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Conservation of the Cylindrical and Elliptic Cylindrical K-P Equations

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Abstract: We study the invariance properties and exact solutions of the Kadomtsev-Petviashvili equation and construct its conservation laws and that of its transformed elliptic and elliptic-cylindrical versions. Then, it is shown how the conservation laws and related quantities of the transformed versions may be attained by applying the transformation variables as opposed to independent calculations which are often cumbersome for high order partial differential equations of ‘many’ variables.

Keywords: cylindrical and elliptic cylindrical Kadomtsev-Petviashvili equation ; Lie symmetries; reduction; conservation laws

1 Introduction

The Kadomtsev-Petviashvili (KP) equations originates in the study of the surface wave problems for an incompressible fluid described by the full set of Euler equations with free surface and rigid horizontal bottom boundary conditions. It has been extensively studied in a number of papers ([1,2,3,4,5], inter alia, and references therein). After some well known adjustments, the equation takes the form

$$(u_t + uu_x + u_{xxx})_x + 3s^2 u_{yy} = 0. \qquad (1)$$

The transformation

$$\tau = t, \qquad \chi = x + \frac{y^2}{12s^2 t}, \qquad v = \frac{y}{t} \qquad (2)$$

leads to the cylindrical KP (cKP) equation ([6,7]), with $w = w(\tau, \chi, v)$,

$$(w_\tau + 6ww_\chi + w_{\chi\chi\chi} + \frac{w}{2\tau})_\chi + \frac{3s^2}{\tau^2} w_{vv} = 0 \qquad (3)$$

and the transformation

$$T = t, \qquad \zeta = x + \frac{ty^2}{12s^2(t^2 - a^2)}, \qquad v = \frac{y}{\sqrt{t^2 - a^2}} \qquad (4)$$

with $h = h(T, \zeta, v)$ lead to the elliptic cylindrical KP equation (ecKP)

$$(h_T + 6hh_\zeta + h_{\zeta\zeta\zeta} + \frac{T}{2(T^2-a^2)} h - \frac{a^2 v^2}{12s^2(T^2-a^2)} h_\zeta)_\zeta + \frac{3s^2}{(T^2-a^2)} h_{vv} = 0. \qquad (5)$$

For a detailed account on the nature, application and reasons for the respective transformations, we refer the reader to [6]. The inverse transformations in the respective cases above are

$$t = \tau, \qquad x = \chi - \frac{\tau v^2}{12s^2}, \qquad y = \tau v \qquad (6)$$

and

$$t = T, \qquad x = \zeta - \frac{T v^2}{12s^2}, \qquad y = \sqrt{T^2 - a^2}\, v. \qquad (7)$$

In this paper, we, firstly study the invariance properties of equation (1) and show how this lead to exact solutions. That is, we determine the one parameter Lie groups of transformations (Lie point symmetry generators) to successively reduce the equation. We then construct the conserved vectors of the equation using the method of multipliers and the homotopy operator. Next, we list, independently, the conservation laws of the

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transformed equations cKP and ecKP. As a final emphasis of the study, we show that the conserved vectors of the cKP and ecKP are, in fact, obtainable from the transformed variables. Since, there is a one to one correspondence between the multipliers and conserved vectors, we need only perform the transformation on the multipliers. Thus, all the properties of the KP obtained via a study of the conservation laws are, equivalently, obtainable for the cKP and ecKP. These include integrability, convergence, conserved quantities and so on ([8]).

2 Symmetries, reductions and conservation laws - KP

The Lie symmetry approach on differential equations is well known; for details see e.g., [9,10].

We present some of the definitions and notations below. Intrinsic to a Lie algebraic treatment of differential equations is the universal space $\mathscr{A}$ (see [10]).The space $\mathscr{A}$ is the vector space of all differential functions of all finite orders and forms an algebra. Consider an rth-order system of partial differential equations of n independent variables $x=(x^1,x^2,\dots,x^n)$ and m dependent variables $u=(u^1,u^2,\dots,u^m)$

$$G^{\mu}(x,u,u_{(1)},\dots,u_{(r)})=0,\quad \mu=1,\dots,\tilde{m}, \tag{8}$$

where $u_{(1)},u_{(2)},\dots,u_{(r)}$ denote the collections of all first, second, ..., rth-order partial derivatives, that is, $u_i^{\alpha}=D_i(u^{\alpha}),u_{ij}^{\alpha}=D_jD_i(u^{\alpha}),\dots$ respectively, with the total differentiation operator with respect to x^i given by

$$D_i=\frac{\partial}{\partial x^i}+u_i^{\alpha}\frac{\partial}{\partial u^{\alpha}}+u_{ij}^{\alpha}\frac{\partial}{\partial u_j^{\alpha}}+\dots,\quad i=1,\dots,n, \tag{9}$$

where the summation convention is used whenever appropriate. A current $\mathscr{T}=(\mathscr{T}^1,\dots,\mathscr{T}^n)$ is conserved if it satisfies

$$D_i\mathscr{T}^i=0 \tag{10}$$

along the solutions of (8). It can be shown that every admitted conservation law arises from multipliers $Q_{\mu}(x,u,u_{(1)},\dots)$ such that

$$Q_{\mu}G^{\mu}=D_i\mathscr{T}^i \tag{11}$$

holds identically (that is, off the solution space) for some current $\mathscr{T}$. The conserved vector may then be obtained by the homotopy operator (see [8,11]). Other works on symmetries and conservation laws can be found in [12,13].

Definition A Lie-Bäcklund operator is given by

$$X=\xi^i\frac{\partial}{\partial x^i}+\eta^{\alpha}\frac{\partial}{\partial u^{\alpha}}+\sum_{s\geq 1}\eta^{\alpha}_{i_1\dots i_s}\frac{\partial}{\partial u^{\alpha}_{i_1\dots i_s}}, \tag{12}$$

where $\xi^i,\eta^{\alpha}\in\mathscr{A}$ and the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned}\eta_i^{\alpha}&=D_i(W^{\alpha})+\xi^j u_{ij}^{\alpha},\\ \eta^{\alpha}_{i_1\dots i_s}&=D_{i_1}\dots D_{i_s}(W^{\alpha})+\xi^j u^{\alpha}_{ji_1\dots i_s},\quad s>1.\end{aligned} \tag{13}$$

In (13), W^{α} is the Lie characteristic function given by

$$W^{\alpha}=\eta^{\alpha}-\xi^j u_j^{\alpha}\,. \tag{14}$$

A Lie symmetry generator of (8) is a one parameter Lie group transformation that leaves the given differential equation invariant under the transformation of all independent variables and dependent variables. In this paper, we will assume that X is a Lie point operator, i.e., ξ and η are functions of x and u and are independent of derivatives of u. A Lie-Bäcklund operator of the form $\tilde{X}=\eta^{\alpha}\partial/\partial u^{\alpha}+\cdots$ is called a *canonical* or *evolutionary* representation of X.

2.1 Symmetries and reductions

A one parameter Lie group of transformations that leave invariant (1) will be written as a vector field

$$\begin{aligned}X=&\tau(t,x,y,u)\partial_t+\xi(t,x,y,u)\partial_x+\eta(t,x,y,u)\partial_u\\ &+\phi(t,x,y,u)\partial_u.\end{aligned} \tag{15}$$

This would be a generator of point symmetry of the system. The tedious calculations reveal the following point symmetry generators

$$\begin{aligned}&X_1=6\mathscr{F}_1(t)\partial_x+\mathscr{F}_1'\partial_u,\\ &X_2=-36\mathscr{F}_2(t)\partial_y+\tfrac{6}{s^2}y\mathscr{F}_2'\partial_x+\tfrac{1}{s^2}y\mathscr{F}_2''\partial_u,\\ &X_3=108\mathscr{F}_3(t)\partial_t+(36x\mathscr{F}_3'-\tfrac{6}{s^2}y^2\mathscr{F}_3'')\partial_x+72y\mathscr{F}_3'\partial_y\\ &+(6x\mathscr{F}_3''-72u\mathscr{F}_3'-\tfrac{1}{s^2}y^2\mathscr{F}_3''')\partial_u\end{aligned} \tag{16}$$

where the $\mathscr{F}_i$s are arbitrary functions of t. In X_3, for e.g., $\mathscr{F}_3=t$ leads to the scaling transformation $3t\partial_t+x\partial_x+2y\partial_y-2u\partial_u$. Also, we can easily show that a linear combination of the appropriate choice of $\mathscr{F}_i$s lead to the translations

$$\mathscr{X}_1=\partial_t,\qquad \mathscr{X}_2=\partial_x,\qquad \mathscr{X}_3=\partial_y.$$

The first two, in the first instance yield the transformation $\alpha=x-ct$, $y=y$ and $u=u$, so that equation (1) become

$$-cu_{\alpha\alpha}+6u_{\alpha}^2+6uu_{\alpha\alpha}+u_{\alpha\alpha\alpha\alpha}+3s^2u_{yy}=0, \tag{17}$$

which admits a Lie point symmetry generator $X_1=\partial_{\alpha}$ and $X_2=\partial_y$. These symmetries yield the transformation $\gamma=y-k\alpha$ and $u=u$. The equation (17) under new invariant transformation becomes

$$(-ck^2+3s^2+6k^2u)u_{\gamma\gamma}+6k^2u_{\gamma}^2+k^4u_{\gamma\gamma\gamma\gamma}=0. \tag{18}$$

By integrating equation (18) with respect to γ twice, we have

$$(-ck^2+3s^2)u+3k^2u^2+k^4u''=b\gamma+d, \qquad (19)$$

where b and d are integrating constant. For $b\neq 0$, equation (19) have no symmetry generator, which implies no further reduction is possible. To this end, we consider $b=0$ with $d\neq 0$. Suppose $\mathscr{H}=u$, $\mathscr{K}=u'$ and hence

$$\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}\mathscr{H}}=\frac{u''}{u'}=\frac{d-(-ck^2+3s^2)\mathscr{H}-3k^2\mathscr{H}^2}{k^4\mathscr{K}},$$

implies

$$\frac{\mathrm{d}u}{\mathrm{d}\gamma}=\frac{\sqrt{2}}{k^2}\sqrt{du-\frac{1}{2}(-ck^2+3s^2)u^2-k^2u^3},$$

and consequently the solution of equation (1) is given as

$$\begin{aligned}&\int\frac{1}{\sqrt{du-\frac{1}{2}(-ck^2+3s^2)u^2-k^2u^3}}\mathrm{d}u\\&=\frac{\sqrt{2}}{k^2}[y-k(x-ct)]\end{aligned} \qquad (20)$$

2.2 Conservation Laws

The conserved vectors of (1) will be written $(\mathscr{T}^t,\mathscr{T}^x,\mathscr{T}^y)$, i.e.,

$$D_t\mathscr{T}^t+D_x\mathscr{T}^x+D_y\mathscr{T}^y=0 \qquad (21)$$

along the solutions of the differential equation. By omitting the calculation details, we directly write below the set of multipliers Q and their corresponding conserved vectors for the system (1).

(i) $Q_1=-\frac{1}{18s^2}y(y^2F_t^1-18s^2xF^1(t))$

$$\begin{aligned}\mathscr{T}_1^t&=-\tfrac{1}{36s^2}y(y^2F^{1'}u_x+18s^2F^1(u-xu_x)),\\ \mathscr{T}_1^x&=-\tfrac{1}{36s^2}y(u(-y^2F^{1''}+6F^{1'}(3s^2x+2y^2u_x))\\&\quad+y^2F^{1'}(u_t+2u_{xxx})\\&\quad+18s^2F^1(6u^2-xu_t-12xuu_x+2u_{xx}-2xu_{xxx})),\\ \mathscr{T}_1^y&=-3s^2xF^1(u-yu_y)-\tfrac{1}{6}y^2F^{1'}(-3u+yu_y)\end{aligned}$$

(ii) $Q_2=-\frac{1}{6s^2}(y^2F_t^2-6s^2xF^2(t))$

$$\begin{aligned}\mathscr{T}_2^t&=\tfrac{1}{12}(-\tfrac{y^2F^{2'}u_x}{s^2}-6F^2(u-xu_x)),\\ \mathscr{T}_2^x&=-\tfrac{1}{12s^2}[u(-y^2F^{2''}+6F^{2'}(s^2x+2y^2u_x))\\&\quad+y^2F^{2'}(u_t+2u_{xxx})\\&\quad+6s^2F^2(6u^2-xu_t-12xuu_x+2u_{xx}-2xu_{xxx})],\\ \mathscr{T}_2^y&=yuF^{2'}+\tfrac{1}{2}(6s^2xF^2-y^2F^{2'})u_y\end{aligned}$$

(iii) $Q_3=yF^3(t)$

$$\begin{aligned}\mathscr{T}_3^t&=\tfrac{1}{2}yF^3u_x,\\ \mathscr{T}_3^x&=\tfrac{1}{2}y(-u(F^{3'}-12F^3u_x)+F^3(u_t+2u_{xxx})),\\ \mathscr{T}_3^y&=-3s^2F^3(u-yu_y)\end{aligned}$$

(iv) $Q_4=F^4(t)$

$$\begin{aligned}\mathscr{T}_4^t&=\tfrac{1}{2}F^4u_x,\\ \mathscr{T}_4^x&=u(-\tfrac{F^{4'}}{2}+6F^4u_x)+\tfrac{1}{2}F^4(u_t+2u_{xxx})3s^2F^4u_y,\\ \mathscr{T}_4^y&=3s^2F^4u_y\end{aligned}$$

3 Conservation laws - cKP and ecKP

Here, firstly, we use a combination of the multiplier and homotopy approach, as in the previous section, to obtain the multipliers and their corresponding conservation laws for cKP and ecKP, respectively. As would be expected, they admit four independent conservation laws given below.

The CKP admits the following four conserved vectors (i)

$$\begin{aligned}\mathscr{T}_1^\tau&=\tfrac{1}{72s^2}[-2v^3\tau^2G^{1'}w_\chi\\&\quad+G^1(-36s^2vw+v(v^2\tau+36s^2\chi)w_\chi)],\\ \mathscr{T}_1^\chi&=\tfrac{1}{72s^2\tau}v(\tau(w(2v^2\tau^2G^{1''}+G^{1'}(v^2\tau-36s^2\chi\\&\quad-24v^2\tau^2w_\chi))-2v^2\tau^2G^{1'}(w_\tau+2w_{\chi\chi\chi}))\\&\quad+G^1(-216s^2\tau w^2+12w(3s^2\chi+\tau(v^2\tau+36s^2\chi)w_\chi)\\&\quad+\tau((v^2\tau+36s^2\chi)w_\tau\\&\quad-72s^2w_{\chi\chi}+2(v^2\tau+36s^2\chi)w_{\chi\chi\chi}))),\\ \mathscr{T}_1^v&=\tfrac{1}{12\tau^2}[-2v^2\tau^2G^{1'}(-3w+vw_v)\\&\quad+G^1(-3(v^2\tau+12s^2\chi)w+v(v^2\tau+36s^2\chi)w_v)].\end{aligned}$$

(ii)

$$\begin{aligned}\mathscr{T}_2^\tau&=\tfrac{1}{24s^2}[-2v^2\tau^2G^{2'}w_\chi\\&\quad+G^2(-12s^2w+(v^2\tau+12s^2\chi)w_\chi)],\\ \mathscr{T}_2^\chi&=\tfrac{1}{24s^2\tau}(\tau(w(2v^2\tau^2G^{2''}\\&\quad+G^{2'}(v^2\tau-12s^2\chi-24v^2\tau^2w_\chi))\\&\quad-2v^2\tau^2G^{2'}(w_\tau+2w_{\chi\chi\chi}))\\&\quad+G^2(-72s^2\tau w^2+12w(s^2\chi+\tau(v^2\tau+12s^2\chi)w_\chi)\\&\quad+\tau((v^2\tau+12s^2\chi)w_\tau\\&\quad-24s^2w_{\chi\chi}+2(v^2\tau+12s^2\chi)w_{\chi\chi\chi}))),\\ \mathscr{T}_2^v&=\tfrac{1}{4\tau^2}[-2v\tau^2G^{2'}(-2w+vw_v)\\&\quad+G^2(-2v\tau w+(v^2\tau+12s^2\chi)w_v)].\end{aligned}$$

(iii)

$$\begin{aligned}\mathscr{T}_3^\tau&=\tfrac{1}{2}vG^3w_\chi,\\ \mathscr{T}_3^\chi&=\tfrac{1}{2\tau}[v(-\tau wG^{3'}+G^3(w(1+12\tau w_\chi)\\&\quad+\tau(w_\tau+2w_{\chi\chi\chi})))],\\ \mathscr{T}_3^v&=-\tfrac{1}{\tau^2}(3s^2G^3(w-vw_v)).\end{aligned}$$

(iv)

$$\begin{aligned}\mathscr{T}_4^\tau&=\tfrac{1}{2}G^4w_\chi,\\ \mathscr{T}_4^\chi&=\tfrac{1}{2\tau}[-\tau wG^{4'}+G^4(w(1+12\tau w_\chi)\\&\quad+\tau(w_\tau+2w_{\chi\chi\chi}))],\\ \mathscr{T}_4^v&=\tfrac{1}{\tau^2}(3s^2G^4w_v).\end{aligned}$$

The corresponding multipliers are

$$\begin{aligned}\bar{Q}_1&=\tfrac{1}{36s^2}v[36G^1(\tau)\chi s^2-2v^2\tau^2G_\tau^1+v^2\tau G^1],\\ \bar{Q}_2&=\tfrac{1}{12s^2}[12G^2(\tau)\chi s^2-2v^2\tau^2G_\tau^2+v^2\tau G^2]\\ \bar{Q}_3&=G^3(\tau)v,\\ \bar{Q}_4&=G^4(\tau)\end{aligned}.$$

Similarly, the corresponding multipliers and conserved vectors for ecKP are

(i) $\mathscr{Q}_1 = \frac{1}{36s^2}v[-2T^2H^1_Tv^2+2a^2H^1_Tv^2+36H^1(T)\zeta s^2+TH^1v^2]$

$$\begin{aligned}\mathscr{T}_1^T &= \tfrac{1}{72s^2}[-36s^2vhH^1+v((36s^2\zeta+Tv^2)H^1\\&+2(a^2-T^2)v^2H^{1'})h_\zeta],\\ \mathscr{T}_1^\zeta &= \tfrac{1}{432s^4(a^2-T^2)}v(-1296a^2s^4h^2H^1\\&+1296s^4T^2h^2H^1-6s^2h(H^1(36s^2T\zeta\\&+7a^2v^2-12(a^2-T^2)(36s^2\zeta+Tv^2)h_\zeta)\\&-(a^2-T^2)(2(-a^2+T^2)v^2H^{1''}\\&+H^{1'}(-36s^2\zeta+Tv^2+24(a^2-T^2)v^2h_\zeta)))\\&+2(a^2-T^2)v^2H^{1'}(6s^2(a^2-T^2)h_T+a^2v^2h_\zeta\\&+12s^2(a^2-T^2)h_{\zeta\zeta\zeta})\\&+H^1(6s^2(a^2-T^2)(36s^2\zeta\\&+Tv^2)h_T+a^2v^2(36s^2\zeta\\&+Tv^2)h_\zeta+12s^2(a^2-T^2)(-36s^2h_{\zeta\zeta}\\&+(36s^2\zeta+Tv^2)h_{\zeta\zeta\zeta}))),\\ \mathscr{T}_1^v &= \tfrac{1}{12(a^2-T^2)}(3h((12s^2\zeta+Tv^2)H^1\\&+2(a^2-T^2)v^2H^{1'})-v((36s^2\zeta\\&+Tv^2)H^1+2(a^2-T^2)v^2H^{1'})h_v)\end{aligned}$$

(ii)

$$\begin{aligned}\mathscr{Q}_2 &= \tfrac{1}{12s^2}[-2T^2H^2_Tv^2+2a^2H^2_Tv^2+12H^2(T)\zeta s^2\\&-TH^2v^2]\end{aligned}$$

$$\begin{aligned}\mathscr{T}_2^T &= \tfrac{1}{24s^2}[-12s^2hH^2+((12s^2\zeta+Tv^2)H^2\\&+2(a^2-T^2)v^2H^{2'})h_\zeta],\\ \mathscr{T}_2^\zeta &= \tfrac{1}{144s^4(a^2-T^2)}(-432a^2s^4h^2H^2+432s^4T^2h^2H^2\\&-6s^2h(3H^2(4s^2T\zeta+a^2v^2-4(a^2-T^2)(12s^2\zeta\\&+Tv^2)h_\zeta)-(a^2-T^2)(2(-a^2+T^2)v^2H^{2''}\\&+H^{2'}(-12s^2\zeta+Tv^2+24(a^2-T^2)v^2h_\zeta)))\\&+2(a^2-T^2)v^2H^{2'}(6s^2(a^2-T^2)h_T+a^2v^2h_\zeta\\&+12s^2(a^2-T^2)h_{\zeta\zeta\zeta})\\&+H^2(6s^2(a^2-T^2)(12s^2\zeta+Tv^2)h_T+a^2v^2(12s^2\zeta\\&+Tv^2)h_\zeta+12s^2(a^2-T^2)(-12s^2h_{\zeta\zeta}\\&+(12s^2\zeta+Tv^2)h_{\zeta\zeta\zeta}))),\\ \mathscr{T}_2^v &= \tfrac{1}{4(a^2-T^2)}[2vh(TH^2+2(a^2-T^2)H^{2'})\\&-((12s^2\zeta+Tv^2)H^2+2(a^2-T^2)v^2H^{2'})h_v]\end{aligned}$$

(iii) $\mathscr{Q}_3 = vH^3(T)$

$$\begin{aligned}\mathscr{T}_3^T &= \tfrac{1}{2}vH^3h_\zeta,\\ \mathscr{T}_3^\zeta &= \tfrac{1}{12s^2(a^2-T^2)}[v(-6s^2h((a^2-T^2)H^{3'}\\&+H^3(T-12(a^2-T^2)h_\zeta))+H^3(6s^2(a^2-T^2)h_T\\&+a^2v^2h_\zeta+12s^2(a^2-T^2)h_{\zeta\zeta\zeta}))],\\ \mathscr{T}_3^v &= \tfrac{1}{a^2-T^2}(3s^2H^3(h-vh_v))\end{aligned}$$

(iv) $\mathscr{Q}_4 = H^4(T)$

$$\begin{aligned}\mathscr{T}_4^T &= \tfrac{1}{2}H^4h_\zeta,\\ \mathscr{T}_4^\zeta &= \tfrac{1}{12s^2(a^2-T^2)}[-6s^2h((a^2-T^2)H^{4'}\\&+H^4(T-12(a^2-T^2)h_\zeta))+H^4(6s^2(a^2-T^2)h_T\\&+a^2v^2h_\zeta+12s^2(a^2-T^2)h_{\zeta\zeta\zeta})],\\ \mathscr{T}_4^v &= -\tfrac{1}{a^2-T^2}(3s^2H^4h_v)\end{aligned}$$

3.1 Conservation laws via transformations

The transformation (2) and transformation (4) transform the equation (1) into cKP and ecKP, respectively. By exploiting this fact, we show that one can avoid the lengthly procedure and can directly obtain the conservation laws by using these transformations. It is sufficient to calculate the multipliers via the transformations and as there is a one to one corresponding between the conserve vector and the multiplier, the conserved vectors can be constructed directly from the homotopy integral. For illustration, we transform some multipliers.

(a) Transformation of multipliers from KP to cKP

(i) The transformation of Q_1 is

$$\begin{aligned}Q_1 &= -\tfrac{1}{18s^2}y(y^2F_t^1-18s^2xF^1(t))\\&= -\tfrac{1}{18s^2}(\tau v)[\tau^2v^2F_\tau^1-18s^2(\chi+\tfrac{\tau v^2}{12s^2})F^1(\tau)]\\&= -\tfrac{1}{36s^2}(\tau v)(2\tau^2v^2F_\tau^1-36s^2\chi F^1-3\tau v^2F^1).\end{aligned}$$

If we let $\tau F^1(\tau) = G^1(\tau)$, we get

$$\begin{aligned}Q_1 &= -\tfrac{1}{36s^2}(\tau v)(2\tau^2v^2(\tfrac{1}{\tau}G_\tau^1-\tfrac{1}{\tau^2}G^1)\\&-\tfrac{36s^2}{\tau}\chi G^1-3v^2G^1)\\&= \tfrac{1}{36s^2}v[36G^1\chi s^2-2v^2\tau^2G_\tau^1+v^2\tau G^1],\end{aligned}$$

which matches $\bar{Q}_1$ of cKP.

(ii) The transformation of Q_2 is

$$\begin{aligned}Q_2 &= -\tfrac{1}{6s^2}(y^2F_t^2-6s^2xF^2(t))\\&= -\tfrac{1}{6s^2}((\tau v)^2F_\tau^2-6s^2(\chi+\tfrac{\tau v^2}{12s^2})F^2(\tau))\\&= \tfrac{1}{12s^2}[-2\tau^2v^2F_\tau^2+12s^2(\chi+\tfrac{\tau v^2}{12s^2})F^2]\end{aligned}$$

where, if we let $F^2(\tau) = G^2(\tau)$, we obtain

$$Q_2 = \frac{1}{12s^2}[12G^2(\tau)\chi s^2-2v^2\tau^2G_\tau^2+v^2\tau G^2],$$

which is the same as $\bar{Q}_2$ of cKP.

(b) Transformation of multipliers from KP to ecKP

The multiplier Q_2 of KP leads to

$$\begin{aligned}Q_2 &= -\tfrac{1}{6s^2}(y^2F_t^2-6s^2xF^2(t))\\&= -\tfrac{1}{6s^2}[(\sqrt{T^2-a^2}v)^2F_T^2-6s^2(\zeta-\tfrac{Tv^2}{12s^2})F^2(T)]\\&= \tfrac{1}{12s^2}(-2T^2v^2F_T^2+2a^2v^2F_T^2\\&+12s^2\zeta F^2-Tv^2F^2)\end{aligned}$$

Again, if we let $F^2(T) = H^2(T)$, we obtain

$$Q_2 = \frac{1}{12s^2}[-2T^2H_T^2v^2 + 2a^2H_T^2v^2 + 12H^2(T)\zeta s^2 - TH^2v^2],$$

which is $\mathscr{Q}_2$ of ecKP.

It is clear, therefore, that one can obtain the other multipliers and hence the conservation laws for cKP and ecKP from that of the KP.

4 Conclusion

In this study, via a knowledge of the Lie symmetry generators, we have successively reduced the fourth order KP to a first order ODE and further obtained an exact solution. The conservation laws for KP, cKP, and ecKP have been obtained independently. We have demonstrated that the transformations used to transform KP to cKP and ecKP can, in fact, be used to obtain the conservation laws for cKP and ecKP. This fact is significant in the sense that the interesting features, well known or otherwise, of the KP, like the exact solutions, multipliers and conservation laws can easily be achieved for the cKP and ecKP.

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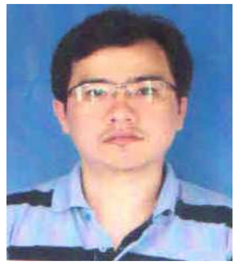

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