Quantum Physics Letters
An International Journal

http://dx.doi.org/10.18576/qpl/110301

Dynamics of Two-Level Laser Coupled to Vacuum Reservoir

Menisha Alemu

Department of Physics, Wolkite University, P. O. Box 07, Wolkite, Ethiopia

Received: 2 May 2022, Revised: 12 Jun. 2022, Accepted: 18 Aug. 2022
Published online: 1 Dec. 2022

Abstract: We analyze the quantum properties of the light generated by a two-level laser with an open cavity and coupled to a vacuum reservoir via a single-port mirror. The two-level laser consists of two-level atoms available in an open cavity and pumped from the lower to the upper level by means of electron bombardment. We seek to carry out our analysis by putting the noise operators associated with the vacuum reservoir in normal order. Applying the large-time approximation scheme, we have obtained the steady-state solutions of the equations of evolution for the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators. Using the resulting steady-state solutions, we have calculated the mean photon number, the variance of the photon number, and the quadrature variance for cavity light. It is found that the two-level laser generates coherent light for $\gamma + \gamma_c \ll r_a$ and chaotic light for $\gamma + \gamma_c = r_a$. Moreover, we have established that a large part of the total mean photon number is confined in a relatively small frequency interval. We have also established that the mean photon number in the presence of spontaneous emission is less than that in the absence of spontaneous emission. In other words, the effect of spontaneous emission is to decrease the mean photon number.

Keywords: Photon statistics, Power spectrum, Quadrature variance, Spontaneous emission.

1 Introduction

A two-level laser is a source of coherent or chaotic light emitted by two-level atoms inside a cavity coupled to a vacuum reservoir via a single-port mirror. In one model of such a laser, two-level atoms initially in the upper level are injected at a constant rate into a cavity and removed after they have decayed due to spontaneous emission [1,2]. In another model the two-level atoms available in a cavity are pumped to the upper level by some convenient means such as electron bombardment [3].

There has been a considerable interest to study the quantum properties of the light generated by a two-level laser [4,5,6,7,8,9,10,11,12]. It is found that the light generated by this laser operating well above threshold is coherent and the light generated by the same laser operating below threshold is chaotic [3,8]. In the quantum theory of a laser, one usually takes into consideration the interaction of the atoms inside the cavity with the vacuum reservoir outside the cavity. There may be some justification for the possibility of such interaction for a laser with an open cavity into which and from which atoms are injected and removed. However, there cannot be any valid justification for leaving open the laser cavity in which the available atoms are pumped to the upper level by means of electron bombardment. Therefore, the aforementioned interaction is not feasible for a laser in which the atoms available in a closed cavity are pumped to the upper level by electron bombardment.

Moreover, Beyene Bashu. and Fesseha Kassahun [13] have studied the squeezing and the statistical properties of the light produced by a two-level laser with the atoms placed in a closed cavity and driven by coherent light. They have shown that the maximum quadrature squeezing of the light generated by the laser, operating below threshold, is found to be 50% below the vacuum-state level.

We seek here to analyze the quantum properties of the light emitted by two-level atoms available in an open cavity and pumped to the upper level at a constant rate by electron bombardment. Thus taking into account the interaction of the two-level atoms with a resonant cavity mode and the damping of the cavity mode by a vacuum reservoir, we obtain the photon statistics, the quadrature variance, and the power spectrum for the light emitted by the atoms. We carry out this analysis by putting the noise operators associated with the vacuum reservoir in normal

* Corresponding author e-mail: minisha.alemu@gmail.com

© 2022 NSP
Natural Sciences Publishing Cor.

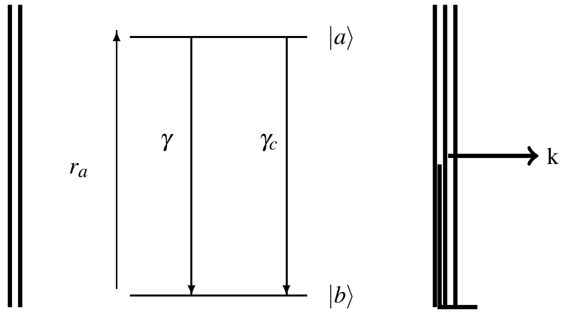

Fig. 1: Schematic representation of a two-level atom coupled to a vacuum reservoir, where γ is the rate of spontaneous emission decay constant, γ_c is the stimulated emission, and r_a is the rate at which a single atom is pumped from the lower level to the upper level by means of electron bombardment.

order and considering the interaction of the two-level atoms with the vacuum reservoir outside the cavity.

2 Operator dynamics

We consider here the case in which N two-level atoms are available in an open cavity. Then the interaction of the cavity mode with one of the atoms can be described at resonance by the Hamiltonian [8]

$$\hat{H} = ig\left(\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a^k\right), \tag{1}$$

where

$$\hat{\sigma}_a^k = |b\rangle_{kk}\langle a| \tag{2}$$

is lowering atomic operator, $\hat{a}$ is the annihilation operator for the cavity mode, g is the coupling constant between the atom and the cavity mode. We assume that the laser cavity is coupled to a vacuum reservoir via a single-port mirror. In addition, we carry out our calculation by putting the noise operators associated with the vacuum reservoir in normal order. Thus the noise operators will not have any effect on the dynamics of the cavity mode operators. We can therefore drop the noise operator and write the quantum Langevin equation for the operator $\hat{a}$ as [3]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - i[\hat{a},\hat{H}], \tag{3}$$

where κ is the cavity damping constant . With the aid of Eqs. (1) and (3), we easily establish that [3]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_a^k, \tag{4}$$

this holds for free cavity mode with photons.

Furthermore, the master equation for a two-level atom interacting with a vacuum reservoir is given by [10]

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H},\hat{\rho}] + \frac{\gamma}{2}\left[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\right], \tag{5}$$

where γ is the spontaneous emission decay constant and $\hat{\rho}$ is the density operator. We can rewrite Eq. (5) as

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H},\hat{\rho}] + \frac{\gamma}{2}\left[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\eta}_a^k\hat{\rho} - \hat{\rho}\hat{\eta}_a^k\right], \tag{6}$$

where

$$\hat{\eta}_a^k = |a\rangle_{kk}\langle a|. \tag{7}$$

Using Eq. (1), we can put Eq. (6) in the form

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= g\left[\hat{\sigma}_a^{\dagger k}\hat{a}\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a^k\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^k\right] \\ &+ \frac{\gamma}{2}\left[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\eta}_a^k\hat{\rho} - \hat{\rho}\hat{\eta}_a^k\right].\end{aligned} \tag{8}$$

Now applying the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr(\frac{d\rho}{dt}\hat{A}) \tag{9}$$

along with Eq. (8) as well as employing the cyclic properties of the trace operation with the assumption that the cavity mode operators and atomic mode operators are commute , we can easily establish that [16,17]

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -\frac{\gamma}{2}\langle\hat{\sigma}_a^k\rangle + g\left[\langle\hat{\eta}_b^k\hat{a}\rangle - \langle\hat{\eta}_a^k\hat{a}\rangle\right], \tag{10}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -\gamma\langle\hat{\eta}_a^k\rangle + g\left[\langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle\right], \tag{11}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = \gamma\langle\hat{\eta}_a^k\rangle - g\left[\langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle\right], \tag{12}$$

where

$$\hat{\eta}_b^k = |b\rangle_{kk}\langle b|. \tag{13}$$

We see that Eqs. (10)-(12) are nonlinear differential equations and hence it is not possible to find exact time-dependent solutions of these equations. We intend to over come this problem by applying the large-time approximation [8]. Then using this approximation scheme, we get from Eq. (4) the approximately valid relation

$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a^k. \tag{14}$$

Now substitution of Eq. (14) into the aforementioned equations yields

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -\frac{1}{2}\left[\gamma+\gamma_c\right]\langle\hat{\sigma}_a^k\rangle, \tag{15}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -\left[\gamma+\gamma_c\right]\langle\hat{\eta}_a^k\rangle, \tag{16}$$

© 2022 NSP
Natural Sciences Publishing Cor.

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = [\gamma+\gamma_c]\,\langle\hat{\eta}_a^k\rangle, \tag{17}$$

where

$$\gamma_c = \frac{4g^2}{\kappa} \tag{18}$$

is the stimulated emission decay constant. The two-level atoms available in the cavity are pumped from the lower to the upper level by means of electron bombardment. The pumping process must surely affect the dynamics of $\langle\hat{\eta}_a^k\rangle$ and $\langle\hat{\eta}_b^k\rangle$. If r_a represents the rate at which a single atom is pumped from the lower to the upper level, then $\langle\hat{\eta}_a^k\rangle$ increases at the rate of $r_a\langle\hat{\eta}_b^k\rangle$ and $\langle\hat{\eta}_b^k\rangle$ decreases at the same rate. In view of this, we rewrite Eqs. (16) and (17) as

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -[\gamma+\gamma_c]\,\langle\hat{\eta}_a^k\rangle + r_a\langle\hat{\eta}_b^k\rangle, \tag{19}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = [\gamma+\gamma_c]\,\langle\hat{\eta}_a^k\rangle - r_a\langle\hat{\eta}_b\rangle. \tag{20}$$

In order to include the contribution of all the atoms to the dynamics of the two-level laser, we next sum Eqs. (15), (19), and (20) over the N two-level atoms, so that

$$\frac{d}{dt}\langle\hat{m}_a\rangle = -\frac{1}{2}[\gamma+\gamma_c]\,\langle\hat{m}_a\rangle, \tag{21}$$

$$\frac{d}{dt}\langle\hat{N}_a\rangle = -[\gamma+\gamma_c]\,\langle\hat{N}_a\rangle + r_a\langle\hat{N}_b\rangle, \tag{22}$$

$$\frac{d}{dt}\langle\hat{N}_b\rangle = [\gamma+\gamma_c]\,\langle\hat{N}_a\rangle - r_a\langle\hat{N}_b\rangle, \tag{23}$$

in which

$$\hat{m}_a = \sum_{k=1}^{N}\hat{\sigma}_a^k, \tag{24}$$

$$\hat{N}_a = \sum_{k=1}^{N}\hat{\eta}_a^k, \tag{25}$$

$$\hat{N}_b = \sum_{k=1}^{N}\hat{\eta}_b^k, \tag{26}$$

with the operators $\hat{N}_a$ and $\hat{N}_b$ representing the number of atoms in the upper and lower levels. In addition, employing the completeness relation

$$\hat{\eta}_a^k + \hat{\eta}_b^k = \hat{I}, \tag{27}$$

we easily arrive at

$$\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle = N. \tag{28}$$

Now taking into account Eq. (28), one can put Eq. (22) in the form

$$\frac{d}{dt}\langle\hat{N}_a\rangle = -[\gamma+\gamma_c+r_a]\,\langle\hat{N}_a\rangle + r_aN. \tag{29}$$

We immediately see that the steady-state solution of this equation is

$$\langle\hat{N}_a\rangle = \frac{r_aN}{\gamma+\gamma_c+r_a} \tag{30}$$

and the steady-state solution of Eq. (23) turns out to be

$$\langle\hat{N}_b\rangle = \frac{\gamma+\gamma_c}{r_a}\langle N_a\rangle. \tag{31}$$

For $r_a = 0$, we see that $\langle\hat{N}_a\rangle = 0$ and $\langle\hat{N}_b\rangle = N$. This result holds whether the atoms are initially in the upper or lower level.

Furthermore, applying the definition given by Eq. (2) and setting for any k

$$\hat{\sigma}_a^k = |b\rangle\langle a|, \tag{32}$$

we have

$$\hat{m}_a = N|b\rangle\langle a|. \tag{33}$$

We therefore find that

$$\hat{m}_a^\dagger\hat{m}_a = N\hat{N}_a, \tag{34}$$

in which

$$\hat{N}_a = N|a\rangle\langle a|. \tag{35}$$

Following the same procedure, one can also establish that

$$\hat{m}_a\hat{m}_a^\dagger = N\hat{N}_b, \tag{36}$$

with

$$\hat{N}_b = N|b\rangle\langle b|. \tag{37}$$

In the presence of N two-level atoms, we rewrite Eq. (4) as [10]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \lambda\hat{m}_a, \tag{38}$$

in which λ is a constant whose value remains to be fixed. Eq. (38) represents the quantum Langevin equation for cavity mode operator when the cavity mode is interacting with N two-level atoms. Using Eq. (14), we get

$$[\hat{a},\hat{a}^\dagger]_k = \frac{4g^2}{\kappa^2}(\hat{\eta}_b^k - \hat{\eta}_a^k) \tag{39}$$

and on summing over all atoms, we have

$$[\hat{a},\hat{a}^\dagger] = \frac{4g^2}{\kappa^2}(\hat{N}_b - \hat{N}_a), \tag{40}$$

where

$$[\hat{a},\hat{a}^\dagger] = \sum_{k=1}^{N}[\hat{a},\hat{a}^\dagger]_k \tag{41}$$

stands for the commutator of $\hat{a}$ and $\hat{a}^\dagger$ when light mode a is interecting with all the N two-level atoms. On the other

© 2022 NSP
Natural Sciences Publishing Cor.

hand, applying the large-time approximation to Eq. (38), one can easily verify that

$$[\hat{a},\hat{a}^{\dagger}] = N\frac{4\lambda^2}{\kappa^2}(\hat{N}_b - \hat{N}_a). \tag{42}$$

Thus on account of Eqs. (40) and (42), we see that

$$\lambda = \pm\frac{g}{\sqrt{N}}. \tag{43}$$

In view of this result, Eq. (38) can be written as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \frac{g}{\sqrt{N}}\hat{m}_a \tag{44}$$

Furthermore, in order to include the effect of pumping process, we rewrite Eq. (21) as

$$\frac{d}{dt}\hat{m}_a = -\frac{\mu}{2}\hat{m}_a + \hat{G}(t), \tag{45}$$

in which $\hat{G}(t)$ is a noise operator with vanishing mean and μ is a parameter whose value remains to be determined. Employing the relation

$$\frac{d}{dt}\left\langle \hat{m}_a^{\dagger}\hat{m}_a\right\rangle = \left\langle \frac{d\hat{m}_a^{\dagger}}{dt}\hat{m}_a\right\rangle + \left\langle \hat{m}_a^{\dagger}\frac{d\hat{m}_a}{dt}\right\rangle \tag{46}$$

along with Eq. (45), we easily find

$$\frac{d}{dt}\langle\hat{m}_a^{\dagger}\hat{m}_a\rangle = -\mu\langle\hat{m}_a^{\dagger}\hat{m}_a\rangle + \langle\hat{m}_a^{\dagger}\hat{G}(t)\rangle + \langle\hat{G}^{\dagger}(t)\hat{m}_a\rangle, \tag{47}$$

from which follows

$$\frac{d}{dt}\langle\hat{N}_a\rangle = -\mu\langle\hat{N}_a\rangle + \frac{1}{N}\left[\langle\hat{m}_a^{\dagger}\hat{G}(t)\rangle + \langle\hat{G}^{\dagger}(t)\hat{m}_a\rangle\right]. \tag{48}$$

Hence comparison of Eqs. (29) and (48) shows that

$$\mu = \gamma + \gamma_c + r_a \tag{49}$$

and

$$\left\langle\hat{m}_a^{\dagger}\hat{G}(t)\right\rangle + \left\langle\hat{G}^{\dagger}(t)\hat{m}_a\right\rangle = r_a N^2. \tag{50}$$

We observe that Eq. (50) is equivalent to

$$\left\langle\hat{G}^{\dagger}(t)\hat{G}(t')\right\rangle = r_a N^2\delta(t-t'). \tag{51}$$

One can also easily verify that

$$\left\langle\hat{G}(t)\hat{G}^{\dagger}(t')\right\rangle = (\gamma+\gamma_c)N^2\delta(t-t'). \tag{52}$$

3 Photon statistics

In this section we seek to determine the mean and photon number variance for the cavity light. To this end, the mean photon number for the cavity light, represented by the operators a and $a^{\dagger}$, is defined by

$$\bar{n} = \langle\hat{a}^{\dagger}\hat{a}\rangle. \tag{53}$$

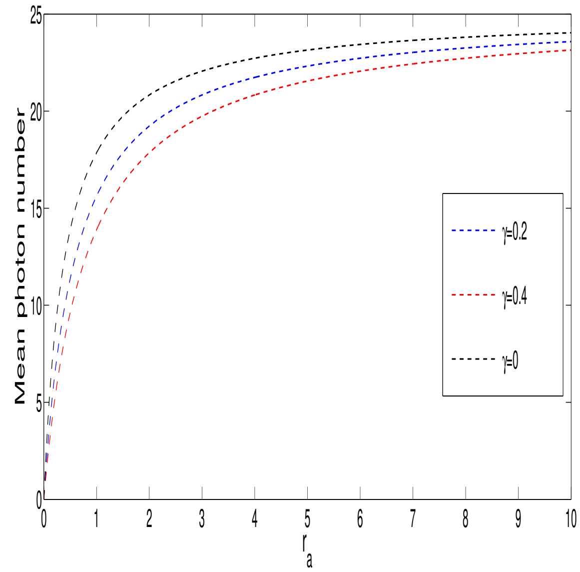

Fig. 2: Plots of the mean photon number for the cavity light at steady state,
Eq. (56) versus r_a for $\kappa = 0.8$, $\gamma_c = 0.4$, and $N = 50$.

We note that the steady-state solution of Eq. (44) is

$$\hat{a} = \frac{2g}{\sqrt{N}\kappa}\hat{m}_a, \tag{54}$$

so that introducing Eq. (54) and its adjoint into Eq. (53), we see that

$$\bar{n} = \frac{\gamma_c}{\kappa N}\langle\hat{m}_a^{\dagger}\hat{m}_a\rangle. \tag{55}$$

Now on account of Eq. (34) along with Eq. (30), Eq. (55) becomes

$$\bar{n} = \frac{\gamma_c}{\kappa}\left(\frac{r_a}{\gamma+\gamma_c+r_a}\right)N. \tag{56}$$

We see from this expression that the mean photon number in the cavity is zero in the absence of the pumping process. This result also shows that the mean photon number in the cavity increases with the number of atoms.

We note that for the two-level laser operating well above threshold ($\gamma+\gamma_c \ll r_a$), Eq. (56) reduces to

$$\bar{n} = \frac{\gamma_c}{\kappa}N. \tag{57}$$

And for the same laser operating at threshold ($\gamma+\gamma_c = r_a$), we have

$$\bar{n} = \frac{\gamma_c}{2\kappa}N. \tag{58}$$

In the absence of spontaneous emission($\gamma = 0$), Eq. (56) becomes

$$\bar{n} = \frac{\gamma_c}{\kappa}\left[\frac{Nr_a}{\gamma_c+r_a}\right]. \tag{59}$$

© 2022 NSP
Natural Sciences Publishing Cor.

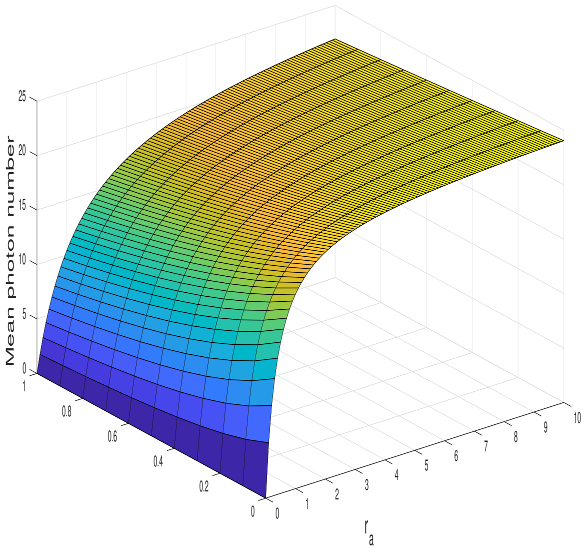

Fig. 3: Plots of the mean photon number for the cavity light at steady state,
Eq. (56) versus r_a and γ for $\kappa = 0.8$, $\gamma_c = 0.4$, and $N = 50$.

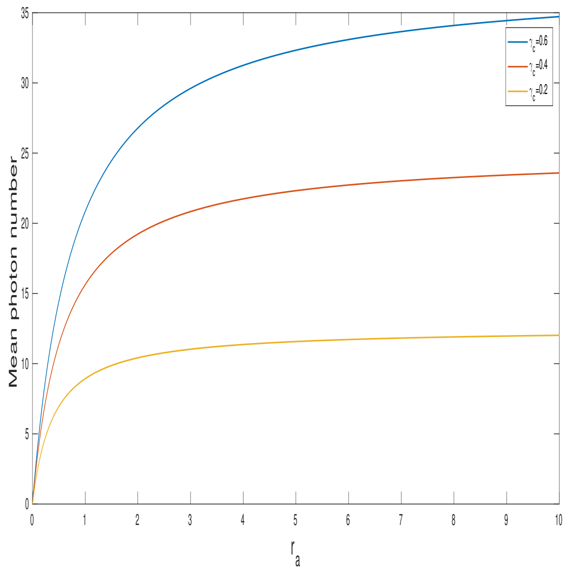

Fig. 4: Plots of the mean photon number for the cavity light at steady state,
Eq. (56) versus r_a for $\kappa = 0.8$, $\gamma = 0.2$, $N = 50$, and different values of γ_c.

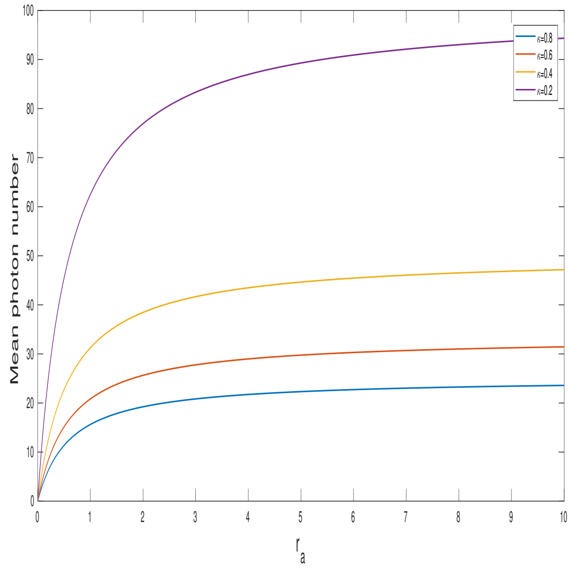

Fig. 5: Plots of the mean photon number for the cavity light at steady state,
Eq. (56) versus r_a for $\gamma_c = 0.4$, $\gamma = 0.2$, $N = 50$, and different values of κ.

We would like to point out that this result is in complete agreement with the one obtained in [8]. The plots in Fig. 2 and 3 indicates that the mean photon number for the cavity light is greater for absence of spontaneous emission than the presence of spontaneous emission. Moreover, the plot in Figure 2 indicates that the mean photon number of the cavity light increases with r_a . In Figure 4, we plot the mean photon number of the cavity light versus r_a for different values of γ_c . It is not difficult to see from this figure that the mean photon number of the cavity light increases with γ_c. In addition from figure 5, for fixed γ_c, γ, and N, the mean photon number of the cavity light decreases with values of κ.

Furthermore, employing Eq. (54) along with its adjoint, we readily find

$$\langle \hat{a}\hat{a}^{\dagger} \rangle = \frac{\gamma_c}{\kappa N} \langle \hat{m}_a \hat{m}_a^{\dagger} \rangle. \tag{60}$$

With the help of Eq. (36) , Eq. (60) takes the form

$$\langle \hat{a}\hat{a}^{\dagger} \rangle = \frac{\gamma_c}{\kappa} \langle \hat{N}_b \rangle. \tag{61}$$

Using Eq. (54) and its adjoint, one can easily establish that

$$[\hat{a}, \hat{a}^{\dagger}] = \frac{\gamma_c}{\kappa} \left[\langle \hat{N}_b \rangle - \langle \hat{N}_a \rangle \right]. \tag{62}$$

Furthermore, employing Eqs. (54) and (33), we readily gets

$$\langle \hat{a}^2(t) \rangle = 0. \tag{63}$$

© 2022 NSP
Natural Sciences Publishing Cor.

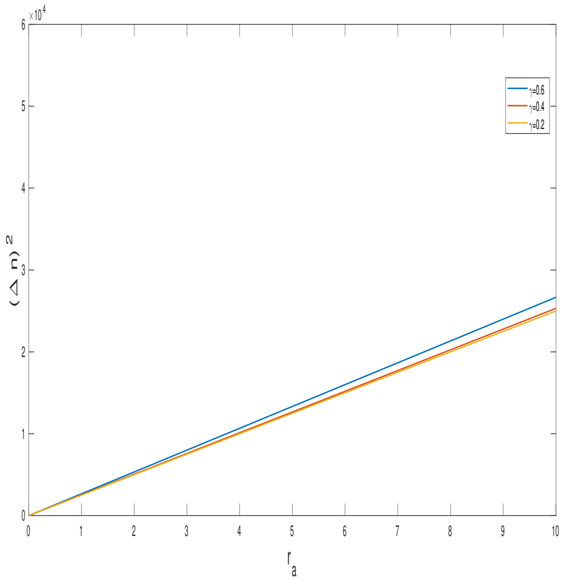

Fig. 6: Plots of the variance of photon number for the cavity light at steady state,
Eq. (68) versus r_a for $\kappa = 0.8$, $\gamma_c = 0.4$, $N = 50$, and different values of γ.

On the other hand, assuming the atoms to be initial in the lower level, the expectation value of the solution of Eq. (45) happens to be

$$\langle \hat{m}_a(t) \rangle = 0. \tag{64}$$

Hence the expectation value of the solution of Eq. (44) turns out to be

$$\langle \hat{a}(t) \rangle = 0. \tag{65}$$

In view of Eqs. (44) and (65), we claim that $\hat{a}(t)$ is a Gaussian variable with zero mean. Moreover, the variance of the photon number for the cavity light can be written as

$$(\Delta n)^2 = \langle (\hat{a}^\dagger \hat{a})^2 \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2. \tag{66}$$

Using the fact that $\hat{a}$ is a Gaussian variable with zero mean, we readily get

$$(\Delta n)^2 = \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a}\hat{a}^\dagger \rangle + \langle \hat{a}^{\dagger 2} \rangle \langle \hat{a}^2 \rangle. \tag{67}$$

On account of Eqs. (56), (61), (65), and (31), we arrive at

$$(\Delta n)^2 = \frac{\gamma + \gamma_c}{r_a} \bar{n}^2. \tag{68}$$

Therefore, for $\gamma + \gamma_c \ll r_a$, the variance of the photon number turns out to be

$$(\Delta n)^2 = 0. \tag{69}$$

This represents the normally-ordered variance of the photon number for coherent light [3]. On the other hand, for the same laser operating at $\gamma + \gamma_c = r_a$, we see that the variance of the photon number is

$$(\Delta n)^2 = \bar{n}^2, \tag{70}$$

which represents the normally-ordered variance of the photon number for chaotic light. In Figure 6, we plot the variance of photon number of the cavity light versus r_a for different values of γ. It is not difficult to see from this figure that the variance of the photon number of the cavity light increases with γ.

4 Power spectrum

It is also interesting to consider the power spectrum of the cavity light. The power spectrum of a single-mode light with central frequency ω_0 is expressible as

$$P(\omega) = \frac{1}{\pi} Re \int_0^\infty d\tau e^{i(\omega-\omega_0)\tau} \langle \hat{a}^\dagger(t)\hat{a}(t+\tau) \rangle_{ss}. \tag{71}$$

Upon integrating both sides of Eq. (71) over ω, we readily get

$$\int P(\omega) d\omega = \bar{n}, \tag{72}$$

in which $\bar{n}$ is the steady-state mean photon number. From this result, we observe that $P(\omega)d\omega$ is the steady-state mean photon number in the interval between ω and $\omega + d\omega$ [2].

We now proceed to calculate the two-time correlation function that appears in Eq. (71) for the cavity light. To this end, we realize that the solution of Eq. (44) can be written as

$$\hat{a}(t+\tau) = \hat{a}(t) e^{-\frac{\kappa}{2}\tau} + \int_0^\tau e^{-\frac{\kappa}{2}(\tau-\tau')} \frac{g}{\sqrt{N}} \hat{m}_a(t+\tau') d\tau'. \tag{73}$$

On the other hand, the solution of Eq. (45) is expressible as

$$\hat{m}_a(t+\tau) = \hat{m}_a(t) e^{-\frac{\mu}{2}\tau} + \int_0^\tau e^{-\frac{\mu}{2}(\tau-\tau')} \hat{G}(t+\tau') d\tau', \tag{74}$$

so that on introducing this into Eq. (73), we have

$$\hat{a}(t+\tau) = \hat{a}(t) e^{-\frac{\kappa}{2}\tau} + \frac{g}{\sqrt{N}} \hat{m}_a(t) e^{-\frac{\kappa}{2}\tau} \int_0^\tau d\tau' e^{\frac{1}{2}(\kappa-\mu)\tau'}$$

$$+ \frac{g}{\sqrt{N}} e^{-\frac{\kappa}{2}\tau} \int_0^\tau d\tau' e^{\frac{1}{2}(\kappa-\mu)\tau'} \int_0^\tau d\tau'' e^{\frac{1}{2}\mu\tau''} \hat{G}(t+\tau''). \tag{75}$$

Thus on carrying out the first integration, we find

$$\hat{a}(t+\tau) = \hat{a}(t) e^{-\frac{\kappa}{2}\tau} + \frac{2g\hat{m}_a(t)}{\sqrt{N}(\kappa-\mu)} \left[e^{-\frac{1}{2}\mu\tau} - e^{-\frac{1}{2}\kappa\tau} \right]$$

$$+ \frac{g}{\sqrt{N}} e^{-\frac{\kappa}{2}\tau} \int_0^\tau d\tau' e^{\frac{1}{2}(\kappa-\mu)\tau'} \times \int_0^\tau d\tau'' e^{\frac{1}{2}\mu\tau''} \hat{G}(t+\tau'') \tag{76}$$

© 2022 NSP
Natural Sciences Publishing Cor.

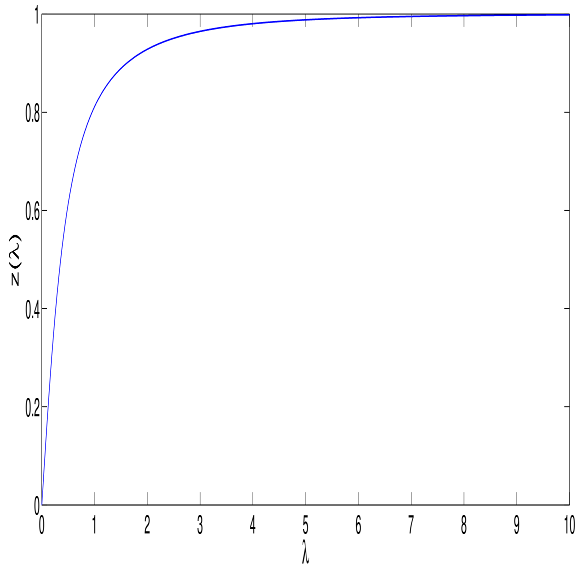

Fig. 7: Plot of Eq. (83) for $\kappa = 0.8$, $\mu = 5$

Now multiplying this equation on the left by $\hat{a}^\dagger(t)$ and taking the expectation value of the resulting expression, we get

$$\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle e^{-\frac{\kappa}{2}\tau}$$

$$+\frac{2g}{\sqrt{N}(\kappa-\mu)}\langle\hat{a}^\dagger(t)\hat{m}_a(t)\rangle\left[e^{-\frac{1}{2}\mu\tau}-e^{-\frac{1}{2}\kappa\tau}\right]$$

$$+\frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\int_0^\tau d\tau' e^{\frac{1}{2}(\kappa-\mu)\tau'}$$

$$\times\int_0^\tau d\tau'' e^{\frac{1}{2}\mu\tau''}\langle\hat{a}^\dagger(t)\hat{G}(t+\tau'')\rangle. \tag{77}$$

Furthermore, applying Eq. (54) along with its adjoint and taking into account Eq. (34), we have

$$\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \bar{n}\left[\frac{\kappa}{\kappa-\mu}e^{-\frac{1}{2}\mu\tau}-\frac{\mu}{\kappa-\mu}e^{-\frac{1}{2}\kappa\tau}\right]. \tag{78}$$

Hence on substituting this into Eq. (71) and carrying out the integration, we get

$$P(\omega) = \frac{\bar{n}\kappa}{\kappa-\mu}\left[\frac{\mu/2\pi}{(\omega-\omega_0)^2+(\frac{\mu}{2})^2}\right]-\frac{\bar{n}\mu}{\kappa-\mu}\left[\frac{\kappa/2\pi}{(\omega-\omega_0)^2+(\frac{\kappa}{2})^2}\right]$$

$$-\frac{\bar{n}\mu}{\kappa-\mu}\left[\frac{\kappa/2\pi}{(\omega-\omega_0)^2+(\frac{\kappa}{2})^2}\right]. \tag{79}$$

We realize that the mean photon number in the interval between $\omega'= -\lambda$ and $\omega'= \lambda$ is expressible as

$$\bar{n}_{\pm\lambda} = \int_{-\lambda}^{\lambda} P(\omega')d\omega', \tag{80}$$

in which $\omega'= \omega-\omega_0$. Therefore, upon inserting Eq. (79) into Eq. (80) and carrying out the integration, applying the relation described by

$$\int_{-\lambda}^{\lambda}\frac{dx}{x^2+a^2} = \frac{2}{a}tan^{-1}(\frac{\lambda}{a}), \tag{81}$$

we easily obtain

$$\bar{n}_{\pm\lambda} = \bar{n}z(\lambda) \tag{82}$$

where $z(\lambda)$ is given by

$$z(\lambda) = \frac{2\kappa/\pi}{\kappa-\mu}tan^{-1}\left(\frac{2\lambda}{\mu}\right)-\frac{2\mu/\pi}{\kappa-\mu}tan^{-1}\left(\frac{2\lambda}{\kappa}\right). \tag{83}$$

From the plot in Figure 7, we easily find $z(0.5) = 0.66$, $z(1) = 0.86$, $z(2) = 0.96$. Then combination of these results with Eq. (82) yields $\bar{n}_{\pm 0.5} = 0.66\bar{n}$, $\bar{n}_{\pm 1} = 0.86\bar{n}$, $\bar{n}_{\pm 2} = 0.96\bar{n}$. We therefore observe that a large part of the total mean photon number is confined in a relatively small frequency interval.

5 Quadrature variance

Here we seek to calculate the variance of the plus and minus quadrature operators defined by

$$\hat{a}_+ = \hat{a}^\dagger+\hat{a} \tag{84}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger-\hat{a}). \tag{85}$$

It can be readily established that [14],

$$[\hat{a}_-,\hat{a}_+] = 2i\frac{\gamma_c}{\kappa}(\langle\hat{N}_a\rangle-\langle\hat{N}_b\rangle). \tag{86}$$

It then follows that [15,16,17],

$$\Delta a_+\Delta a_- \geq \frac{\gamma_c}{\kappa}\left|\langle\hat{N}_a\rangle-\langle\hat{N}_b\rangle\right|, \tag{87}$$

which takes for $\gamma+\gamma_c \ll r_a$ the form

$$\Delta a_+\Delta a_- \geq \bar{n}, \tag{88}$$

where Δa_+ and Δa_- are the uncertainties in the plus and minus quadratures.

The quadrature variance of the laser light beam is expressible as

$$(\Delta a_\pm)^2 = \pm\langle(\hat{a}^\dagger\pm\hat{a})^2\rangle\mp[\langle\hat{a}^\dagger+\hat{a}\rangle]^2. \tag{89}$$

© 2022 NSP
Natural Sciences Publishing Cor.

On account of Eqs. (63) and (65), Eq. (89) takes the form

$$(\Delta a_{\pm})^2 = \langle \hat{a}^{\dagger}\hat{a}\rangle + \langle \hat{a}\hat{a}^{\dagger}\rangle. \tag{90}$$

Now employing Eqs. (56), (61), and (90), we arrive at

$$(\Delta a_{\pm})^2 = \bar{n} + \frac{\gamma+\gamma_c}{r_a}\bar{n}. \tag{91}$$

This reduces to

$$(\Delta a_{\pm})^2 = \bar{n} \tag{92}$$

for $\gamma+\gamma_c \ll r_a$ and to

$$(\Delta a_{\pm})^2 = 2\bar{n} \tag{93}$$

for $\gamma+\gamma_c = r_a$. This represents the normally-ordered quadrature variance for chaotic light. On the basis of Eqs. (88) and (92), we assert that the light generated by the two-level laser is in a coherent state for $\gamma+\gamma_c \ll r_a$. This is due to the fact that, a light mode is said to be coherent state, if the uncertainties in the two quadratures are equal and satisfy the minimum uncertainty relation [10].

6 Conclusion

In this paper we analyze the quantum properties of the light emitted by two-level atoms available in an open cavity and pumped to the upper level at a constant rate. We have carried out our analysis by putting the noise operators associated with the vacuum reservoir in normal order. Taking into account the interaction of the two-level atoms with a resonant cavity mode and the damping of the cavity mode by a vacuum reservoir, we obtain the photon statistics, the quadrature variance, and the power spectrum for the light emitted by the atoms.

We have seen that the mean photon number for the cavity light is greater for absence of spontaneous emission than the presence of spontaneous emission. In addition, our results show that the light generated by the two-level laser is in chaotic state for $\gamma+\gamma_c = r_a$ and in coherent state for $\gamma+\gamma_c \ll r_a$.

References

[1] C. C. Gerry and P. L. Knight, *Introductory Quantum Optics*(Cambridge University Press, Cam- bridge, 2004).
[2] Fesseha Kassahun, *Fundamentals of Quantum Optics*, (Lulu Press Inc., North Carolina, 2010).
[3] Fesseha Kassahun, Three-Level Laser Dynamics with the Atoms Pumped by Electron Bombardment, *arXiv:1105.1438* [quant-ph], (2012).
[4] D. Erenso and R. Vyas, Two-level atom coupled to a squeezed vacuum inside a coherently driven cavity, *Phys. Rev. A* **65**, 063808, (2002).
[5] Cirac and J. I. Sanchez-Soto, Suppression of spontaneous emission by squeezed light in a cavit, *Phys. Rev. A* **44**, 1948, (1991).
[6] Jin and S. Xiao, Extra intracavity squeezing of a degenerate optical parametric oscillator coupling with N two-level atoms, *Phys. Rev. A* **49**, 499, (1994).
[7] Geoffery New, *Introduction to Nonlinear Optics*. (Cambridge University Press, New York, 2011).
[8] Fesseha Kassahun, Stimulated emission by two-level atoms pumped to the upper level, *Opt. Commun.* **284**, 1357-1363, (2011).
[9] Rice and P.R. Baird, Interaction of a two-level atom with a squeezed vacuum: Photon statistics and spectra, *Phys. Rev. A* **53**, 3633, (1996).
[10] Fesseha Kassahun, *Refined Quantum Analysis of Light,* Revised Edition (CreateSpace Independent Publishing Platform, 2016) .
[11] J. I. Cirac, Interaction of a two-level atom with a cavity mode in the bad-cavity limit, *Phys. Rev. A* **46**, 4354, (1992).
[12] Eyob Alebachew, A coherently driven two-level atom inside a parametric oscillator, *J. Mod. Opt.* **55**, 1159, (2008).
[13] Beyene Bashu and Fesseha Kassahun, Quadrature Squeezing in the Cavity Mode Driven by Coherent Light and Interacting with N Two-Level Atoms, *arXiv:1812.04879* [quant-ph], (2019).
[14] Fesseha Kassahun, The Commutation Relation for Cavity Mode Operators. *arXiv:1611.01003* [quant-ph], (2018).
[15] Tewodros Yirgashewa, Coherently Driven Three-Level Laser with Parametric Amplifier,*PhD Dissertation* Addis Ababa University, Ethiopia,(2010).
[16] C. Gerry and P.L. Knight “Introductory Quantum Optics” (Cambridge: Cambridge University Press),(2005).
[17] Menisha Alemu, Spontaneous Emission by Three-level Atoms Pumped by Electron Bombardment, *Universal Journal of Physics and Application* 14(1): 11-22, (2020).

© 2022 NSP
Natural Sciences Publishing Cor.