\ I)$ developed by [21] is explained here for the construction of MCPNBDs.
Let $S_j = [q_{j1}, q_{j2}, ..., q_{j(k-1)}]$ be $i$ sets, where $j = 1, 2, ..., i, 1 \leq q_{ju} \leq v-1, u = 1, 2, ..., k-1$.

- If $1, 2, ..., v-1$ appears exactly once in $S^*$ then designs will be minimal CNBD.
- If most of the $1, 2, ..., v-1$ appears once but remaining ones do not appear in $S^*$ then designs will be MCPNBD.

Where $S^*$ contains:
(i) Each element of all $S_j$.
(ii) Sum of all elements $(mod\ v)$ in each $S_j$.
(iii) Complements of all elements in $(i)$ and $(ii)$, here complement of "$a$" is "$v-a$".

- Logic behind the $(Rule\ I)$ expresses that:

Let $m = (v-2)/2$ and $v$ is even. $S = [1, 2, ..., m-1]$ will provide MCPNBDs-I if sum of elements in $S$ is divisible by $v$. Otherwise, replace one or more elements with their complements to make the sum divisible by $v$. Here complement of "$a$" is "$v-a$".

**Example** 2.1: $S_1 = [4, 6, 7]$ and $S_2 = [2, 3]$ generate MCPNBD-I for $v = 18, k_1 = 4, k_2 = 3$.

**Proof**: $S^* = [4, 6, 7, 17, 2, 3, 5, 14, 12, 11, 1, 16, 15, 13]$ here each of $1, 2, ..., 17$ appears once except $8, 9$ and 10 which do not appear. Hence $S_1 = [4, 6, 7]$ and $S_2 = [2, 3]$ generate MCPNBD-I for $v = 18$, $k_1 = 4$ and $k_2 = 3$. Take $v$ blocks to get the blocks from $S_1$. Consider first unit elements as $0, 1, ..., v-1$. Add $4\,(mod\ v)$ to each first unit element, to obtain second unit elements. Add 5 $(mod\ 18)$ to second unit elements to obtain third unit elements, and so on, see Table 1.

Table 1: Blocks generated from $S_1$.

Blocks

| **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** | **13** | **14** | **15** | **16** | **17** | **18** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 0 | 1 | 2 | 3 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 17 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |

Take $v$ more blocks for $S_2$ and obtain the design, see Table 2.

Table 2: Blocks generated from $S_2$.

Blocks

| **19** | **20** | **21** | **22** | **23** | **24** | **25** | **26** | **27** | **28** | **29** | **30** | **31** | **32** | **33** | **34** | **35** | **36** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 2 | 3 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 0 | 1 | 2 | 3 | 4 |

Table 1 and Table 2 jointly present the MCPNBD-I for $v = 18$, $k_1 = 4$ and $k_2 = 3$.

In all next construction, elements of S given in each generator will be divided in $(i)$ $i$ groups of size $k_1$ and two groups of size $k_2$ for MCPNBDs-I in blocks of two different sizes, $(ii)$ $i$ groups of size $k_1$, one group of size $k_2$ and two groups of size $k_3$ for MCPNBDs-I in blocks of three different sizes, such that the sum of each group should be divisible by $v$. Sets of shifts to generate MCPNBDs-I will be obtained by deleting any one element from each group.

## 3 Construction of MCPNBDs-I for *m* = 4*j*

In this Section, MCPNBDs-I are constructed in two and three different blocks sizes for $m = 4j$, where $m = (v-2)/2$.
**Construction** 3.1: If $m = 4j$ then $i$ sets of shifts derived from
$S = [1,2,...,(m/2),(m+4)/2,(m+6)/2,...,(m-1),(3m+2)/2]$
will produce MCPNBDs-I for $v = 2ik+4$.

### *3.1 MCPNBDs-I in two different blocks sizes for m = 4j*

Here, MCPNBDs-I are constructed from $(i+2)$ sets for $v = 2ik_1+4k_2+4$ in two different blocks sizes, using construction 3.1. These sets will be taken from $S = [1,2,...,(m/2),(m+4)/2,(m+6)/2,...,(m-1),(3m+2)/2]$.

**Generator** 3.1: MCPNBDs-I can be generated from $i$ sets for $k_1$ and two for $k_2$ obtained from $S = [1,2,...,(m/2),(m+4)/2,(m+6)/2,...,(m-1),(3m+2)/2]$ for $v = 2ik_1+4k_2+4$ with $m = 4j; j, l, u$ integer and:

• $k_2 = 3$:
$k_1 = 4u+1, k_1 > 1, i = 4l+1$.
$k_1 = 4u+3, k_1 > 3, i = 4l+3$.
**Example** 3.1
MCPNBD-I is generated from following sets for
$v = 26, k_1 = 5, k_2 = 3, S_1 = [3,4,8,9], S_2 = [6,19], S_3 = [10,11]$
• $k_2 = 4$:
$k_1 = 4u+1, k_1 > 1, i = 4l+3$.
$k_1 = 4u+3, k_1 > 3, i = 4l+1$.
• $k_2 = 5$:
$k_1 = 4u+1, k_1 > 1, i = 4l+1$.
$k_1 = 4u+3, k_1 > 3, i = 4l+1$.

### *3.2 MCPNBDs-I in three different blocks sizes for m = 4j*

Here, MCPNBDs-I are constructed from $(i+3)$ sets for $v = 2ik_1+2k_2+4k_3+4$ in three different blocks sizes, using construction 3.1. These sets will be obtained from $S = [1,2,...,(m/2),(m+4)/2,(m+6)/2,...,(m-1),(3m+2)/2]$.

**Generator** 3.2: MCPNBDs-I can be generated from $i$ sets for $k_1$, one set for $k_2$ and two for $k_3$ obtained from
$S = [1,2,...,(m/2),(m+4)/2,(m+6)/2,...,(m-1),(3m+2)/2]$ for $v = 2ik_1+2k_2+4k_3+4$
with $m = 4j; j, l, u$ integer and:
• $k_3 = 3$:
$k_1 = 4u+2, k_2 = k_1-1$, $i$ even, l integer.
$k_1(odd) > 3, k_2 = k_1-1, i = 4l+1$.
$k_1 = 4u+1, k_1 > 5, k_2 = k_1-2, i = 4l+2$.
$k_1 = 4u+3, k_1 > 3, k_2 = k_1-2, i = 4l$.

• $k_3 = 4$:
$k_1 = 4u, l(integer) > 1, k_2 = k_1-1$, i integer.
$k_1 = 4u+2, k_2 = k_1-1$, $i$ odd.
$k_1(odd) > 5, k_2 = k_1-1, i = 4l+3$.
$k_1 = 4u+1, k_1 > 5, k_2 = k_1-2, i = 4l$.
$k_1 = 4u+3, k_1 > 3, k_2 = k_1-2, i = 4l+2$.

• $k3 = 5$:
$k_1 = 4u+2, k_2 = k_1 - 1$, $i$ even.
$k_1(odd) > 5, k2 = k_1 - 1, i = 4l+1$.
$k_1 = 4u+1, k_1 > 7, k_2 = k_1 - 2, i = 4l+2$.
$k_1 = 4u+3, k-1 > 7, k_2 = k_1 - 2, i = 4l$.

## 4 Construction of MCPNBDs-I for *m* = 4*j*+1

Here, MCPNBDs-I are constructed in two and three different blocks sizes for $m = 4j+1$, where $m = (v-2)/2$.
**Construction** 4.1: If $m = 4j+1$ then $i$ sets obtained from
$S = [1,2,...,(3m+1)/4,(3m+9)/4,(3m+13)/4,...,(m-1),(5m+3)/4]$ will produce MCPNBDs-I for $v = 2ik+4$.

### *4.1 MCPNBDs-I in two different blocks sizes for m = 4j+1*

Here, MCPNBDs-I are constructed from $(i+2)$ sets for $v = 2ik_1 + 4k_2 + 4$ in blocks of two different sizes, using construction 4.1. These sets will be obtained from
$S = [1,2,...,(3m+1)/4,(3m+9)/4,(3m+13)/4,...,(m-1),(5m+3)/4]$ as:
**Generator** 4.1: MCPNBDs can be generated from $i$ sets for $k_1$ and two for $k_2$ obtained from
$S = [1,2,...,(3m+1)/4,(3m+9)/4,(3m+13)/4,...,(m-1),(5m+3)/4]$ for
$v = 2ik_1 + 4k_2 + 4$ with $m = 4j+1; j,l,u$ are integer and:
• $k_2 = 3$:
$k_1 = 4u+2$, $i$ odd.
$k_1(odd) > 3, i = 4l+2$.

**Example** 4.1. MCPNBD-I is generated from following sets for
$v = 28, k_1 = 6, k_2 = 3, S_1 = [3,4,5,6,8], S_2 = [9,12], S_3 = [10,17]$
• $k_2 = 4$:
$k_1 = 4u, u > 1$ and $i$ integer.
$k_1 = 4u+2$, $i$ even.
$k_1(odd) > 3, i = 4l$.

• $k_2 = 5$:
$k_1 = 4u+2$, $i$ odd.
$k_1(odd) > 5, i = 4l+2$

### *4.2 MCPNBDs-I in three different blocks sizes for m = 4j+1*

Here, MCPNBDs-I are constructed from $(i+3)$ sets for $v = 2ik_1 + 2k_2 + 4k_3 + 4$ in three different blocks sizes, using construction 4.1. These sets will be obtained from $S = [1,2,...,(3m+1)/4,(3m+9)/4,(3m+13)/4,...,(m-1),(5m+3)/4]$.
**Generator** 4.2: MCPNBDs-I can be generated from $i$ sets for $k_1$, one set for $k_2$ and two for $k_3$ obtained from $S = [1,2,...,(3m+1)/4,(3m+9)/4,(3m+13)/4,...,(m-1),(5m+3)/4]$ for $v = 2ik_1 + 2k_2 + 4k_3 + 4$ with $m = 4j+1$; $j$, l, $u$ are integer and:
• $k_3 = 3$:
$k_1 = 4u+1, k_1 > 1, k_2 = k_1 - 1, i = 4l+2$.

$k_1 = 4u+3, k_1 > 3, k_2 = k_1 - 1, i = 4l.$
$k_1 = 4u, u > 1, k_2 = k_1 - 2$, $i$ integer.
$k_1 = 4u+2, k_2 = k_1 - 2$, $i$ odd.
$k_1(odd) > 5, k_2 = k_1 - 2, i = 4l+3.$

**Example** 4.2. MCPNBD-I is generated from following sets for $v = 36, k_1 = 6, k_2 = 4, k_3 = 3$.
$S_1 = [1,2,3,6], S_2 = [4,7,10],\ S_3 = [5,9],\ S_4 = [11,12]$

• $k_3 = 4$:
$k_1 = 4u+1, k_1 > 5, k_2 = k_1 - 1, i = 4l.$
$k_1 = 4u+3, k_1 > 3, k_2 = k_1 - 1, i = 4l+2.$
$k_1 = 4u+2, k_2 = k_1 - 2, i\,even, l > 1.$
$k_1(odd) > 5, k_2 = k_1 - 2, i = 4l+1.$

• $k_3 = 5$:
$k_1 = 4u+1, k_1 > 5, k_2 = k_1 - 1, i = 4l+2.$
$k_1 = 4u+3, k_1 > 3, k_2 = k_1 - 1, i = 4l.$
$k_1 = 4u, u > 1, k_2 = k_1 - 2$, $i$ integer.
$k_1 = 4u+2, k2 = k_1 - 2, i\,odd, u > 1.$
$k_1(odd) > 7, k_2 = k_1 - 2, i = 4l+3.$

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## Appendix A

MCPNBDs in two different block sizes for $v \leq 100$

| $v$ | $k_1$ | $k_2$ | Set of Shifts |
|---|---|---|---|
| 26 | 5 | 3 | [1,3,4,8]+[2,5]+[6,9] |
| 28 | 6 | 3 | [2,3,4,5,6]+[7,9]+[1,10] |
| 36 | 5 | 3 | [1,3,4,6]+[2,7,8,9]+[11,12]+[5,15] |
| 66 | 5 | 3 | [3,4,5,25]+[7,8,9,11]+[6,10,13,15]+[14,19,20,30]+ [1,2,16,23]+[18,21]+[12,26] |
| 76 | 5 | 3 | [3,4,5,31]+[6,8,9,17]+[1,13,15,22]+[10,11,16,19]+ [2,12,18,21]+[24,26,27,28]+[14,30]+[7,34] |
| 28 | 6 | 3 | [2,3,4,5,6]+[7,9]+[1,10] |
| 52 | 6 | 3 | [3,4,5,6,13]+[2,8,9,10,11]+[14,15,16,17,18]+ [1,19]+[7,22] |
| 76 | 6 | 3 | [2,3,4,5,15]+[21,22,23,24,27]+ [6,7,10,16,18]+ [8,9,12,13,14]+[1,26,28,30,31]+[11,32]+[17,25] |
| 100 | 6 | 3 | [2,3,4,6,39]+[5,7,10,12,23]+[13,14,15,16,17]+[1,9,20,21,22]+ [24,26,28,29,31]+[30,32,33,34,35]+ [18,19,37,40,41]+[11,42,47]+[8,44] |
| 50 | 5 | 4 | [1,3,4,5]+[17,18,20,22]+[2,10,11,12]+[6,9,16]+[7,8,14] |
| 60 | 5 | 4 | [3,4,5,11]+[6,8,9,10]+[1,13,14,15]+[21,22,24,25]+ [7,16,18]+[2,12,20] |
| 90 | 5 | 4 | [3,4,5,37]+[7,8,11,29]+[2,12,15,18]+ [14,16,17,19]+ [1,20,21,22]+[9,10,13,28]+[31,33,34,40]+[36,38,39]+[6,25,27] |
| 100 | 5 | 4 | [25,39,44,45]+[5,6,8,33]+[9,13,15,22]+[4,14,18,30]+ [1,17,23,24]+[3,16,26,27]+[7,12,21,29]+[19,37,40,42]+[10,11,36]+[2,20,32] |
| 44 | 6 | 4 | [1,2,3,5,6]+[10,11,12,16,19]+[8,9,13]+[4,7,15] |

| $v$ | $k_1$ | $k_2$ | Set of Shifts |
|---|---|---|---|
| 68 | 6 | 4 | [2,3,4,7,20]+[8,9,10,11,13]+[14,15,24,25,28]+ [16,18,21,23,27]+ [5,12,22]+[1,6,19] |
| 92 | 6 | 4 | [3,4,6,19,24]+[8,9,10,11,13]+[17,18,26,39,40]+ [1,5,20,21,22]+ [16,25,30,34,37]+[2,29,31,33,32]+ [12,15,27]+[7,14,28] |
| 34 | 7 | 4 | [3,4,5,6,11,14]+[2,7,10]+[1,8,12] |
| 90 | 7 | 4 | [3,4,5,6,7,22]+[8,9,10,11,13,14]+[15,16,17,18,33,40]+ [12,21,24,26,28,32]+ [2,19,29,31,34,30]+ [36,38,39]+[1,20,27] |
| 36 | 6 | 5 | [2,3,4,5,6]+[1,7,8,9]+[10,12,13,15] |
| 60 | 6 | 5 | [3,4,5,6,20]+[7,8,9,11,12]+[10,14,15,16,28]+ [1,2,17,19]+[18,24,25,26] |
| 74 | 6 | 5 | [2,3,13,26]+[6,7,9,24]+[4,14,15,20]+[16,18,25,34]+ [8,10,11,22]+[1,5,12,27]+[17,31,32,33] |
| 84 | 6 | 5 | [2,4,5,12,21]+[1,7,8,10,24]+[9,13,14,15,16]+ [3,6,11,19,22]+[25,26,27,28,33]+ [20,30,31,35]+[18,36,37,38] |
| 52 | 7 | 5 | [1,2,3,4,5,13]+[9,11,14,16,17,18]+[7,8,10,12]+ [6,21,22,23] |

# Appendix B

MCPNBDs in two different block sizes for $v \leq 100$

| $v$ | $k_1$ | $k_2$ | $k_3$ | Set of Shifts |
|---|---|---|---|---|
| 34 | 5 | 4 | 3 | [1,2,4,12,15]+[5,8,10,11]+[3,6,25]+[7,13,14] |
| 36 | 5 | 4 | 3 | [1,2,3,8,22]+[4,9,11,12]+[6,7,10,13]+[5,15,16] |
| 42 | 5 | 4 | 3 | [1,2,3,5,31]+[7,8,9,18]+[4,10,12,16]+[13,14,15]+[6,17,19] |
| 44 | 5 | 4 | 3 | [1,3,4,16,20]+[5,6,8,11,14]+[9,10,12,13]+[2,15,27]+[7,18,19] |
| 52 | 5 | 4 | 3 | [1,3,4,12,32]+[6,7,8,9,22]+[2,13,16,21]+[5,14,15,18]+[11,17,24]+[10,19,23] |
| 74 | 5 | 4 | 3 | [1,2,4,12,55]+[3,7,9,21,34]+[10,11,13,16,24]+[27,28,30,31,32]+ [5,14,15,17,23]+[8,18,22,26]+[20,25,29]+[6,33,35] |
| 76 | 5 | 4 | 3 | [2,3,4,33,34]+[6,8,10,25,27]+[11,12,13,14,26]+ [22,23,28,32]+[1,17,18,19]+[7,15,24] |
| 82 | 5 | 4 | 3 | [3,4,24,25,26]+[7,8,9,27,31]+[10,14,16,20,22]+ [13,15,17,18,19]+ [1,6,12,30,33]+ [2,23,28,29]+[32,34,37,61]+[11,35,36]+[5,38,39] |
| 84 | 5 | 4 | 3 | [2,5,20,28,29]+[6,9,21,23,25]+[4,14,15,16,35]+ [12,13,17,18,24]+ [30,31,33,34,40]+ [1,8,22,26,27]+ [3,10,19,52]+[11,36,37]+[7,38,39] |
| 92 | 5 | 4 | 3 | [10,12,13,15,42]+[7,8,9,11,57]+[3,4,23,30,32]+[16,17,18,19,22]+[2,20,21,24,25]+ [33,34,37,39,41]+[6,26,29,31]+[1,27,28,36]+[14,38,40]+[5,43,44] |
| 36 | 6 | 4 | 3 | [2,3,4,5,6,16]+[7,8,9,12]+[10,11,15]+[1,13,22] |
| 44 | 6 | 4 | 3 | [1,2,3,4,7,27]+[6,9,11,18]+[8,10,12,14]+[5,19,20]+[13,15,16] |
| 50 | 6 | 4 | 3 | [2,4,6,10,12,16]+[3,5,7,9,11,15]+[19,21,23,37]+[1,14,17,18]+[8,20,22] |
| 60 | 6 | 4 | 3 | [1,2,3,5,22,27]+[7,8,9,10,12,14]+ [13,15,16,19,20,37]+ [4,17,18,21]+[11,24,25]+[6,26,28] |
| 68 | 6 | 4 | 3 | [2,3,4,6,23,30]+[7,8,9,10,12,22]+ [15,16,18,20,25,42]+ [11,17,19,21]+ [1,14,24,29]+[13,27,28]+[5,31,32] |
| 74 | 6 | 4 | 3 | [1,2,3,4,31,33]+[6,7,8,10,13,30]+[15,16,17,18,27,55]+ [21,22,23,24,26,32]+ [12,14,20,28]+[9,11,25,29]+[5,34,35] |
| 84 | 6 | 4 | 3 | [1,2,3,5,34,39]+[7,8,10,11,12,36]+ [9,13,14,15,16,17]+ [20,21,22,23,30,52]+ [25,26,27,28,33,29]+[4,19,24,37]+[18,31,35]+[6,38,40] |
| 92 | 6 | 4 | 3 | [2,3,4,5,37,41]+[7,8,9,10,19,39]+ [12,14,15,16,17,18]+ [25,32,31,33,29,34]+ [23,27,28,30,36,40]+[20,22,24,26]+[1,13,21,57]+[11,38,43]+[6,42,44] |
| 98 | 6 | 4 | 3 | [1,2,3,4,42,46]+[8,9,10,11,29,31]+ [13,14,16,17,18,20]+ [19,21,36,38,39,43]+ [7,26,27,28,35,73]+[12,32,33,34,41,44]+[15,22,24,37]+[5,23,30,40]+[6,45,47] |
| 42 | 6 | 5 | 3 | [1,2,3,4,14,18]+[10,12,15,16,31]+[5,7,8,9,13]+[6,17,19] |
| 50 | 6 | 5 | 3 | [1,2,4,6,16,21]+[3,7,8,9,11,12]+[10,15,18,20,37]+[14,17,19]+[5,22,23] |
| 66 | 6 | 5 | 3 | [1,2,3,5,6,49]+[11,21,22,26,23,29]+ [4,9,10,12,15,16]+ [7,8,14,18,19]+[20,24,27,30,31]+[13,25,28] |
| 74 | 6 | 5 | 3 | [1,3,4,5,6,55]+[7,8,9,10,11,29]+[15,18,22,26,32,35]+ [16,20,23,27,28,34]+ [2,12,14,21,25]+[13,30,31]+[17,24,33] |
| 90 | 6 | 5 | 3 | [1,3,4,6,35,41]+[8,9,11,13,22,27]+[10,14,15,16,17,18]+[12,20,24,26,31,67]+ [19,25,29,33,34,40]+[2,7,21,28,32]+[30,36,37,38,39]+[5,42,43] |
| 98 | 6 | 5 | 3 | [1,2,3,19,27,46]+[7,8,9,12,21,41]+ [13,14,15,16,17,23]+ [4,10,18,20,22,24]+ [5,26,28,29,35,73]+[30,31,32,33,34,36]+[37,38,39,40,42]+[11,43,44]+[6,45,47] |
| 42 | 6 | 5 | 4 | [2,4,5,6,10,15]+[7,9,18,19,31]+[3,12,13,14]+[1,8,16,17] |
| 44 | 6 | 6 | 4 | [1,2,3,4,7,27]+[6,8,9,10,11]+[15,16,18,19,20]+[5,12,13,14] |
| 50 | 7 | 5 | 4 | [1,5,6,7,8,11,12]+[2,3,4,9,10,22]+[14,15,16,17,18,20]+[19,21,23,37]+ |
| 52 | 6 | 5 | 4 | [1,3,4,5,7,32]+[8,9,10,12,13]+[16,19,22,23,24]+[2,15,17,18]+[6,11,14,21] |
| 66 | 6 | 5 | 4 | [1,2,4,5,23,31]+[7,8,9,10,11,21]+[13,14,15,16,25,49]+ [22,24,27,29,30]+[3,18,19,26]+[6,12,20,28] |

| $v$ | $k_1$ | $k_2$ | $k_3$ | Set of Shifts |
|---|---|---|---|---|
| 68 | 6 | 5 | 4 | [2,5,6,9,14,32]+[4,8,10,11,12,23]+[13,17,19,28,29,30]+ [3,7,15,21,22]+[18,24,25,27,42]+[1,16,20,31] |
| 76 | 6 | 5 | 4 | [2,4,5,15,22,28]+[8,9,10,11,12,26]+ [16,17,18,32,34,35]+ [21,23,25,36,47]+ [3,7,19,20,27]+[1,14,30,31]+[6,13,24,33] |
| 90 | 6 | 5 | 4 | [1,2,3,8,37,39]+[6,7,9,12,15,41]+[10,13,14,16,17,20]+[18,19,21,24,31,67]+ [26,27,28,29,30,40]+[33,34,35,36,42]+[4,5,38,43]+[11,22,25,32] |
| 92 | 6 | 5 | 4 | [2,4,6,23,24,33]+[8,9,10,11,15,39]+[12,13,14,16,17,20]+[25,26,29,32,34,38]+ [5,7,18,19,21,22]+[1,3,27,30,31]+[28,36,37,40,43]+[41,42,44,57] |
| 100 | 6 | 5 | 4 | [1,3,5,23,24,44]+[7,9,10,11,21,42]+ [15,25,36,37,39,48]+ [6,8,13,19,22,32]+[28,30,33,34,35,40]+ [2,14,26,27,31]+[16,17,18,20,29]+[4,12,41,43]+[45,46,47,62] |
| 44 | 7 | 5 | 4 | [1,2,3,5,7,12,14]+[10,15,16,20,27]+[6,8,11,19]+[4,9,13,18] |
| 100 | 7 | 5 | 4 | [4,5,6,7,18,19,41]+[8,9,10,11,13,14,35]+ [2,3,15,17,20,21,22]+ [23,24,25,26,27,33,42]+ [12,28,29,30,31,34,36]+[32,37,39,44,48]+[1,16,40,43]+[45,46,47,62] |
| 74 | 7 | 6 | 4 | [1,3,4,5,10,20,31]+[7,8,9,11,12,13,14]+[17,18,30,33,34,35,55]+ [22,23,24,25,26,28]+[2,16,27,29]+[6,15,21,32] |
| 50 | 7 | 6 | 5 | [2,3,4,5,6,14,16]+[8,9,11,12,23,37]+[1,7,10,15,17]+[18,19,20,21,22] |
| 52 | 7 | 6 | 5 | [1,2,3,4,11,12,19]+[5,7,8,9,10,13]+[14,15,16,17,18,24]+[6,21,22,23,32] |
| 76 | 7 | 6 | 5 | [3,4,5,6,8,24,26]+[7,9,10,11,12,13,14]+[17,21,22,25,31,36]+ [1,2,15,16,19,23]+[20,27,28,30,47]+[18,32,33,34,35] |
| 52 | 8 | 6 | 5 | [1,2,3,4,5,6,8,23]+[13,14,16,18,21,22]+[7,9,10,11,15]+[12,17,19,24,32] |
| 68 | 8 | 6 | 5 | [3,4,5,6,7,12,15,16]+[1,2,8,9,10,11,13,14]+ [17,19,20,27,32,21]+[22,23,24,25,42]+[18,28,29,30,31] |
| 84 | 8 | 6 | 5 | [3,4,5,6,7,12,14,33]+[1,2,8,9,10,11,13,30]+ [17,18,19,20,21,22,23,28]+ [24,25,26,27,29,37]+[16,31,34,35,52]+[15,36,38,39,40] |
| 100 | 8 | 6 | 5 | [1,2,5,6,7,8,9,62]+[3,10,12,13,14,15,16,17]+ [11,19,20,22,23,27,30,48]+ [4,21,24,25,29,31,32,34]+ [26,28,33,35,36,42]+[37,39,40,41,43]+[18,44,45,46,47] |
| 58 | 8 | 7 | 5 | [1,2,3,4,6,7,14,21]+[9,10,11,12,13,18,43]+[5,8,17,19,20,22,25]+[16,23,24,26,27]+[16,23,24,26,27] |
| 68 | 8 | 7 | 5 | [2,3,4,5,6,7,10,31]+[1,8,9,11,12,13,14]+[15,21,27,16,20,18,19]+[22,23,24,25,42]+[17,28,29,30,32] |
| 74 | 8 | 7 | 5 | [1,2,3,4,6,7,23,28]+[10,11,12,13,14,15,18,55]+[5,16,20,21,22,30,34]+ [8,9,24,25,26,27,29]+[17,31,32,33,35] |
| 84 | 8 | 7 | 5 | [2,3,5,6,7,11,21,29]+[1,4,9,12,13,14,15,16]+[10,19,20,23,34,22,40]+ [8,24,25,26,27,28,30]+[17,31,33,35,52]+[18,36,37,38,39] |
| 90 | 8 | 7 | 5 | [1,2,4,5,7,8,22,41]+[6,9,10,11,12,13,14,15]+[17,18,19,20,21,24,29,32]+ [3,25,26,28,30,31,37]+[27,33,34,35,36,38,67]+[16,39,40,42,43] |
| 100 | 8 | 7 | 5 | [3,4,5,6,7,8,31,36]+[9,10,11,12,13,14,15,16]+[18,47,30,19,23,20,21,22]+ [24,26,27,28,29,32,34]+[1,2,25,33,37,40,62]+[35,39,41,42,43]+[17,44,45,46,48] |