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The Degasperis-Procesi Lagrangian Density:Functional Fractional Formulation

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Abstract: In this article, we developed the Hamiltonian formulation of third-order continuous field systems. A fractional variational principle based on the combined Riemann–Liouville fractional derivative operator is established. The fractional variational principle is used to obtain fractional Euler equations and fractional Hamilton equations. The Hamilton equations of motion are also found to agree with the Euler-Lagrange equations for these systems. Finally, we study an example to elucidate the results.
Keywords: Functional fractional derivatives, Degasperis-Procesi lagrangian, Hamiltonian formulation, Euler-Lagrange equations.

1 Introduction

Fractional calculus is a branch of mathematics that deals with generalization of well-known operations of differentiations and integrations to an arbitrary non-integer order, which can be a real non-integer or even an imaginary number.

Nowadays, physicists are increasingly using this powerful instrument to deal with some problems that in the classical sense have not been solvable. In this context, in describing and explaining some physically complex systems, the fractional calculus has emerged as one of the most powerful and commonly useful tools. The fractional Lagrangian, fractional Hamiltonian, and fractional mechanics have been developed by Riewe [1,2]. Lagrangian mechanics, with a fractional derivative, leads directly to equations of motion with non-conservative classical forces, according to his findings. In physics, engineering and applied mathematics, fractional derivatives have played an important role [3–9].

In the meantime, the Euler-Lagrange equations for unrestricted and restricted fractional variational problems were presented by Agrawal [10,11]. The resulting equations are found to be similar to those which contain integral-order derivatives for variational problems. In other words, when the derivative of fractional order is substituted by the derivative of integral order, the results of the fractional calculus of variations are reduced to those obtained from the traditional fractional calculus of variations. This approach is extended to classical fractional derivative fields [12].
New studies of systems with higher-order fractional derivatives have been discussed in references [13-17] over the past decades, and an integral quantization path has been discovered for both conservative and non-conservative systems.

Treatment for third- order derivative systems was presented in this paper; the researchers generalized the Hamiltonian formulation for third-order derivative continuous systems and applied this formulation to Lee-generalized Wick's electrodynamics.

Recently, Muslih and El-Zalan have suggested novel formulations to treat discrete systems using higher-order Lagrangian equations through the path of integral quantization to obtain canonical coordinates without the need for integration over higher-order derivatives [18-20]. More specifically, to Lagrangian equations of the second order, they have applied formalism. The application of fractional calculus in continuum mechanics, physics, electromagnetics, and signal processing has been emphasized in recent symposium proceedings and monographs. We are specifying some of these applications here.

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In order to elucidate viscous interactions between fluid and solid structures, the first, fractional calculus, is used. Using the theory of Blot, for a slab of cancellous bone in the elastic frame [21], we derive reflection and transmission scattering operators. In the theory of viscoelasticity, the method of fractional derivatives is helpful because it allows the possibility of obtaining constitutive equations for the elastic complex modulus of viscoelastic materials using limited parameters determined experimentally. Furthermore, for several models of viscoelastic substances, the fractional derivative method has been used to study impedances [22]. The application of fractional calculus to the viscous-diffusion equation within a semi-infinite space has revealed explicit analytical (fractional) solutions for fluid velocity and shear stress throughout the domain in conjunction with the Laplace transformation method. The fractional methodology has been shown to be much simpler and more powerful than existing techniques after validation after comparing the fractional results for boundary shear-stress and fluid velocity with the current analytical results for the first and second Stokes problems [23].
The method of functional derivatives has been successfully used to solve different types of integer and fractional equations. This work aims to a generalization of the aforementioned work on Hamilton's equation for Degasperis-Procesi Field using functional fractional derivatives. The remainder of this paper is structured as follows: The definitions of fractional derivatives are briefly discussed in Section 2. Fractional Degasperis-Procesi Lagrangian density will be presented in Section 3. Fractional Euler- Lagrange equation for Degasperis-Procesi Lagrangian density in Section 4. The fractional form of the Euler-Lagrangian equation in the Degasperis-Procesi field functional derivative terms is investigated in Section 5. Subsequently, the fractional form of the Euler-Lagrange equation in terms of momentum density for the Degasperis-Procesi field is obtained in Section 6. Then, we derived the Hamilton Formulation for Degasperis-Procesi field with fractional derivatives of the third order in section 7. In Section 8, one example is described. The work concludes with some concluding observations.

2 Basic definitions

In this part of study, we briefly present some fundamental definitions used in this work. The left and right Riemann-Liouville fractional derivatives are defined as follows: The left Riemann- Liouville fractional derivative [3]

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha+1} f(\tau)d\tau \tag{1}$$

The right Riemann- Liouville fractional derivative

$$ {}_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_a^x (\tau-x)^{n-\alpha+1} f(\tau)d\tau. \tag{2}$$

where Γ denotes the Gamma function, and α is the order of the derivative such that $n-1<\alpha<n$. If α is an integer, these derivatives are defined in the usual sense, i.e.

$$ {}_aD_x^\alpha f(x) = \left(\frac{d}{dx}\right)^n f(x) \tag{3}$$

$$ {}_aD_t^\alpha f(x) = \left(\frac{d}{dx}\right)^n f(t) \qquad \alpha = 1,2,.. \tag{4}$$

3 Fractional Degasperis-Procesi lagrangian densities

The Lagrangian density is [24]:

$$ L(\psi_x,\psi_{xx},\psi_t,\psi_{xxt},\psi_{xxx}) = -\frac{1}{2}\,\psi_x\psi_t + \frac{1}{2}\,\psi_x\psi_{xxt} - \frac{1}{3}\psi_x^4 + \frac{5}{8}\psi_x\psi_{xx}^2 - \frac{1}{16}\psi_x^2\,\psi_{xxx}$$

Using the definition of the fractional derivative of Riemann-Liouville, the fractional density of Degasperis-Process Lagrangian takes the form of:

$$L = \begin{bmatrix} -\frac{1}{2}\left[{}_aD_x^{\alpha}\psi(x,t)\ {}_aD_t^{\alpha}\psi(x,t) + {}_xD_b^{\beta}\psi(x,t)\ {}_tD_b^{\beta}\psi(x,t)\right] + \\ \frac{1}{2}\left[{}_aD_x^{\alpha}\psi(x,t)\ {}_aD_x^{2\alpha}\ {}_aD_t^{\alpha}\psi(x,t) + {}_xD_b^{\beta}\psi(x,t)\ {}_xD_b^{2\beta}\ {}_tD_b^{\beta}\psi(x,t)\right] \\ -\frac{1}{3}\left[\left({}_aD_x^{\alpha}\psi(x,t)\right)^4 + \left({}_xD_b^{\beta}\psi(x,t)\right)^4\right] + \\ \frac{5}{8}\left[{}_aD_x^{\alpha}\psi(x,t)\left({}_aD_x^{2\alpha}\psi(x,t)\right)^2 + {}_xD_b^{\beta}\psi(x,t)\left({}_xD_b^{2\beta}\psi(x,t)\right)^2\right] \\ -\frac{1}{16}\left[\left({}_aD_x^{\alpha}\psi(x,t)\right)^2\ {}_aD_x^{3\alpha}\psi(x,t) + \left({}_xD_b^{\beta}(x,t)\right)^2\ {}_xD_b^{3\alpha}\psi(x,t)\right] \end{bmatrix} \quad (5)$$

If we take α =β = 1, then we obtain the corresponding equations of the classical case of integer order.

4 Fractional Euler – Lagrange equation for Degasperis-Procesi lagrangian density

Consider the action function of the form**:**

$$\mathcal{L} = \mathcal{L}\begin{pmatrix} \psi(x,t),\ {}_aD_t^{\alpha}\psi(x,t),\ {}_tD_b^{\beta}\psi(x,t),\ {}_aD_x^{\alpha}\psi(x,t),\ {}_xD_b^{\beta}\psi(x,t) \\ {}_aD_x^{\alpha}\ {}_aD_x^{\alpha}\psi(x,t),\ {}_xD_b^{\beta}\ {}_xD_b^{\beta}\psi(x,t),\ {}_aD_x^{\alpha}\ {}_aD_x^{\alpha}\ {}_aD_t^{\alpha}\psi(x,t) \\ {}_xD_b^{\beta}\ {}_xD_b^{\beta}\ {}_tD_b^{\beta}\psi(x,t),\ {}_aD_x^{\alpha}\ {}_aD_x^{\alpha}\ {}_aD_x^{\alpha}\psi(x,t) \\ {}_xD_b^{\beta}\ {}_xD_b^{\beta}\ {}_xD_b^{\beta}\psi(x,t) \end{pmatrix} \quad (6)$$

Recently,(**Agrawal)**obtained the fractional Euler–Lagrange equations for variational problems. Consider the action function of the form:

δj

$$= \int \mathcal{L}\begin{bmatrix} \frac{\partial\mathcal{L}}{\partial\psi(x,t)}\delta\left(\psi(x,t)\right) + \\ \frac{\partial\mathcal{L}}{\partial\ {}_aD_t^{\alpha}\psi(x,t)}\delta\left({}_aD_t^{\alpha}\psi(x,t)\right) + \frac{\partial\mathcal{L}}{\partial\ {}_tD_b^{\beta}\psi(x,t)}\delta\left({}_tD_b^{\beta}\psi(x,t)\right) + \\ \frac{\partial\mathcal{L}}{\partial\ {}_aD_x^{\alpha}\psi(x,t)}\delta\left({}_aD_x^{\alpha}\psi(x,t)\right) + \frac{\partial\mathcal{L}}{\partial\ {}_xD_b^{\beta}(x,t)}\delta\left({}_xD_b^{\beta}(x,t)\right) + \\ \frac{\partial\mathcal{L}}{\partial\ {}_aD_x^{\alpha}\ {}_aD_x^{\alpha}\psi(x,t)}\delta\left({}_aD_x^{\alpha}\ {}_aD_x^{\alpha}\psi(x,t)\right) + \frac{\partial\mathcal{L}}{\partial\ {}_xD_b^{\beta}\ {}_xD_b^{\beta}\psi(x,t)}\delta\left({}_xD_b^{\beta}\ {}_xD_b^{\beta}\psi(x,t)\right) + \\ \frac{\partial\mathcal{L}}{\partial\ {}_aD_x^{\alpha}\ {}_aD_x^{\alpha}\ {}_aD_t^{\alpha}\psi(x,t)}\delta\left({}_aD_x^{\alpha}\ {}_aD_x^{\alpha}\ {}_aD_t^{\alpha}\psi(x,t)\right) + \\ \frac{\partial\mathcal{L}}{\partial\ {}_xD_b^{\alpha}\ {}_xD_b^{\alpha}\ {}_tD_b^{\alpha}\psi(x,t)}\delta\left({}_xD_b^{\beta}\ {}_xD_b^{\beta}\ {}_tD_b^{\beta}\psi(x,t)\right) + \\ \frac{\partial\mathcal{L}}{\partial\ {}_aD_x^{\alpha}\ {}_aD_x^{\alpha}\ {}_aD_x^{\alpha}\psi(x,t)}\delta\left({}_aD_x^{\alpha}\ {}_aD_x^{\alpha}\ {}_aD_x^{\alpha}\psi(x,t)\right) + \\ \frac{\partial\mathcal{L}}{\partial\ {}_xD_b^{\alpha}\ {}_xD_b^{\alpha}\ {}_xD_b^{\alpha}\psi(x,t)}\delta\left({}_xD_b^{\beta}\ {}_xD_b^{\beta}\ {}_xD_b^{\beta}\psi(x,t)\right) + \end{bmatrix} dx^3 \quad (7)$$

The Euler Lagrange equations for the fractional calculus of variations problem with third order derivatives are obtained as:

$$\left[\begin{array}{c}
\frac{\partial \mathcal{L}}{\partial \psi(x,t)} - {}_{a}D_{x}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{x}^{\alpha}\psi(x,t)} - {}_{x}D_{b}^{\beta}\frac{\partial \mathcal{L}}{\partial\, {}_{x}D_{b}^{\beta}(x,t)} + \\
{}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\psi(x,t)} + {}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\frac{\partial \mathcal{L}}{\partial\, {}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\psi(x,t)} - \\
{}_{a}D_{t}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{t}^{\alpha}\psi(x,t)} - {}_{x}D_{b}^{\beta}\frac{\partial \mathcal{L}}{\partial\, {}_{x}D_{b}^{\beta}\psi(x,t)} - {}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\,{}_{a}D_{t}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi(x,t)} - \\
{}_{x}D_{b}^{\alpha}\,{}_{x}D_{b}^{\alpha}\,{}_{t}D_{t}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\psi(x,t)} - {}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\psi(x,t)} - \\
{}_{x}D_{b}^{\alpha}\,{}_{x}D_{b}^{\alpha}\,{}_{x}D_{b}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\psi(x,t)}
\end{array}\right] = 0 \qquad (8)$$

For $\alpha = \beta = 1$, Eq. (9) become

$$\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \psi_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \psi_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xx}}\right) - \frac{\partial^3}{\partial x^2 \partial t}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xxt}}\right) - \frac{\partial^3}{\partial x^3}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xxx}}\right) = 0$$

5 Fractional form of the Euler-Lagrangian equation in terms of functional derivative for Degasperis-Procesi field

Integrating by parts the indicated terms in Eq. (7) with respect to space, we obtain the following:

$$\int dt \int \left[\begin{array}{c}
\frac{\partial \mathcal{L}}{\partial \psi(x,t)} - {}_{a}D_{x}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{x}^{\alpha}\psi(x,t)} - {}_{x}D_{b}^{\beta}\frac{\partial \mathcal{L}}{\partial\, {}_{x}D_{b}^{\beta}(x,t)} + \\
{}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\psi(x,t)} + {}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\frac{\partial \mathcal{L}}{\partial\, {}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\psi(x,t)} - \\
{}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\,{}_{a}D_{t}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi(x,t)} + {}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\,{}_{t}D_{b}^{\beta}\frac{\partial \mathcal{L}}{\partial\, {}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\psi(x,t)} + \\
{}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\psi(x,t)} + {}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\frac{\partial \mathcal{L}}{\partial\, {}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\psi(x,t)}
\end{array}\right] d\tau\delta\psi$$

$$+ \int dt \int \left[\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{t}^{\alpha}\psi(x,t)}\, {}_{a}D_{t}^{\alpha}\psi(x,t) + \frac{\partial \mathcal{L}}{\partial\, {}_{t}D_{b}^{\beta}\psi(x,t)}\, {}_{t}D_{b}^{\beta}\psi(x,t)\right] d\tau = 0 \qquad (9)$$

We can take the integration over space $d\tau$ in the previous equation and convert it into summation:

$$\sum_i \left[\begin{array}{c}
\frac{\partial \mathcal{L}}{\partial \psi(x,t)} - {}_{a}D_{x}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{x}^{\alpha}\psi(x,t)} - {}_{x}D_{b}^{\beta}\frac{\partial \mathcal{L}}{\partial\, {}_{x}D_{b}^{\beta}(x,t)} + \\
{}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\psi(x,t)} + {}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\frac{\partial \mathcal{L}}{\partial\, {}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\psi(x,t)} - \\
{}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\,{}_{a}D_{t}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi(x,t)} - \\
{}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\,{}_{t}D_{b}^{\beta}\frac{\partial \mathcal{L}}{\partial\, {}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\,{}_{t}D_{b}^{\beta}\psi(x,t)} - \\
{}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\,{}_{a}D_{x}^{\alpha}\psi(x,t)} - \\
{}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\frac{\partial \mathcal{L}}{\partial\, {}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\,{}_{x}D_{b}^{\beta}\psi(x,t)}
\end{array}\right]_i \delta\psi_i\delta\tau_i +$$

$$\sum_i \left[\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{t}^{\alpha}\psi(x,t)}\right]_i \delta\left({}_{a}D_{t}^{\alpha}\psi_i(x,t)\right)_i \delta\tau_i + \sum_i \left[\frac{\partial \mathcal{L}}{\partial\, {}_{t}D_{b}^{\beta}\psi(x,t)}\right]_i \delta\left({}_{t}D_{b}^{\beta}\psi(x,t)\right)_i \delta\tau_i = 0 \qquad (10)$$

We can rewrite Eq. (10)in terms of the Lagrangian density as:

$$\sum_i [\delta\mathcal{L}]_i\, \delta\tau_i = 0 \tag{11}$$

where the left-hand side in Eqs. (10) and (11)represents the variation of L (i.e., δL), which is now produced by independent variations in $\delta\psi_i, \delta\left({}_aD_t^\alpha\psi\right)_i and\ \delta\left({}_tD_b^\beta\psi\right)_i$. Suppose all $\delta\psi_i, \delta(\psi_t)_i$ are zeros except for a particular $\delta\psi_j$. It is natural to define the functional derivative of the Lagrangian (∂L)with respect to $\delta\psi_i, \delta\left({}_aD_t^\alpha\psi\right)_i and\ \delta\left({}_tD_b^\beta\psi\right)$ for a point in the j-th cell to the ratio of δL to $\delta\psi_j$ [25].

$$\frac{\partial\ L}{\partial\ \psi_t} = \lim_{\delta\tau_j\to 0}\frac{\delta L}{\delta\psi_{t_j}\delta_{\tau j}}$$

Using Eq. (10), and noting that the left-hand side represents $\delta\mathcal{L}$, we obtain the following:

$$\frac{\partial\ L}{\partial\ \psi} = \left[\begin{array}{c} \frac{\partial\mathcal{L}}{\partial\psi(x,t)} - {}_aD_x^\alpha\frac{\partial\mathcal{L}}{\partial\ {}_aD_x^\alpha\psi(x,t)} - {}_xD_b^\beta\frac{\partial\mathcal{L}}{\partial\ {}_xD_b^\beta(x,t)} + \\ {}_aD_x^\alpha\ {}_aD_x^\alpha\frac{\partial\mathcal{L}}{\partial\ {}_aD_x^\alpha\ {}_aD_x^\alpha\psi(x,t)} + {}_xD_b^\beta\ {}_xD_b^\beta\frac{\partial\mathcal{L}}{\partial\ {}_xD_b^\beta\ {}_xD_b^\beta\psi(x,t)} - \\ {}_aD_x^\alpha\ {}_aD_x^\alpha\ {}_aD_t^\alpha\frac{\partial\mathcal{L}}{\partial\ {}_aD_x^\alpha\ {}_aD_x^\alpha\ {}_aD_t^\alpha\psi(x,t)} - \\ {}_xD_b^\beta\ {}_xD_b^\alpha\ {}_tD_b^\beta\frac{\partial\mathcal{L}}{\partial\ {}_xD_b^\alpha\ {}_xD_b^\alpha\ {}_tD_b^\beta\psi(x,t)} - \\ {}_aD_x^\alpha\ {}_aD_x^\alpha\ {}_aD_x^\alpha\frac{\partial\mathcal{L}}{\partial\ {}_aD_x^\alpha\ {}_aD_x^\alpha\ {}_aD_x^\alpha\psi(x,t)} - \\ {}_xD_b^\beta\ {}_xD_b^\beta\ {}_xD_b^\beta\frac{\partial\mathcal{L}}{\partial\ {}_xD_b^\beta\ {}_xD_b^\beta\ {}_xD_b^\beta\psi(x,t)} \end{array}\right] \tag{12}$$

$$\frac{\partial\ L}{\partial\ {}_aD_t^\alpha\psi} = \lim_{\delta\tau_j\to 0}\frac{\delta L}{\delta\tau_j\ {}_aD_t^\alpha\psi_j} = \frac{\partial\mathcal{L}}{\partial\ {}_aD_t^\alpha\psi} \tag{13}$$

$$\frac{\partial\ L}{\partial\ {}_tD_b^\beta\psi} = \lim_{\delta\tau_j\to 0}\frac{\delta L}{\delta\tau_j\ {}_tD_b^\beta\psi_j} = \frac{\partial\mathcal{L}}{\partial\ {}_tD_b^\beta\psi} \tag{14}$$

Using Eqs. (12) and (13) and (14), we can rewrite the Eq. (8)Euler-Lagrange equation in terms of the Lagrangian L using the functional derivative in the form of the following:

$$\frac{\partial\ L}{\partial\ \psi} - {}_aD_t^\alpha\left(\frac{\partial\ L}{\partial\ {}_aD_t^\alpha\psi}\right) - {}_tD_b^\beta\left(\frac{\partial\ L}{\partial\ {}_tD_b^\beta\psi}\right) = 0 \tag{15}$$

For α, β →1, Eq. (15) reduces to the usual Euler-Lagrange equation for the classical fields[25]. With the help of Eqs. (12) and (15), we can write the variation of the Lagrangian in terms of functional derivatives and variations of $\psi,\ {}_aD_t^\alpha\psi\ and\ \ {}_tD_b^\beta\psi$.

$$\delta L = \int \left[\frac{\partial L}{\partial \psi} \delta\psi + \frac{\partial L}{\partial \ {}_aD_t^{\alpha}\psi} \delta \ {}_aD_t^{\alpha}\psi + \frac{\partial L}{\partial \ {}_tD_b^{\beta}\psi} \delta \ {}_tD_b^{\beta}\psi \right] \mathrm{d}^3\mathrm{r} \tag{16}$$

6 Fractional form of the Euler-Lagrange equation in terms of momentum density for Degasperis-Procesi field

The right side fractional form of momentum can be written as follows[26] :

$$P_j^{a} = \frac{\delta L}{\delta \ {}_aD_t^{\alpha}\psi_j} \tag{17}$$

Using Eqs. (12) and (14), we obtain the following:

$$\pi_j = \frac{\partial L}{\partial \ {}_aD_t^{\alpha}\psi_j} = \frac{\partial \mathcal{L}}{\partial \ {}_aD_t^{\alpha}\psi_j} \tag{18}$$

From Eq. (18), we can define the right-side form of momentum density as follows:

$${}_aD_t^{\alpha}(\pi_{\alpha})_j = {}_aD_t^{\alpha}\left[\frac{\partial L}{\partial \ {}_aD_t^{\alpha}\psi_j}\right] \tag{19}$$

Repeating the same steps above for the left side fractional form of momentum density π_{β} , we obtain the following:

$$(\pi_{\beta})_j = \frac{\partial L}{\partial \ {}_tD_b^{\beta}\psi_j} = \frac{\partial \mathcal{L}}{\partial \ {}_tD_b^{\beta}\psi_j} \tag{20}$$

Take the right fractional derivative for Eq. (20) We get the following:

$${}_tD_b^{\beta}(\pi_{\beta})_j = {}_aD_t^{\alpha}\left[\frac{\partial L}{\partial \ {}_tD_b^{\beta}\psi_j}\right] \tag{21}$$

Substituting Eqs. (19)and (21)into Eq. (15), we obtain the following:

$$\frac{\partial L}{\partial \psi} = -\left[{}_aD_t^{\alpha}\pi_{\alpha} + {}_tD_b^{\beta}\pi_{\beta}\right] \tag{22}$$

The above equation represents the Euler-Lagrange equation's fractional form in terms of momentum density and Lagrangian's functional derivative.

7 Hamilton formulation for Degasperis-Procesi field with third order fractional derivatives

We start by the general definition of the Hamiltonian density $\mathcal{H}$ in fractional form as:

$$\mathcal{H} = \pi_{\alpha} \ {}_aD_t^{\alpha}\psi(x,t) + \pi_{\beta} \ {}_xD_b^{\beta}\psi(x,t) - \mathcal{L} \tag{23}$$

Hamiltonian H can be also written in terms of Hamiltonian density $\mathcal{H}$ as:

$$H = \sum_i \mathcal{H}_i \delta\tau_i \tag{24}$$

Substituting Eqs. (23) into Eq. (24), one gets:

$$H = \sum_i \left(\pi_\alpha \; {}_aD_t^\alpha \psi(x,t) + \pi_\beta \; {}_xD_b^\beta \psi(x,t)\right)\delta\tau_i - \sum_i \mathcal{L}_i \delta\tau_i \tag{25}$$

We write H as the hamiltonian density H volume integral and assume that the cells are small enough to ignore the difference between a volume integral and the corresponding summation of the cell; then we will have:

$$H = \int \left[\pi_\alpha \; {}_aD_t^\alpha \psi(x,t) + \pi_\beta \; {}_xD_b^\beta \psi(x,t)\right] \, \mathrm{d}^3\mathrm{r} - \int \mathcal{L} \; \mathrm{d}^3\mathrm{r} \tag{26}$$

We can rewrite Eq. (26) as:

$$H = \int \left[\pi_\alpha \; {}_aD_t^\alpha \psi(x,t) + \pi_\beta \; {}_xD_b^\beta \psi(x,t)\right] \, \mathrm{d}^3\mathrm{r} - L \tag{27}$$

Now, take the variation of H, we get:

$$\delta H = \int \delta\left[\pi_\alpha \; {}_aD_t^\alpha \psi(x,t) + \pi_\beta \; {}_tD_b^\beta \psi(x,t)\right] \, \mathrm{d}^3\mathrm{r} - \delta L \tag{28}$$

Using Eq. (19) and Eq. (21),we rewrite the variation of Lagrangian given by Eq. (22) as:

$$\delta L = \int \left[\left[{}_aD_t^\alpha \pi_\alpha \dot{+} \; {}_tD_b^\beta \pi_\beta\right]\delta\psi + \pi_\alpha \delta \; {}_aD_t^\alpha \psi + \pi_\beta \delta \; {}_tD_b^\beta \psi\right] \, \mathrm{d}^3\mathrm{r} \tag{29}$$

The above equation can be arranged as:

$$\delta L = \int \begin{pmatrix} \left[{}_aD_t^\alpha \pi_\alpha + \; {}_tD_b^\beta \pi_\beta\right]\delta\psi + \delta\left[\pi_\alpha \; {}_aD_t^\alpha \psi + \pi_\beta \; {}_tD_b^\beta \psi\right] \\ \left[-\left[{}_aD_t^\alpha \psi + \; {}_tD_b^\beta \psi\right]\delta\pi\right] \end{pmatrix} \, \mathrm{d}^3\mathrm{r} \tag{30}$$

Substituting Eq. (17) into Eq. (19), one gets:

$$\delta H = \int \left[-\left[{}_aD_t^\alpha \pi_\alpha + \; {}_tD_b^\beta \pi_\beta\right]\delta\psi + \left[{}_aD_t^\alpha \psi + \; {}_tD_b^\beta \psi\right]\delta\pi\right] \, \mathrm{d}^3\mathrm{r} \tag{31}$$

As with the Lagrangian density, we now assume that H depends on ψ and $\pi_\alpha$and π_β. We can then follow the same procedure we used earlier to get expressions for the functional derivatives of the Lagrangian in terms of ordinary derivatives of the Lagrangian density as follows in cases 1 and 2.

Case1: All variables are independent ψ , π_α, π_β.

$$\delta H = \int \left[\frac{\partial H}{\partial \psi}\delta\psi + \frac{\partial H}{\partial \pi_\alpha}\delta\pi_\alpha + \frac{\partial H}{\partial \pi_\beta}\delta\pi_\beta\right] \, \mathrm{d}^3\mathrm{r} \tag{32}$$

Comparing Eq. (31)with Eq. (32),we get Hamilton's equations of motion in a form of fractional functional derivatives:

$$\frac{\partial H}{\partial \psi} = {}_aD_t^{\alpha}\pi_{\alpha} + {}_tD_b^{\beta}\pi_{\beta} \tag{33}$$

$$\frac{\partial H}{\partial \pi_{\alpha}} = {}_aD_t^{\alpha}\psi \tag{34}$$

$$\frac{\partial H}{\partial \pi_{\beta}} = {}_tD_b^{\beta}\psi \tag{35}$$

By analogy with Eq. (12) for functional derivative of the Lagranian in terms of fractional derivative of the Lagrangian density, we can simply define the functional derivative of *H* in terms of the fractional derivative of Hamiltonian density with respect to the general variable field φ [26] as :

$$\frac{\partial H}{\partial \varphi(x,t)} = \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \varphi(x,t)} - {}_aD_x^{\alpha}\frac{\partial \mathcal{H}}{\partial\, {}_aD_x^{\alpha}\varphi(x,t)} - {}_xD_b^{\beta}\frac{\partial \mathcal{H}}{\partial\, {}_xD_b^{\beta}\varphi(x,t)} + \\ {}_aD_x^{\alpha}\,{}_aD_x^{\alpha}\frac{\partial \mathcal{H}}{\partial\, {}_aD_x^{\alpha}\,{}_aD_x^{\alpha}\varphi(x,t)} + {}_xD_b^{\beta}\,{}_xD_b^{\beta}\frac{\partial \mathcal{H}}{\partial\, {}_xD_b^{\beta}\,{}_xD_b^{\beta}\varphi(x,t)} - \\ {}_aD_x^{\alpha}\,{}_aD_x^{\alpha}\,{}_aD_t^{\alpha}\frac{\partial \mathcal{H}}{\partial\, {}_aD_x^{\alpha}\,{}_aD_x^{\alpha}\,{}_aD_t^{\alpha}\varphi(x,t)} - \\ {}_xD_b^{\beta}\,{}_xD_b^{\beta}\,{}_tD_b^{\beta}\frac{\partial \mathcal{H}}{\partial\, {}_xD_b^{\beta}\,{}_xD_b^{\beta}\,{}_tD_b^{\beta}\varphi(x,t)} - \\ {}_aD_x^{\alpha}\,{}_aD_x^{\alpha}\,{}_aD_x^{\alpha}\frac{\partial \mathcal{H}}{\partial\, {}_aD_x^{\alpha}\,{}_aD_x^{\alpha}\,{}_aD_x^{\alpha}\varphi(x,t)} - \\ {}_xD_b^{\beta}\,{}_xD_b^{\beta}\,{}_xD_b^{\beta}\frac{\partial \mathcal{H}}{\partial\, {}_xD_b^{\beta}\,{}_xD_b^{\beta}\,{}_xD_b^{\beta}\varphi(x,t)} \end{bmatrix} \tag{36}$$

Using the definition given in Eq. (36) above, we can rewrite the equations of motion (i.e. Eqs. (33), (34), (35)) in terms of the Hamiltonian density such that:

$$\begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \psi(x,t)} - {}_aD_x^{\alpha}\frac{\partial \mathcal{H}}{\partial\, {}_aD_x^{\alpha}\psi(x,t)} - {}_xD_b^{\alpha}\frac{\partial \mathcal{H}}{\partial\, {}_xD_b^{\beta}\psi(x,t)} + \\ {}_aD_x^{\alpha}\,{}_aD_x^{\alpha}\frac{\partial \mathcal{H}}{\partial\, {}_aD_x^{\alpha}\,{}_aD_x^{\alpha}\psi(x,t)} + {}_xD_b^{\beta}\,{}_xD_b^{\beta}\frac{\partial \mathcal{H}}{\partial\, {}_xD_b^{\beta}\,{}_xD_b^{\beta}\psi(x,t)} - \\ {}_aD_x^{\alpha}\,{}_aD_x^{\alpha}\,{}_aD_t^{\alpha}\frac{\partial \mathcal{H}}{\partial\, {}_aD_x^{\alpha}\,{}_aD_x^{\alpha}\,{}_aD_t^{\alpha}\psi(x,t)} - \\ {}_xD_b^{\beta}\,{}_xD_b^{\beta}\,{}_tD_b^{\beta}\frac{\partial \mathcal{H}}{\partial\, {}_xD_b^{\beta}\,{}_xD_b^{\beta}\,{}_tD_b^{\beta}\psi(x,t)} - \\ {}_aD_x^{\alpha}\,{}_aD_x^{\alpha}\,{}_aD_x^{\alpha}\frac{\partial \mathcal{H}}{\partial\, {}_aD_x^{\alpha}\,{}_aD_x^{\alpha}\,{}_aD_x^{\alpha}\psi(x,t)} - \\ {}_xD_b^{\beta}\,{}_xD_b^{\beta}\,{}_xD_b^{\beta}\frac{\partial \mathcal{H}}{\partial\, {}_xD_b^{\beta}\,{}_xD_b^{\beta}\,{}_xD_b^{\beta}\psi(x,t)} \end{bmatrix} = -\,{}_aD_t^{\alpha}\pi_{\alpha} - {}_tD_b^{\beta}\pi_{\alpha} \tag{37}$$

$$\begin{bmatrix} \frac{\partial\mathcal{H}}{\partial\pi_\alpha} - {}_aD_x^\alpha \frac{\partial\mathcal{H}}{\partial\, {}_aD_x^\alpha\pi_\alpha} - {}_xD_b^\beta \frac{\partial\mathcal{H}}{\partial\, {}_xD_b^\beta\pi_\alpha} + \\ {}_aD_x^\alpha\ {}_aD_x^\alpha \frac{\partial\mathcal{H}}{\partial\, {}_aD_x^\alpha\ {}_aD_x^\alpha\pi_\alpha} + {}_xD_b^\beta\ {}_xD_b^\beta \frac{\partial\mathcal{H}}{\partial\, {}_xD_b^\beta\ {}_xD_b^\beta\pi_\alpha} - \\ {}_aD_x^\alpha\ {}_aD_x^\alpha\ {}_aD_t^\alpha \frac{\partial\mathcal{H}}{\partial\, {}_aD_x^\alpha\ {}_aD_x^\alpha\ {}_aD_t^\alpha\pi_\alpha} - \\ {}_xD_b^\beta\ {}_xD_b^\beta\ {}_tD_b^\beta \frac{\partial\mathcal{H}}{\partial\, {}_xD_b^\beta\ {}_xD_b^\beta\ {}_tD_b^\beta\pi_\alpha} - \\ {}_aD_x^\alpha\ {}_aD_x^\alpha\ {}_aD_x^\alpha \frac{\partial\mathcal{H}}{\partial\, {}_aD_x^\alpha\ {}_aD_x^\alpha\ {}_aD_x^\alpha\pi_\alpha} - \\ {}_xD_b^\beta\ {}_xD_b^\beta\ {}_xD_b^\beta \frac{\partial\mathcal{H}}{\partial\, {}_xD_b^\beta\ {}_xD_b^\beta\ {}_xD_b^\beta\pi_\alpha} \end{bmatrix} = {}_aD_t^\alpha\psi \tag{38}$$

$$\begin{bmatrix} \frac{\partial\mathcal{H}}{\partial\pi_\beta} - {}_aD_x^\alpha \frac{\partial\mathcal{H}}{\partial\, {}_aD_x^\alpha\pi_\beta} - {}_xD_b^\beta \frac{\partial\mathcal{H}}{\partial\, {}_xD_b^\beta\pi_\beta} + \\ {}_aD_x^\alpha\ {}_aD_x^\alpha \frac{\partial\mathcal{H}}{\partial\, {}_aD_x^\alpha\ {}_aD_x^\alpha\pi_\beta} + {}_xD_b^\beta\ {}_xD_b^\beta \frac{\partial\mathcal{H}}{\partial\, {}_xD_b^\beta\ {}_xD_b^\beta\pi_\beta} - \\ {}_aD_x^\alpha\ {}_aD_x^\alpha\ {}_aD_t^\alpha \frac{\partial\mathcal{H}}{\partial\, {}_aD_x^\alpha\ {}_aD_x^\alpha\ {}_aD_t^\alpha\pi_\beta} - \\ {}_xD_b^\beta\ {}_xD_b^\beta\ {}_tD_b^\beta \frac{\partial\mathcal{H}}{\partial\, {}_xD_b^\beta\ {}_xD_b^\beta\ {}_tD_b^\beta\pi_\beta} - \\ {}_aD_x^\alpha\ {}_aD_x^\alpha\ {}_aD_x^\alpha \frac{\partial\mathcal{H}}{\partial\, {}_aD_x^\alpha\ {}_aD_x^\alpha\ {}_aD_x^\alpha\pi_\beta} - \\ {}_xD_b^\beta\ {}_xD_b^\beta\ {}_xD_b^\beta \frac{\partial\mathcal{H}}{\partial\, {}_xD_b^\beta\ {}_xD_b^\beta\ {}_xD_b^\beta\pi_\beta} \end{bmatrix} = {}_tD_b^\beta\psi \tag{39}$$

In many cases, $\pi_\beta = 0$ is adopted because, in the Lagrangian density and in the Hamiltonian density, the time derivative side is defined as ${}_xD_b^\alpha\psi(x,t)$, yielding $\pi_\beta = \frac{\partial\mathcal{L}}{\partial\, {}_tD_b^\beta\psi} = 0.$ Therefore, $\pi_\beta = 0$ is adopted in the present study.

Case 2*: As π_α depends on (ψ), only the variation in the independent variables ψ is relevant, allowing Eq. (36) to be expressed as:*

$$\delta \mathrm{H} = \int \left[\frac{\partial\, \mathrm{H}}{\partial\, \psi}\delta\psi\right] d^3r \tag{40}$$

To state the equations of motion presented in Eq. (32), $\pi_\alpha = R(\psi)$ is adopted, due to which the variation is given by:

$$\delta\pi = \frac{\partial R}{\partial\psi}\delta\psi \tag{41}$$

After substituting Eq. (41) into Eq. (32) and comparing the results with Eq. (40), then the general equations of the Hamiltonian density for this case is obtained:

$$\left[\begin{array}{c} \frac{\partial \mathcal{H}}{\partial \psi(x,t)} - {}_aD_x^\alpha \frac{\partial \mathcal{H}}{\partial\, {}_aD_x^\alpha \psi(x,t)} - {}_xD_b^\beta \frac{\partial \mathcal{H}}{\partial\, {}_xD_b^\beta \psi(x,t)} + \\ {}_aD_x^\alpha\, {}_aD_x^\alpha \frac{\partial \mathcal{H}}{\partial\, {}_aD_x^\alpha\, {}_aD_x^\alpha \psi(x,t)} + {}_xD_b^\beta\, {}_xD_b^\beta \frac{\partial \mathcal{H}}{\partial\, {}_xD_b^\alpha\, {}_xD_b^\alpha \psi(x,t)} - \\ {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_t^\alpha \frac{\partial \mathcal{H}}{\partial\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_t^\alpha \psi(x,t)} - \\ {}_xD_b^\beta\, {}_xD_b^\beta\, {}_tD_b^\beta \frac{\partial \mathcal{H}}{\partial\, {}_xD_b^\beta\, {}_xD_b^\beta\, {}_tD_b^\beta \psi(x,t)} - \\ {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_x^\alpha \frac{\partial \mathcal{H}}{\partial\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_x^\alpha \psi(x,t)} - \\ {}_xD_b^\beta\, {}_xD_b^\beta\, {}_xD_b^\beta \frac{\partial \mathcal{H}}{\partial\, {}_xD_b^\beta\, {}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t)} \end{array}\right]$$

$$= -\, {}_aD_t^\alpha \pi_\alpha - {}_tD_b^\beta \pi_\beta + \left({}_aD_t^\alpha \psi + {}_tD_b^\beta \psi \right) \frac{\partial R}{\partial \psi} \tag{42}$$

8 Examples

In this section, we study one example as applications on the formalism presented above.

Example **Degasperis-Procesi equation**

Lagrangian density in fractional form:

$$L = \left[\begin{array}{c} -\frac{1}{2}\left[{}_aD_x^\alpha \psi(x,t)\, {}_aD_t^\alpha \psi(x,t) + {}_xD_b^\beta \psi(x,t)\, {}_tD_b^\beta \psi(x,t) \right] + \\ \frac{1}{2}\left[{}_aD_x^\alpha \psi(x,t)\, {}_aD_x^{2\alpha}\, {}_aD_t^\alpha \psi(x,t) + {}_xD_b^\beta \psi(x,t)\, {}_xD_b^{2\beta}\, {}_tD_b^\beta \psi(x,t) \right] \\ -\frac{1}{3}\left[\left({}_aD_x^\alpha \psi(x,t) \right)^4 + \left({}_xD_b^\beta \psi(x,t) \right)^4 \right] + \\ \frac{5}{8}\left[{}_aD_x^\alpha \psi(x,t) \left({}_aD_x^{2\alpha} \psi(x,t) \right)^2 + {}_xD_b^\beta \psi(x,t) \left({}_xD_b^{2\beta} \psi(x,t) \right)^2 \right] \\ -\frac{1}{16}\left[\left({}_aD_x^\alpha \psi(x,t) \right)^2 {}_aD_x^{3\alpha} \psi(x,t) + \left({}_xD_b^\beta (x,t) \right)^2 {}_xD_b^{3\alpha} \psi(x,t) \right] \end{array}\right] \tag{43}$$

Substituting the Lagrangian of the fractional Degasperis-Procesi equation (43)into Euler-Lagrange formula (8)obtains:

$$\left[\begin{array}{c} {}_aD_x^\alpha\, {}_aD_t^\alpha \psi(x,t) + {}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t) \\ -\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_t^\alpha \psi(x,t) - {}_xD_b^\beta\, {}_xD_b^\beta\, {}_xD_b^\beta\, {}_tD_b^\alpha \psi(x,t) \\ +4\left({}_aD_x^\alpha \psi(x,t) \right)^2 {}_aD_x^\alpha\, {}_aD_x^\alpha \psi(x,t) + 4\left({}_xD_b^\beta \psi(x,t) \right)^2 {}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t) \\ +3\, {}_aD_x^\alpha\, {}_aD_x^\alpha \psi(x,t)\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_x^\alpha \psi(x,t) + 3\, {}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t)\, {}_xD_b^\beta\, {}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t) \\ +\frac{2}{3}\, {}_aD_x^\alpha \psi(x,t)\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_x^\alpha \psi(x,t) + \frac{2}{3}\, {}_xD_b^\beta \psi(x,t)\, {}_xD_b^\beta\, {}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t) \end{array}\right] = 0 \tag{44}$$

Now we want to derive Eq. (43)using the Hamiltonian density equations of motion. First we determine π_β and π_α using Eqs.((45)and (46)):

$$\pi_\alpha = \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^\alpha \psi(x,t)} = \frac{1}{2}\left[\, {}_aD_x^\alpha \psi(x,t)\right] \quad (45)$$

$$\pi_\beta = \frac{\partial \mathcal{L}}{\partial\, {}_tD_b^\beta \psi(x,t)} = \frac{1}{2}\left[\, {}_xD_b^\beta \psi(x,t)\right]$$

Thus, the Hamiltonian reads as

$$\mathcal{H} = \begin{pmatrix} -\frac{1}{2}\, {}_aD_x^\alpha \psi(x,t)\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_t^\alpha \psi(x,t) - \frac{1}{2}\, {}_xD_b^\beta \psi(x,t)\, {}_xD_b^\beta\, {}_xD_b^\beta\, {}_tD_b^\beta \psi(x,t) \\ +\frac{1}{3}\left({}_aD_x^\alpha \psi(x,t)\right)^4 + \frac{1}{3}\left({}_xD_b^\beta \psi(x,t)\right)^4 - \frac{5}{8}\, {}_aD_x^\alpha \psi(x,t)\left({}_aD_x^\alpha\, {}_aD_x^\alpha \psi(x,t)\right)^2 \\ -\frac{5}{8}\, {}_xD_b^\beta \psi(x,t)\left({}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t)\right)^2 + \frac{1}{16}\left({}_aD_x^\alpha \psi(x,t)\right)^2 {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_x^\alpha \psi(x,t) \\ +\frac{1}{16}\left({}_xD_b^\beta \psi(x,t)\right)^2 {}_xD_b^\beta\, {}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t) \end{pmatrix} \quad (47)$$

Now, because and π_α variable is dependent on ψ ,respectively, then we have to use the equations of motion for case2. Applying Eq. (42), we get:

$$\begin{bmatrix} {}_aD_x^\alpha\, {}_aD_t^\alpha \psi(x,t) + {}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t) \\ -\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_t^\alpha \psi(x,t) - {}_xD_b^\beta\, {}_xD_b^\beta\, {}_xD_b^\beta\, {}_tD_b^\alpha \psi(x,t) \\ +4\left({}_aD_x^\alpha \psi(x,t)\right)^2 {}_aD_x^\alpha\, {}_aD_x^\alpha \psi(x,t) + 4\left({}_xD_b^\beta \psi(x,t)\right)^2 {}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t) \\ +3\, {}_aD_x^\alpha\, {}_aD_x^\alpha \psi(x,t)\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_x^\alpha \psi(x,t) + 3\, {}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t)\, {}_xD_b^\beta\, {}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t) \\ +\frac{2}{3}\, {}_aD_x^\alpha \psi(x,t)\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_x^\alpha \psi(x,t) + \frac{2}{3}\, {}_xD_b^\beta \psi(x,t)\, {}_xD_b^\beta\, {}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t) \end{bmatrix} = 0 \quad (48)$$

The above equation is exactly the same as that derived by Euler- Lagrange (Eq. (44)). For If α = β = 1, then Eqs. (44)and (48)become:

$$\begin{aligned} &\psi_{xt}(x,t) - \psi_{xxxt}(x,t) + 4\psi_x^2(x,t)\psi_{xx}(x,t) \\ &+3\psi_{xx}(x,t)\psi_{xxx}(x,t) + \frac{3}{2}\psi_x(x,t)\psi_{xxxx}(x,t) = 0 \end{aligned} \quad (49)$$

This is the known Degasperis-Procesi equation. If we do not consider the dependency of π_α on ψ, and apply Eq. (37)in case1, then we get:

$$\begin{bmatrix} \frac{1}{2}\, {}_aD_x^\alpha\, {}_aD_t^\alpha \psi(x,t) + \frac{1}{2}\, {}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t) \\ -\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_t^\alpha \psi(x,t) - {}_xD_b^\beta\, {}_xD_b^\beta\, {}_xD_b^\beta\, {}_tD_b^\alpha \psi(x,t) \\ +4\left({}_aD_x^\alpha \psi(x,t)\right)^2 {}_aD_x^\alpha\, {}_aD_x^\alpha \psi(x,t) + 4\left({}_xD_b^\beta \psi(x,t)\right)^2 {}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t) \\ +3\, {}_aD_x^\alpha\, {}_aD_x^\alpha \psi(x,t)\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_x^\alpha \psi(x,t) + 3\, {}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t)\, {}_xD_b^\beta\, {}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t) \\ +\frac{2}{3}\, {}_aD_x^\alpha \psi(x,t)\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_x^\alpha\, {}_aD_x^\alpha \psi(x,t) + \frac{2}{3}\, {}_xD_b^\beta \psi(x,t)\, {}_xD_b^\beta\, {}_xD_b^\beta\, {}_xD_b^\beta \psi(x,t) \end{bmatrix} = 0 \quad (50)$$

If α = β = 1, then Eq. (50)becomes:

$$\frac{1}{2}\psi_{xt}(x,t) - \psi_{xxxt}(x,t) + 4\psi_x^2(x,t)\psi_{xx}(x,t)$$
$$+3\psi_{xx}(x,t)\psi_{xxx}(x,t) + \frac{3}{2}\psi_x(x,t)\psi_{xxxx}(x,t) = 0 \quad (51)$$

This is not equivalent to Degasperis-Procesi given by Eq. (44).This means that equations of motion mentioned in case1 do not represent the general case for equations of motion in terms of the Hamiltonian density, but they only represent a special case for independency of π_α on ψ.

9 Conclusion

The Hamiltonian formulation of continuous field systems was developed. Our results are identical to those obtained using the Euler-Lagrange model. The conjugate momenta are studied in two cases: field dependent and field independent. The findings of the equations of motion are found in accord with the Lagrangian formulation of continuous systems only for derivatives of integer orders. For derivatives of integer orders only for example (α=1), the classical results are found as a specific case of the fractional formulation

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