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Approximate Solutions of Fuzzy Fractional Differential Equations via Homotopy Analysis Method

Dulfikar Jawad Hashim[1], Ali Fareed Jameel[2,3,] and Teh Yuan Ying[1]*

[1]School of Quantitative Sciences, College of Arts and Sciences, Universiti Utara Malaysia (UUM), Kedah, Sintok, 06010 Malaysia
[2]Faculty of Education and Arts, Sohar University, Sohar 3111, Sultanate of Oman
[3]Institute of Strategic Industrial Decision Modelling (ISIDM), School of Quantitative Sciences (SQS), Universiti Utara, Malaysia (UUM), Kedah, Sintok, 06010, Malaysia

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Abstract: The Homotopy Analysis Method (HAM) is an approximate-analytical method for solving linear and nonlinear problems. HAM provides the auxiliary or convergence parameter, which considered as a powerful tool to examine and analyze the precision of the approximate series solution and ensure its convergence. In this article, fuzzy set theory properties is introduced to extend and reformulate HAM for the determination of approximate series solutions for fuzzy fractional differential equations involving initial value problems. The extension and reformulation of the method corresponding to Caputo's derivative in the fuzzy domain and the fuzzification of the method followed by the convergence analysis are presented in detail. Consequently, a new HAM for the general fractional differential equation has been developed in fuzzy domain. The difference between other types of approximate-analytical approaches and HAM is that the proposed HAM offers a better way to track the convergence region of the series solution via the convergence control parameter. The capability and accuracy of the method are illustrated by solving two examples involving linear and nonlinear fuzzy fractional differential equations. The obtained results using HAM suggested HAM is effective and simple to use when solving first order initial value problem involving a fuzzy fractional differential equation.

Keywords: Fuzzy numbers, fuzzy level sets, fuzzy fractional derivative, Caputo derivative, initial value problems, fuzzy fractional differential equations, homotopy analysis method.

1 Introduction

Fractional calculus is a generalization of classical calculus, which makes the study of continuous change even more exhaustive. Subsequently, this is not a modern ramification of pure and applied mathematics [1]. Fractional calculus is an emerging field in mathematics with deep applications in all related fields of engineering [2]. At this stage, it is an arena where many models are still to be introduced especially for nonlocality engineering problems such as signal processing [3] and fluid mechanics [4], plus several other interesting applications that can be found in [5,6,7,8,9,10]. Through the application of fractional calculus, fractional differential equations (FDEs) are considered to be a realistic approach to model complex phenomena through their significance in a wide range of engineering, physics, and other application sciences fields [11]. These fields especially involved those problems that have long memory term and cannot be accurately described by using the classical theory of calculus [12]. Such problems include those found in the biological population [13], fluid mechanics [14,15], digital circuit synthesis and legged robots [16]. In addition to many exciting applications of real-world problems based on long memory term and hereditary properties of different materials and processes, which be well predicted by using fractional differential operators [17]. Several approaches were utilized to define the differentiation and integration operators for FDE. For example, the definitions of Grünwald-Letnikov differential and integral operators starts from the classical definitions of derivative and integral based on infinitesimal division and limit. However, one of the disadvantages of this approach is its technical difficulty in computations, proofs and broad restrictions on functions. Fortunately, there are others more agile approaches like the Riemann-Liouville definition, which includes the results of the previous Grünwald-Letnikov definition as a special case. Nevertheless, it turns out that the FDEs that based on Riemann-Liouville derivatives also have

*Corresponding author e-mail: alifareed@uum.edu.my

certain disadvantages when trying to model real-world phenomena. Thus, many researchers preferred FDEs that based on Caputo derivative, which provides more appropriate features when handling real-world problems [18].

The modelling of dynamical problems by means of FDEs is considered incomplete especially, when data deficiencies, measurement errors, or uncertainty occurred in the initial condition [19]. Fuzzy set theory is considered a powerful tool to deal with the uncertainty that arises from a lack of information. An element of uncertainty was first introduced by Agarwal [20] into FDE; in addition to laying the basic concepts for the solution of a fuzzy fractional differential equation (FFDE) based on the notion of differentiability in Riemann-Liouville Hukuhara difference [20]. Therefore, employing FDEs in a fuzzy environment became imperative for generalization in terms of modelling over the last two decades. Many interesting papers can be seen which addressed the existence and uniqueness of the solution for FFDE [21-24], but at the same time we also aware that it is harder to obtain the exact solution for FFDE due to the existence of the fractional kernel under uncertainty. In recent years, researchers have investigated alternative numerical methods that used to approximate FFDEs [25]. We refer the readers to the recent papers [26-30] to explore the utilized methods in finding numerical solutions of FFDEs.

On the other hand, the approximate-analytical techniques offered a powerful tool for researchers and engineers to approximate the linear and nonlinear problems in the form of series solution; in addition, provide a better understanding of these problems. Examples of approximate-analytical techniques are such as the Adomian decomposition method [31] and two-component homotopy perturbation method (MHPM) [32], to name a few. However, the convergence region of the respective results produced by these approximate-analytical techniques is relatively small, and furthermore they do not provide any simple way to deal with this restriction. Only until Liao used the basic idea of homotopy in topology in 1992 to propose a general approximate-analytical method called the Homotopy Analysis Method (HAM) [33] that provide us with an easy and fast way to control and modify the convergence region to ensure the series solution converges. Application of HAM yields approximate-analytical solutions with certain advantages over traditional numerical methods. For instance, HAM does not require the discretization of variables and is thus free from round-off errors, and does not take colossal memory or time in the machine. In view of this, HAM was successfully applied to solve many linear and nonlinear problems of FDEs found in the fields of engineering [34], biology [35], health [36], and so on. However, the existing HAM procedure is still not suitable to approximate the solution of FFDE. Hence, it is necessary to introduce a new kind of HAM procedure called fuzzy fractional HAM in solving FFDEs.

In this paper, we formulate and employ the new approximate-analytical fractional HAM from crisp domain to the fuzzy domain in order to deal with first order fuzzy fractional initial value problems (FFIVPs) in Caputo sense. In particular, the main properties of fuzzy set such as the fuzzy number and fuzzy extension principles have been used to assist the formulation of our proposed method from crisp environment to fuzzy environment. For various perspectives on fuzzy environment, readers can review some relevant definitions such as membership function [37], the α-cut [38], fuzzy numbers [39] and the extension principle [40]. Then, the newly constructed fuzzy fractional HAM is tested on a linear non-homogeneous first order FFIVP and a fuzzy version of the nonlinear fractional Riccati differential equation. The following is the outline for this paper. In Section 2, we looked at several fundamental definitions and advanced ideas of the fractional calculus theory that will be useful in understanding the following parts of this research study. The new general fuzzy fractional HAM formula is presented and analyzed in Section 3. The fuzzy analysis of HAM for solving first order FFIVPs is discussed in Section 4, while the convergence of HAM is illustrated in Section 5. Section 6 demonstrated and discussed the capabilities of the proposed fuzzy fractional HAM in solving two test problems. Finally, this research paper is concluded in Section7.

2 Preliminaries

This section introduces the main definitions and notations of the fractional calculus being used to comprehend the work in the remaining sections. Definition 2.1 and Definition 2.2 illustrated the fractional integral and fractional derivative that are adopted from [28,41].

Definition 2.1 [28,41]. For real number μ, and $x > 0$, the real function $y(x)$ is said to be in the space C_μ, if $\exists\, p \in \mathbb{R}$ satisfy $y(x) = x^{p} y_1(x)$, where $y_1(x) \in C(0,\infty)$ whereas $y(x)$ is said to be in the space C_μ^m iff $y^n \in C_\mu, m \in \mathbb{N}$, where $\mathbb{N}$ represent the set of natural numbers.

Definition 2.2 [28,41]. A function $y(x)$ under the Riemann-Liouville derivative of fractional order β is given as

$$\widehat{D}^{\beta} g(x) = \begin{cases} \frac{1}{\Gamma(m-\beta)} \frac{d^m}{dx^m} \int_a^x \frac{g(z)}{(t-\tau)^{\beta-n+1}}\, dz, & x > a \\ \frac{d^m}{dx^m} g(z), & \beta = m \end{cases} \tag{1}$$

The outstanding features with the Riemann-Liouville fractional derivative can be summarized by the arbitrary function which neither to be continuous at the origin nor differentiable. However, the Caputo derivative is considered more commonly used, especially for modelling the physical and engineering problems because the Caputo derivative of a constant is zero which makes it a practical tool for problems of rate of change [41]. Furthermore, the Caputo fractional derivative allows initial or boundary conditions to be included in the formulation of the problem [42]. Now, it is reasonable to define the Caputo differentiation operator as in Definition 2.3.

Definition 2.3 [28,41]. Caputo fractional derivative of the function $y(x)$ is defined of order $\beta > 0$ in which $m-1 < \beta \leq m$, $m \in \mathbb{N}$, and $y \in C_{-1}^{n}, n \in \mathbb{N}$ with $0 \leq a$ as:

$$D^{\beta} y(x) = \frac{1}{\Gamma(m-\beta)} \int_a^x (x-t)^{(m-1)-\beta}\, y^{(m)}(x)\, dt. \tag{2}$$

Following Definition 2.3, the Caputo fractional derivative given in Eq. (2) possesses the following main properties [11]:

a) Suppose $y \in C_{-1}^{n}, n \in \mathbb{N}$. After this y is well defined and $D^{\beta} y \in C_{-1}$.
b) Suppose $m-1 < \beta \leq m, m \in \mathbb{N}$ and $y \in C_{\mu}^{n}, -1 \leq \mu$. Then

$$\left(J^{\beta} D^{\beta}\right)\left(y(x)\right) = y(x) - \sum_{i=0}^{m-1} y^{(i)}(0^{+})\, {x^{(i)}}/{i!}\,. \tag{3}$$

According to Eq. (3), it is reasonable to define the Riemann-Liouville integral operator as in Definition 2.4.

Definition 2.4 [28,41]. The operator of Riemann-Liouville fractional integral of the function $y(x)$ of order β denote by J_a^{β} and defined as

$$J^{\beta} y(x) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_a^y (x-t)^{\beta-1}\, y(t) dt, x > a, & \beta > 0 \\ y(x), & \beta = 0 \end{cases} \tag{4}$$

where $\Gamma(\beta)$ represents Gamma function which is defined as:

$$\Gamma(\beta) = \int_0^{\infty} x^{\beta-1} e^{-x} dx. \tag{5}$$

Below are some of the important features of J_a^{β} as defined by Eq. (4) in Definition 2.4 [11]:

a) $J^{\beta_1} J^{\beta_2} y(x) = J^{\beta_1+\beta_2} y(x) = J^{\beta_2} J^{\beta_1} y(x)$,
b) $J^{\beta} x^{b} = \frac{\Gamma(b+1)}{\Gamma(b+1+\beta)} x^{b+\beta}$
c) $\forall$ constant $t \in \mathbb{R}$, then $J^{\beta} t = \frac{t}{\Gamma(\beta+1)} x^{\beta}, \beta > 0$.
d) For $g(x) = (x-a)^{k}, -1 < k$, we have $J^{\beta} (x-a)^{k} = \frac{\Gamma(q+1)}{\Gamma(q+1+\beta)} (x-a)^{k+\beta}, \beta > 0, x > a$.

Remark 2.1 [43]. In general, the fractional operators Riemann-Liouville and Caputo do not coincide; that is, $\widehat{D}^{\beta} g(x) \neq D^{\beta} g(x)$. This is because $D^{m} J^{m-\beta} g(x) \neq J^{m-\beta} D^{m} g(x)$.

Corollary 2.1. The Caputo fractional derivative of the power function $g(x) = (x-a)^{k}$, can be represented by

$$D_{a^{+}}^{\beta} g(x) = \begin{cases} \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)} (x-a)^{k-\beta}, & m-1 < \beta \leq m, k > m-1, m \in \mathbb{N}, k \in \mathbb{R} \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

2 General fuzzy fractional HAM

The crisp form of HAM for describing fractional initial value problem (IVP) has been designed in [11], to present the fuzzy dynamic of HAM under the concept of homotopy from topology. We considered the following nonlinear fuzzy fractional equation:

$$\mathcal{F}_{\beta}[\tilde{y}(x;\alpha)] = 0, \tag{7}$$

where $\mathcal{F}_{\beta}$ is a nonlinear fractional differential operator, x is an independent variable, and $\tilde{y}(x;\alpha)$ is an unknown fuzzy fractional function. We can construct the zeroth-order deformation equation according to [24] as follows

$$\mathcal{H}(x;q;\alpha) = (1-q)\mathcal{L}_f\left[\widetilde{\emptyset}(x;q;\alpha) - \tilde{y}_0(x;\alpha)\right] - q\tilde{\hbar}(\alpha) H(x)\, \mathcal{F}_{\beta}\left[\widetilde{\emptyset}(x;q;\alpha)\right], \tag{8}$$

where, $q \in [0,1]$ is an embedding parameter, $\tilde{\hbar}(\alpha) \neq 0$ is a convergence-control parameter for all $\alpha \in [0,1]$, $H(x) \neq 0$ is an auxiliary function, $\mathcal{L}_f$ is an auxiliary linear operator, $\tilde{y}_0(x;\alpha)$ is the initial guess of $\tilde{y}(x;\alpha)$, and $\widetilde{\emptyset}(x;q;\alpha)$ is the auxiliary function which should be satisfied in the initial conditions function. It ought to be noted that the auxiliary function $H(x)$, and the convergence control parameter $\tilde{\hbar}(\alpha)$ are important to control the convergence of the series solution. Clearly, when $q = 0$, and $q = 1$, then Eq. (8) yields [11]

$$\begin{cases} \mathcal{H}(x;0;\alpha) = \mathcal{L}_f\left[\tilde{\emptyset}(x;0;\alpha) - \tilde{y}_0(x;\alpha)\right] = \tilde{y}_0(x;\alpha), \\ \mathcal{H}(x;1;\alpha) = \tilde{\hbar}(\alpha)H(x)\,\mathcal{F}_\beta\left[\tilde{\emptyset}(x;1;\alpha)\right] = \tilde{y}(x;\alpha)\ . \end{cases} \tag{9}$$

Thus, by requiring

$$\mathcal{H}(x;q;\alpha) = 0, \tag{10}$$

we have

$$(1-q)\mathcal{L}_f\left[\tilde{\emptyset}(x;q;\alpha) - \tilde{y}_0(x;\alpha)\right] = q\tilde{\hbar}(\alpha)H(x)\,\mathcal{F}_\beta\left[\tilde{\emptyset}(x;q;\alpha)\right]. \tag{11}$$

As q increases from zero to one, the series solutions will be deformed from the initial guess $\tilde{y}_0(x;\alpha)$ to the exact solution $\tilde{y}(x;\alpha)$. By expanding $\tilde{\emptyset}(x;q;\alpha) = [\overline{\emptyset}(x;q;\alpha)\,, \underline{\emptyset}(x;q;\alpha)]$ as a Taylor series with respect to q, we can obtain the following form of the series solution

$$\tilde{\emptyset}(x;q;\alpha) = \tilde{y}_0(x;\alpha) + \sum_{k=1}^{\infty} \tilde{y}_k(x,\alpha)\, q^k, \tag{12}$$

where

$$\tilde{y}_k(x;\alpha) = \frac{1}{k!}\frac{\partial^k \tilde{y}(x;q;\alpha)}{\partial q^k}. \tag{13}$$

Supposed that, the auxiliary linear operator $\mathcal{L}_f$, the auxiliary parameter $\tilde{\hbar}(\alpha)$, the auxiliary function $H(x)$ and the initial guess $\tilde{y}_0(x;\alpha)$ are chosen in such a way that makes the homotopy series solution is convergent at $q = 1$ as below

$$\tilde{y}(x;1;\alpha) = \tilde{y}_0(x;\alpha) + \sum_{k=1}^{\infty} \tilde{y}_k(x;\alpha), \tag{14}$$

which represents one of the solutions to the specified equation. It ought to be noted that homotopy perturbation method (HPM) can be derived from HAM when $\tilde{\hbar}(\alpha) = -1, \forall\, \alpha \in [0,1]$, and $H(x) = 1$. Therefore, if HPM is considered a special case of HAM [44], then the zeroth order deformation equation given in Eq. (8) takes the following form

$$\mathcal{H}(x;q;\alpha) = (1-q)\mathcal{L}_f\left[\tilde{\emptyset}(x;q;\alpha) - \tilde{y}_0(x;\alpha)\right] + q\,\mathcal{F}_\beta\left[\tilde{\emptyset}(x;q;\alpha)\right], \tag{15}$$

where $\mathcal{L}_f = D^\beta$, now by defining the vectors in Eq. (16), it is possible to deduce the governing equations from the zero-order deformation in Eq. (8) as in

$$\vec{\tilde{y}}_i(x;\alpha) = \{\tilde{y}_0(x;\alpha)\tilde{y}_1(x;\alpha), \dots, \tilde{y}_k(x;\alpha)\}. \tag{16}$$

On differentiating the zeroth order deformation equation in Eq. (8) k times with respect to q and then setting $q = 0$ and after that dividing them by $k!$, we obtain the k^{th}-order deformation equation as follows

$$\mathcal{L}_f[\tilde{y}_k(x;\alpha) - \psi_k \tilde{y}_{k-1}(x;\alpha)] = \tilde{\hbar}(\alpha)\,\mathcal{R}_k\left(\vec{\tilde{y}}_{k-1}(x;\alpha)\right), \tag{17}$$

where

$$\mathcal{R}_k\left(\vec{\tilde{y}}_{k-1}(x;\alpha)\right) = \frac{1}{(k-1)!}\frac{\partial^{k-1}\mathcal{F}_\beta[\tilde{\emptyset}(x;q;\alpha)]}{\partial q^{k-1}}\bigg|_{q=0}, \tag{18}$$

and

$$\psi_k = \begin{cases} 0, & k \le 1 \\ 1, & k > 1 \end{cases} \tag{19}$$

3 Analysis of HAM for first order FFIVPs

This section presents fuzzy fractional HAM for the first order FFIVP as stated below [45]:

$$\begin{cases} D^{\beta}\tilde{y}(x) = \tilde{g}\big(x, \tilde{y}(x)\big),\ \beta \in (0,1] \\ \tilde{y}(x_0) = \tilde{y}_{0,}\ x \in [x_0, X], y_{0,}, y \in \mathbb{R}_{F.} \end{cases} \tag{20}$$

Based on the properties of the defuzzification, we can reformulate Eq. (20) with more generalization $\forall \alpha \in [0,1]$ which yields

$$\begin{cases} D^{\beta}\tilde{y}(x;\alpha) = \tilde{g}(x, \tilde{y}(x;\alpha);\alpha),\ \beta \in (0,1] \\ \tilde{y}(x_0;\alpha) = \tilde{y}_{0,}\ x \in [x_0, X], y_{0,}, y \in \mathbb{R}_F\ , \end{cases} \tag{21}$$

with

$$\begin{cases} \tilde{y}(x) = \tilde{y}(x;\alpha) = \left[\underline{y}(x;\alpha), \overline{y}(x;\alpha)\right], \\ \tilde{g}\big(x, \tilde{y}(x)\big) = \tilde{g}(x, \tilde{y}(x);\alpha) = \left[\underline{g}(x, \tilde{y}(x);\alpha), \overline{g}(x, \tilde{y}(x);\alpha)\right], \\ \tilde{y}(x_0) = \tilde{y}(x_0;\alpha) = \left[\underline{y}_0, \overline{y}_0\right], \end{cases} \tag{22}$$

where, D^{β} is the fuzzy Caputo fractional generalized Hukuhara derivative of order β, g is a fuzzy function of the crisp variable x, and $\tilde{y}_{0,}$ is a fuzzy number. From Section 3, we can set the FFIVP in Eq. (21) as the system of fractional differential equations for all $\alpha \in [0,1]$ as below:

$$\begin{cases} D^{\beta}\underline{y}(x;\alpha) = \underline{g}\left(x, \underline{y}(x;\alpha)\right), \\ D^{\beta}\overline{y}(x;\alpha) = \overline{g}\big(x, \overline{y}(x;\alpha)\big). \end{cases} \tag{23}$$

Now we will work on developing HAM solution from crisp fractional initial value problem described in [24] to fuzzy fractional HAM. From Eq. (21) we can build the zeroth order deformation $\forall\ \alpha \in [0,1]$ as

$$(1-q)\mathcal{L}_f[\tilde{y}(x;q;\alpha) - \tilde{y}_0(x;\alpha)] = q\ \tilde{h}(\alpha)\big[D^{\beta}\tilde{y}(x;q;\alpha) - \tilde{g}(x, \tilde{y}(x;q;\alpha);\alpha)\big]. \tag{24}$$

It is clear, by setting $q = 0$ and $q = 1$, we obtained the initial guess $\tilde{y}_0(x;\alpha)$ and the exact solution $\tilde{y}(x;\alpha)$, respectively, as shown below:

$$\begin{cases} \tilde{y}(x;0;\alpha) = \tilde{y}_0(x;\alpha), \\ \tilde{y}(x;1;\alpha) = \tilde{y}(x;\alpha). \end{cases} \tag{25}$$

When q changes from 0 to 1, the fuzzy solution $\tilde{y}(x;q;\alpha)$ is deforming from the initial guess $\tilde{y}_0(x;\alpha)$ to the exact solution $\tilde{y}(x;\alpha)$. To obtain the HAM series solution, Taylor series will be employed on the solution $\tilde{y}(x;q;\alpha)$ in terms of q in the following form:

$$\tilde{Y}(x;q;\alpha) = \tilde{y}_0(x;\alpha) + \sum_{k=1}^{\infty} \tilde{y}_k(x;\alpha)\, q^k, \tag{26}$$

where

$$\tilde{y}_k(x;\alpha) = \frac{1}{k!}\frac{\partial^k \tilde{y}(x;q;\alpha).}{\partial q^k}. \tag{27}$$

For $q = 1$, we can identify the k^{th}-order deformation equation from Eq. (26) as

$$\tilde{Y}(x;\alpha) = \tilde{y}_0(x;\alpha) + \sum_{k=1}^{\infty} \tilde{y}_k(x;\alpha). \tag{28}$$

We can rewrite the k^{th}-order deformation equation in Eq. (28) as below:

$$D^{\beta}[\tilde{y}_k(x;\alpha) - \psi_k \tilde{y}_{k-1}(x;\alpha)] = \tilde{h}(\alpha)\left[D^{\beta}\tilde{y}_{k-1}(x;\alpha) - \tilde{g}\left(x, \tilde{y}_{k-1}(x;\alpha)\right)\right]. \quad (29)$$

Now, by employing the Riemann-Liouville fractional integral J_a^{β} (the left inverse operator of D^{β}) from Definition 2.4 on both sides of Eq. (29), we get

$$\tilde{y}_k(x;\alpha) = \psi_k \tilde{y}_{k-1}(x;\alpha) + J_a^{\beta}\tilde{h}(\alpha)\left[D^{\beta}\tilde{y}_{k-1}(x;\alpha) - \tilde{g}\left(x, \tilde{y}_{k-1}(x;\alpha)\right)\right] \quad (30)$$

for $k \geq 1$. For instance, from Eq. (30), we can extract the following first order deformation equation by setting $k = 1$ which yields:

$$\tilde{y}_1(x;\alpha) = \tilde{h}(\alpha)\tilde{y}_0(x;\alpha) - \tilde{h}(\alpha)\, J_a^{\beta}\tilde{g}\left(x, \tilde{y}_0(x;\alpha)\right), \quad (31)$$

where $\tilde{y}_0(x;\alpha)$ is the initial guess of Eq. (20). Now, for $k \geq 2$, we have

$$\tilde{y}_k(x;\alpha) = \tilde{y}_{k-1}(x;\alpha) + J_a^{\beta}\tilde{h}(\alpha)\left[D^{\beta}\tilde{y}_{k-1}(x;\alpha) - \tilde{g}\left(x, \tilde{y}_{k-1}(x;\alpha)\right)\right]. \quad (32)$$

Then the series solution by fuzzy fractional HAM is described by the form:

$$\tilde{y}\left(x;\alpha;\tilde{h}(\alpha)\right) = \tilde{y}_0(x;\alpha) + \sum_{k=1}^{\infty}\tilde{y}_k\left(x;\alpha;\tilde{h}(\alpha)\right). \quad (33)$$

So, the final homotopy series solution of Eq. (21) takes the following form

$$\tilde{y}(x;\alpha) = \lim_{k\to\infty}\sum_{k=0}^{\infty}\tilde{y}_k(x;\alpha). \quad (34)$$

4 Fuzzy fractional HAM convergence analysis

The convergence of HAM has been analyzed to obtain the approximate solution of different types of FDEs [37, 46]. It has been found that the convergence region for the solution from HAM relies on the convergence control parameter $\tilde{h}(\alpha)$ values because HAM offers a family of solution expressions in term of these parameters. Thus, the convergence of the approximate solution via HAM depends on the optimal value of $\tilde{h}(\alpha)$. Therefore, the process of determining the optimal value of $\tilde{h}(\alpha)$ plays a pivotal role in obtaining the most accurate approximate series solution for all fuzzy level sets. If we substitute Eq. (33) in the origin Eq. (21), we have the residual form of FFIVP that described as follows:

$$\tilde{R}\left(x;\alpha;\tilde{h}(\alpha)\right) = D^{\beta}\tilde{y}_k\left(x;\alpha;\tilde{h}(\alpha)\right) - \tilde{g}\left(x, \tilde{y}_k\left(x;\alpha;\tilde{h}(\alpha)\right)\right). \quad (35)$$

Then, to obtain the optimal value of $\tilde{h}(\alpha)$, we need to employ the least square method form $\left(\widetilde{RR}\right)$ for the residual formula in Eq. (35) as follows:

$$\widetilde{RR}\left(x;\alpha;\tilde{h}(\alpha)\right) = \int_{x_0}^{X}\left(\tilde{R}\left(x;\alpha;\tilde{h}(\alpha)\right)\right)^2 dx. \quad (36)$$

By setting the values of fractional order $0 < \beta \leq 1$, then the following partial derivatives with respect to $\tilde{h}(\alpha)$ can be obtained for all $\alpha \in [0,1]$ as bellow:

$$\frac{\partial \widetilde{RR}\left(x;\alpha;\tilde{h}(\alpha)\right)}{\partial \tilde{h}(\alpha)} = 0 \rightarrow \begin{cases} \dfrac{\partial \underline{RR}\left(x;\alpha;\underline{h}(\alpha)\right)}{\partial \underline{h}(\alpha)} = 0 \\ \dfrac{\partial \overline{RR}\left(x;\alpha;\overline{h}(\alpha)\right)}{\partial \overline{h}(\alpha)} = 0 \end{cases} \quad (37)$$

Then for each fuzzy level set $\alpha \in [0,1]$, we can determine the optimal value of the convergence control parameter $\tilde{h}(\alpha)$ by solving the system of nonlinear equations in Eq. (37) numerically in term of $\tilde{h}(\alpha)$. Assessment of the optimal value in between the best values of $\tilde{h}(\alpha)$ is aimed to obtain the most accurate HAM series solution $\tilde{y}(x;\alpha)$, for all $\alpha \in [0,1]$ by plotting the $\tilde{h}$-curves in term of upper and lower HAM solutions. These curves define the best region of the $\tilde{h}(\alpha)$ values which are the horizontal line segment with respect $\tilde{y}_t(t_0;\alpha)$ for $0 < t_0 < T$. As familiar in fuzzy environment find the contract h-curves and find the optimal value $\tilde{h}(\alpha)$ for each $\alpha \in [0,1]$. We get to choose the best value of $\tilde{h}(\alpha)$ that provides the best accurate solution of FFIVP with its corresponding fuzzy level set $\tilde{\alpha} = [\underline{\alpha}, \overline{\alpha}]$, then applied $\underline{\alpha}$ for each level set to get the best lower approximate solution. An identical step is applied to $\overline{\alpha}$ to determine the best upper solution.

5 Applications

This section presents the implementation and the analysis of fuzzy fractional HAM in solving FFIVP which described in Eq. (21) of order $0 < \beta \leq 1$ using two examples as follows.

Example 6.1. Consider the following linear nonhomogeneous first order FFIVP [47]:

$$y^{(\beta)}(x) - y(x) = [1+\alpha, 3-\alpha],\ 0 \leq x \leq 1,\ \tilde{y}(0) = \tilde{y}_0(x;\alpha) = 0, \tag{38}$$

where, $y^{(\beta)}(x)$ is the Caputo fractional derivative of order $\beta \in (0,1]$, $[1+\alpha, 3-\alpha]$ are fuzzy numbers defined in [39], $\tilde{y}_0(x;\alpha)$ represents the fuzzy initial condition, and $y(x)$ represent a crisp function of non-fuzzy independent variable x. The exact solution of Eq. (38) when $\beta = 1$ is [47]:

$$[Y(x)]^\alpha = 2e^x + [1-\alpha, \alpha-1](1-e^{-x}). \tag{39}$$

From Section 4, the kth order deformation equation of Eq. (38) is given by:

$$\tilde{y}_k(x;\alpha) = \psi_{k-1}\tilde{y}_{k-1}(x;\alpha) + \tilde{h}(\alpha)\left[\tilde{y}_{k-1}(x;\alpha) - y(x) - (1-\psi_{k-1})[1+\alpha, 3-\alpha]\right] \tag{40}$$

According to Section 4, the selected initial guess of Eq. (38) is $\tilde{y}_0(x;\alpha) = \left[\underline{y}_0(x;\alpha), \overline{y}_0(x;\alpha)\right] = \tilde{0}$. From Section 5, the values of $\tilde{h}(\alpha)$ can be obtained from Eq. (37) and by testing all the values of $\tilde{h}(\alpha)$ at $\beta = 0.8$ for $\alpha \in [0,1]$, the best values of $\tilde{h}(\alpha)$ can be found at $\alpha = 0.8$. Therefore, we can identify the convergence region for $\tilde{h}(\alpha)$ by plotting the $\tilde{h}(\alpha)$-curves of the eighth order fuzzy fractional HAM lower bound solution $\underline{y}(x;\alpha)$ and upper bound solution $\overline{y}(x;\alpha)$ for Eq. (38) in Figure 1.

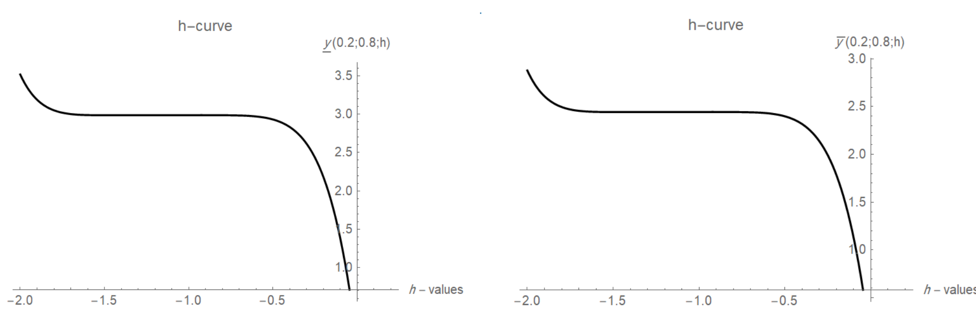


Fig.1: Eighth order fuzzy fractional HAM solution $\tilde{h}(\alpha)$-curves of Eq. (38) with fractional order $\beta = 0.8$ at $\alpha = 0.8$ and $x = 0.2$.

Here, Figure 1 illustrates the valid region of the fuzzy convergence parameters $\tilde{h} = [\underline{h}, \overline{h}]$, and it is obvious, the region of the valid convergence control parameters $\tilde{h}(\alpha)$ values is on $-1.7 \leq \tilde{h} \leq -0.5$ since the line segment of the eighth order fuzzy fractional HAM solution is nearly parallel to the horizontal axis on $-1.7 \leq \tilde{h} \leq -0.5$. Followed by picking the best values of $\underline{h}$ at $\beta = 0.8$ of Eq. (38) as shown in Table 1.

Table1: Optimal values of $\underline{h}(0.8)$ via eighth order fuzzy fractional HAM for lower solution of Eq. (38) when $\beta = 0.8$ at $\alpha = 0.8$.

$\underline{y}(x, \underline{h}(0.8); 0.8)$	$\underline{h}_1 \to -1.1765880300497917$	$\underline{h}_2 \to -1.0948756826863355$
$\underline{h}_3 \to -1.0595694863978484$	$\underline{h}_4 \to -1.0403646323928808$	$\underline{h}_5 \to -1.0280894444733604$

Table 1 consist of the lower optimum convergence control parameters based on the minimum residual error indicated in Section 5 which guarantee the convergence of the series solutions. Figure 2 presents the accuracy of the lower eighth order fuzzy fractional HAM solution from values of $\underline{h}(0.8)$ being extracted from Table 1, which corresponds to the residual error of Eq. (38) as defined in Eq. (41):

$$\widetilde{Er}\big(x;\alpha;\tilde{h}(\alpha)\big) = D^{\beta}\tilde{y}_8\left(x;\alpha;\tilde{h}(\alpha)\right) - \left(\tilde{y}_8\left(x;\alpha;\tilde{h}(\alpha)\right) + [1+\alpha, 3-\alpha]\right). \tag{41}$$

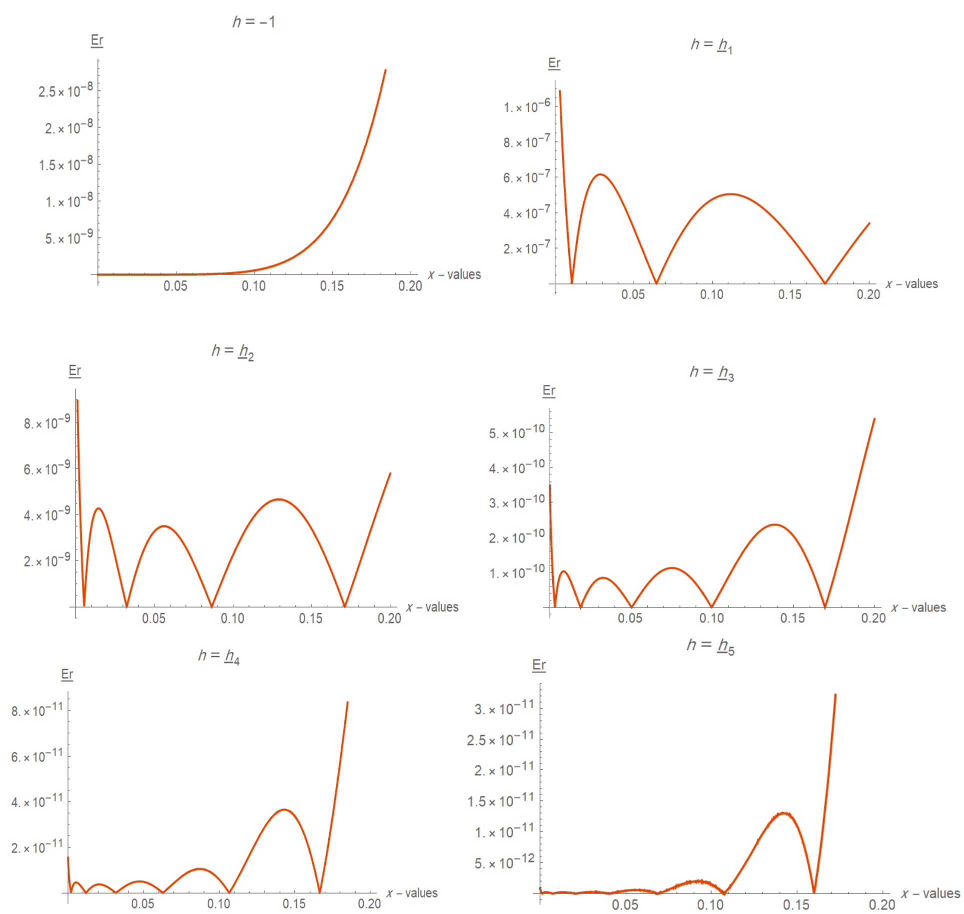


Fig.2: Fuzzy fractional HAM of order eight linked with the lower control parameters values $\underline{h}(0.8)$ when $\beta = 0.8$ for all $x \in [0,0.2]$.

As can be seen from the accuracy of the series solutions depending on each convergence control parameter $\underline{h}$ in Figure 2, one can note that the optimal value $\underline{h}_4 = -1.0403646323928808$ will gives the optimal lower solution $\underline{y}(x;\alpha)$ of Eq. (38) at $\beta = 0.8$ that can be summarized in Table 2.

Table 2: Eighth order fuzzy fractional HAM lower solution and the residual error of Eq. (38) at $\beta = 0.8$, $x = 0.2$ and $h = \underline{h}_4$ for $\alpha \in [0,1]$.

α	$\underline{y}(x;\alpha;\underline{h}_4)$	$\underline{Er}(x,\alpha;\underline{h}_4)$

0	0.35740620806381385	$-7.666089985036706 \times 10^{-11}$
0.2	0.42888744967658066	$-9.200951112120492 \times 10^{-11}$
0.4	0.50036869128933950	$-1.073354738423404 \times 10^{-10}$
0.6	0.57184993290209980	$-1.226840851131783 \times 10^{-10}$
0.8	0.64333117451487580	$-1.380402459005836 \times 10^{-10}$
1	0.71481241612762950	$-1.533217997007341 \times 10^{-10}$

Table 2 illustrate the lower solutions and the series solutions accuracy corresponding to the residual error of Eq. (38) employing the optimum convergence control parameter $\underline{h}_4$. Similarly, the best values of $\overline{h}\,(\alpha)$ for the upper solution via eighth order fuzzy fractional HAM of Eq. (38), when $\alpha = 0.8$ and $\beta = 0.8$ are displayed in Table 3.

Table 3: Optimal values of $\overline{h}\,(0.8)$ via eighth order fuzzy fractional HAM solution of Eq. (38) when $\beta = 0.8$ at $\alpha = 0.8$.

$\overline{y}(x, \overline{h}(0.8); 0.8)$	$\overline{h}_1 \to -1.17658802946494$	$\overline{h}_2 \to -1.094875689109602$
$\overline{h}_3 \to -1.0595694690969126$	$\overline{h}_4 \to -1.0403646588529756$	$\overline{h}_5 \to -1.0280894336074455$

Figure 3 demonstrate the accuracy of the upper eighth order fuzzy fractional HAM solution based on the values of $\overline{h}(0.8)$ being extracted from Table 3 corresponding on the residual error of Eq. (38) that defined in Eq. (41).

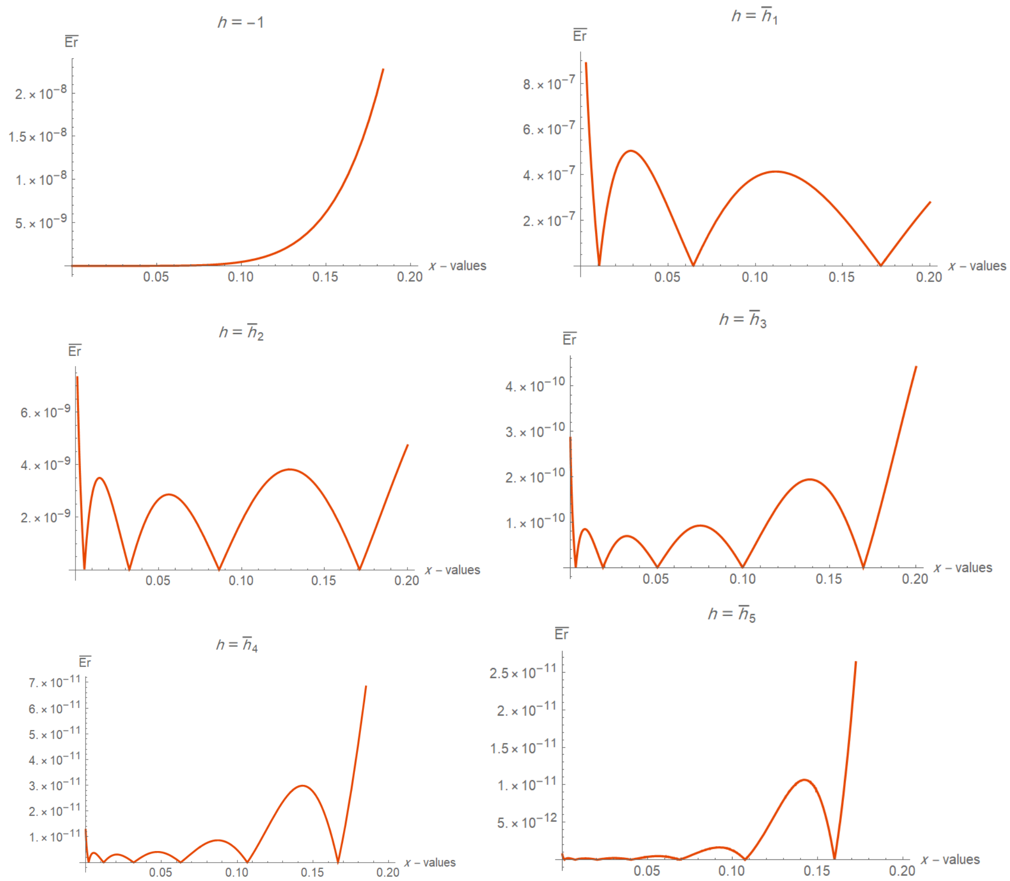


Fig.3: Fuzzy fractional HAM of order eight linked with the optimal upper control parameters values $\overline{h}(0.8)$ at $\beta = 0.8$ for all $x \in [0,0.2]$.

Figure 3 illustrates the accuracy of upper solution of Eq. (38) for each $\overline{h}$ in Table 3. Thus, the best value of $\overline{h}(0.8)$ that can be noticed from Figure 3 is $\overline{h}_5 = -1.0280894336074455$ which gives the best approximate upper solution $\overline{y}(x; \alpha)$ of Eq. (38) at $\beta = 0.8$ and can be displayed in Table 4.

Table 4. Eighth order upper solution via fuzzy fractional HAM and the residual error of Eq. (38) when $\beta = 0.8$ at $x = 0.2$ and $h = \overline{h}_5$ for $\alpha \in [0,1]$.

α	$\overline{y}(x;\alpha;\overline{h}_5)$	$\overline{Er}(x,\alpha;\overline{h}_5)$
0	1.0722186242059966	$2.344044958135782 \times 10^{-10}$
0.2	1.0007373825922476	$2.187503511663635 \times 10^{-10}$
0.4	0.9292561409785227	$2.031139700875428 \times 10^{-10}$
0.6	0.8577748993647978	$1.875051225397328 \times 10^{-10}$
0.8	0.7862936577510702	$1.719078213113789 \times 10^{-10}$
1	0.7148124161373230	$1.562994178527788 \times 10^{-10}$

Table 4 illustrates the upper solutions and the series solutions accuracy corresponding to the residual error of Eq. (38) employing the optimum convergence control parameter $\overline{h}_5$.

Clearly, from Table 2 and Table 4, we can see that the solutions of Eq. (38) with fractional order $\beta = 0.8$ via fuzzy fractional HAM of order eight at $x = 0.2$ fulfilled the properties of fuzzy solutions [39] that can be displyed in Figure 4.

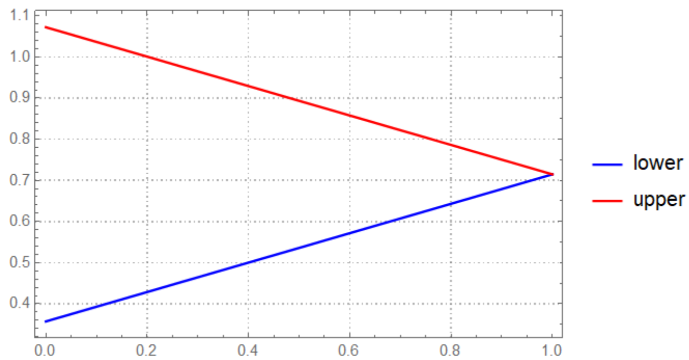


Fig. 4: Eighth order fuzzy fractional HAM corresponding with the best values $\tilde{h} = [\underline{h}_4, \overline{h}_5]$ of Eq. (38) with fractional order $\beta = 0.8$ at $x = 0.2$ and $\alpha \in [0,1]$.

We can also summarize the solutions of eighth order fuzzy fractional HAM over all $x \in [0,0.2]$ and $\alpha \in [0,1]$ corresponding with best values $\tilde{h} = [\underline{h}_4, \overline{h}_5]$ of Eq. (38) in the following three-dimensional figure.

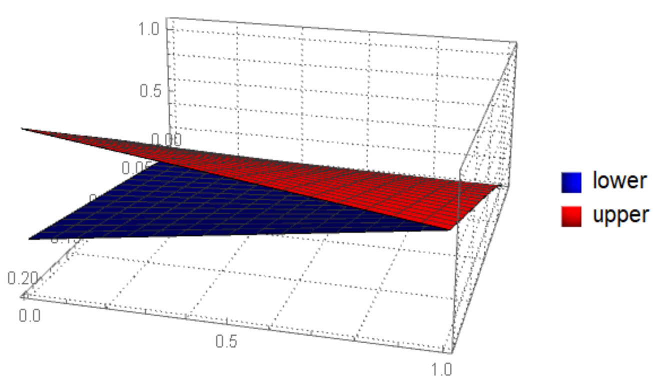


Fig.5: Three-dimensional eighth order fuzzy fractional HAM corresponding with best values $\tilde{h} = [\underline{h}_4, \overline{h}_5]$ of Eq. (38) with fractional order $\beta = 0.8$ at $x \in [0,0.2]$ and $\alpha \in [0,1]$.

From Tables 2 and 4 and Figures 4 and 5, we can also notice that the eighth order fuzzy fractional HAM solutions of Eq. (38) with fractional order $\beta = 0.8$ fulfilled the properties of the fuzzy differential equation in the form of triangular fuzzy numbers for all $\alpha \in [0,1]$ and $x \in [0,0.2]$.

The next step is to compare the eighth order fuzzy fractional HAM and the eighth order fractional residual power method (FRPS) [47] under Hukuhara Caputo differentiability with the fractional derivative order $\beta = 1$ at different values of $\alpha \in [0,1]$ and $x \in [0,1]$. Therefore, convergence region that will gives the best values of $\tilde{h}(\alpha)$ for $\beta = 1$ of the eighth order fuzzy fractional HAM is when $\alpha = 1$. The corresponding $\tilde{h}(\alpha)$-curves for the lower bound and upper bound solution for Eq. (38) as in Figure 6.

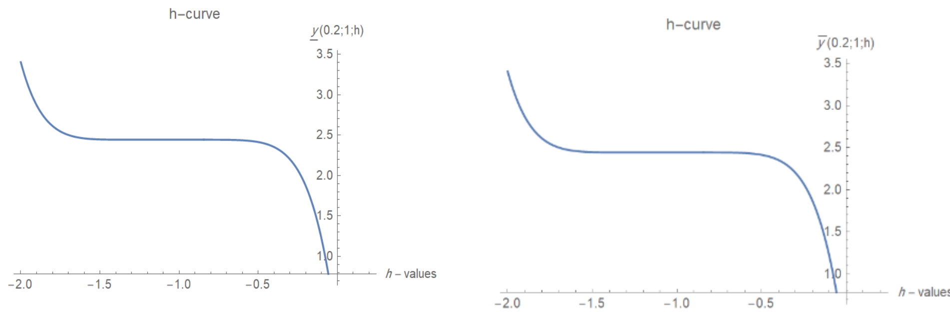


Fig. 6: Eighth order fuzzy fractional HAM solution $\tilde{h}(\alpha)$-curves of Eq. (38) with fractional order $\beta = 1$ at $\alpha = 1$ and $x = 0.2$.

According to Figure 6, clearly the region of the valid convergence control parameters $\tilde{h}(\alpha)$ by eighth order fuzzy fractional HAM is simply on $-1.6 \leq \tilde{h} \leq -0.5$. The best valid values of $\tilde{h} = [\underline{h}, \overline{h}]$ at $\alpha = 1$ and $\beta = 1$ are shown in Table 5.

Table 5: Optimal values of $\underline{h}$ (1) and $\overline{h}(1)$ of the eighth order fuzzy fractional HAM solution of Eq. (38) with order $\beta = 1$ at $\alpha = 1$.

$\overline{y}(x, \overline{h}(1); 1)$	$\overline{h}_1 \to -1.7455492187297452$	$\overline{h}_2 \to -1.4511629931799337$
	$\overline{h}_3 \to -1.0635601570097657$	$\overline{h}_4 \to -1.1204523683849168$
$\underline{y}(x, \underline{h}(1); 1)$	$\underline{h}_1 \to -1.3421191799482968$	$\underline{h}_2 \to -1.1220082931377502$
	$\underline{h}_3 \to -1.0635601570097657$	$\underline{h}_4 \to -1.0386655248309158$

From Table 5, it can be concluded that the best values of $\tilde{h}$ changed according to the order of the fractional derivative $0 < \beta \leq 1$ at the same order of the approximate solution. After testing the values of $\tilde{h}$, we found that the best values $\tilde{h}$ that gives the best solution of Eq. (38) are $\tilde{h}_3 = [\underline{h}_3, \overline{h}_3]$. Figure 7 demonstrated the accuracy of lower and upper solution of eighth order fuzzy fractional HAM solution from the values $\tilde{h}_3$ found in Table 5 and compared with $h = -1$, which corresponds to the absolute error defined in Eq. (42):

$$\widetilde{Err} = \left|\tilde{Y}(x; \alpha) - \tilde{y}_8(x; \alpha; \tilde{h})\right| \tag{42}$$

Figure 7 also illustrated the accuracy of the series solutions for Eq. (38) at $\beta = 1$. Based on Figure 7, the solutions of eighth order fuzzy fractional HAM over all $x \in [0,1]$ and $\alpha \in [0,1]$ with corresponding optimal value $\tilde{h}_3$ of Eq. (38) for $\beta = 1$ can be presented in three-dimensional space as shown in Figure 8.

We can conclude that the results in Figure 8 obtained by eighth order fuzzy fractional HAM solutions of Eq. (38) with fractional order $\beta = 1$ fulfilled the fuzzy differential equation properties in the form of triangular fuzzy numbers for all $\alpha \in [0,1]$ and $x \in [0,1]$. Tables 6 until 8 showed the comparison between eighth order fuzzy fractional HAM with the optimal values $\tilde{h}_3 = [\underline{h}_3, \overline{h}_3]$ and eighth order FRPS of Eq. (38) for different values of x and α when the order of the fractional derivative is fixed for $\beta = 1$

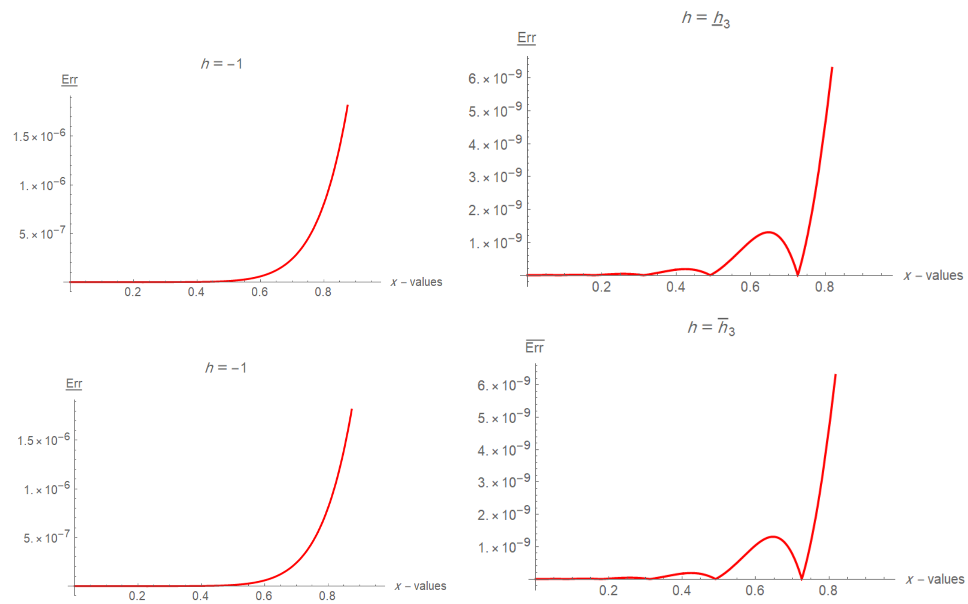


Fig. 7: Fuzzy fractional HAM of order eight linked with the optimal value of $\tilde{h}_3(1)$ the lower and upper bounds solutions of Eq. (38) for $\beta = 1$ and $\alpha = 1, \forall\, x \in [0,1]$.

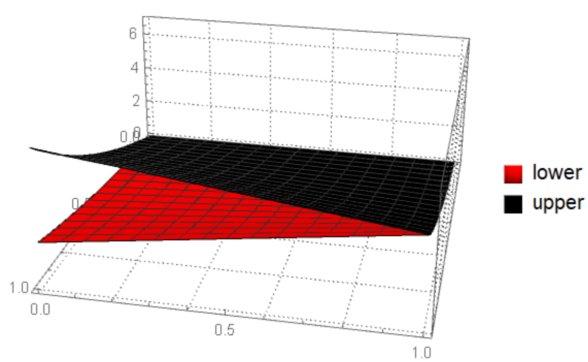


Fig. 8: Eighth order fuzzy fractional HAM corresponding with optimal value $\tilde{h}_3(1)$ of Eq. (38) with $\beta = 1$ for all $x \in [0,1]$ and $\alpha \in [0,1]$.

Table 6: Numerical comparison of approximated solutions of Eq. (38) for different values of x and $\alpha = 1$ when $\beta = 1$.

x	FRPS $\overline{Err}$ [47]	HAM $\overline{Err}$	FRPS $\underline{Err}$ [47]	HAM $\underline{Err}$
0.32	3.0×10^{-10}	5.904×10^{-12}	3.00×10^{-10}	5.892×10^{-12}
0.48	8.00×10^{-9}	4.251×10^{-11}	8.00×10^{-9}	4.246×10^{-11}
0.64	1.06×10^{-7}	5.626×10^{-10}	1.06×10^{-7}	5.625×10^{-10}
0.80	8.04×10^{-7}	3.1590×10^{-9}	8.04×10^{-7}	3.1590×10^{-9}
0.96	4.21×10^{-6}	2.2010×10^{-8}	4.21×10^{-6}	2.2010×10^{-8}

Table 7: Numerical comparison of approximated solutions of Eq. (38) for different values of x and $\alpha = 0.5$ when $\beta = 1$.

x	FRPS $\overline{Err}$ [47]	HAM $\overline{Err}$	FRPS $\underline{Err}$ [47]	HAM $\underline{Err}$
0.32	0	5.816×10^{-13}	0	9.816×10^{-13}
0.48	1.00×10^{-8}	2.717×10^{-11}	6.000×10^{-9}	4.525×10^{-11}
0.64	1.33×10^{-7}	4.630×10^{-10}	7.900×10^{-8}	7.717×10^{-10}

0.80	1.00×10^{-6}	2.864×10^{-8}	6.030×10^{-7}	4.773×10^{-8}
0.96	5.27×10^{-6}	3.300×10^{-7}	3.164×10^{-6}	5.500×10^{-7}

Table 8: Numerical comparison of approximated solutions of Eq. (38) for different values of x and $\alpha = 0$ when $\beta = 1$.

x	FRPS $\overline{Err}$ [47]	HAM $\overline{Err}$	FRPS $\underline{Err}$ [47]	HAM $\underline{Err}$
0.32	4.000×10^{-10}	3.939×10^{-13}	1.000×10^{-10}	4.217×10^{-12}
0.48	1.200×10^{-8}	1.811×10^{-11}	4.000×10^{-9}	2.708×10^{-11}
0.64	1.590×10^{-7}	3.087×10^{-10}	5.300×10^{-8}	6.715×10^{-10}
0.80	1.200×10^{-6}	1.909×10^{-8}	4.020×10^{-7}	6.169×10^{-9}
0.96	6.320×10^{-6}	2.200×10^{-7}	2.109×10^{-6}	2.217×10^{-8}

Based on the findings in Tables 6 until 8, we concluded that the approximate series solutions via fuzzy eight order fractional HAM at the optimal vales of value $\tilde{h}_3 = [\underline{h}_3, \overline{h}_3]$ at different values of x and α gives more accurate solutions compared to the numerical results obtained from the eight order solution of FRPS method [47].

Example 6.2. In this example, the classical crisp nonlinear fractional Riccati differential equation introduced in [48] is treated in its fuzzy form by employing the analysis of fuzzy fractional HAM as introduced in Section 4. The fractional Riccati differential equation is given by:

$$\begin{cases} D^{\beta} y + y^2(x) - 1 = 0, \quad 0 < \beta \leq 1 \\ y(0) = 0 \end{cases} \tag{43}$$

where D^{β} is a fractional derivative of order β in Caputo sense. The applicability of fuzzy fractional HAM to approximate the fuzzy version of Eq. (43) is explained in this section too. Consider the nonlinear first order fuzzy fractional Riccati differential equation:

$$D^{\beta} \tilde{y} + \tilde{y}^2(x) - 1 = 0, \quad 0 < \beta \leq 1 \tag{44}$$

subject to initial condition

$$\tilde{y}(0) = [\tilde{0}] = [\underline{0}, \overline{0}], \forall\, \alpha \in [0,1]. \tag{45}$$

According to the definitions of fuzzy number, we can defuzzify the initial conditions. Let $[\tilde{0}]_{\alpha}$ be triangular fuzzy numbers, such that for all $\alpha \in [0,1]$ we have

$$[\tilde{0}]_{\alpha} = [-1,0,1]_{\alpha} = [\alpha - 1, 1 - \alpha]. \tag{46}$$

Based on Section 4, we can construct the first order deformation equation:

$$\begin{cases} (1-q)\underline{\mathcal{L}_f}\left[\underline{\emptyset}(x;q;\alpha) - \underline{y}_0(x;\alpha)\right] = q\underline{\hbar}(\alpha)\mathfrak{N}_{\beta}\left(\left[\widetilde{\emptyset}(x;q;\alpha)\right]\right) \\ (1-q)\overline{\mathcal{L}_f}\left[\overline{\emptyset}(x;q;\alpha) - \overline{y}_0(x;\alpha)\right] = q\overline{\hbar}(\alpha)\mathfrak{H}_{\beta}\left(\left[\widetilde{\emptyset}(x;q;\alpha)\right]\right) \end{cases} \tag{47}$$

With

$$\mathfrak{N}_{\beta}\left(\left[\widetilde{\emptyset}(x;q;\alpha)\right]\right) = \min\{\mathcal{F}_{\beta}[\underline{\emptyset}(x;q)]_{\alpha}, \mathcal{F}_{\beta}[\overline{\emptyset}(x;q)]_{\alpha} : \delta | \delta \in [\tilde{y}(x)]\},$$
$$\mathfrak{H}_{\beta}\left(\left[\widetilde{\emptyset}(x;q;\alpha)\right]\right) = \max\{\mathcal{F}_{\beta}[\underline{\emptyset}(x;q)]_{\alpha}, \mathcal{F}_{\beta}[\overline{\emptyset}(x;q)]_{\alpha} : \delta | \delta \in [\tilde{y}(x)]\},$$

where δ represents the membership function. Now, we can define the auxiliary linear operator as $\tilde{\mathcal{L}}_f = D^{\beta}$ for any integral constant s satisfying $\tilde{\mathcal{L}}_f(s) = 0$. Based on the expression of the series solution of $\tilde{y}(x;\alpha)$ and according to the initial condition in Eq. (45), we can choose the initial guess $\tilde{y}_0(x;\alpha)$ as

$$\tilde{y}_0(x;\alpha) = \left[\underline{y}_0, \overline{y}_0\right] = [(\alpha-1),(1-\alpha)]. \tag{48}$$

Hence k^{th} order deformation equations are expressed as below:

$$\begin{cases} D^{\beta}\left[\underline{y}_k(x;\alpha) - \chi_k \underline{y}_{k-1}(x;\alpha)\right] = \hbar \mathcal{R}_k\left(\vec{\underline{y}}_{k-1}\right), \\ D^{\beta}\left[\overline{y}_k(x;\alpha) - \chi_k \overline{y}_{k-1}(x;\alpha)\right] = \hbar \mathcal{R}_k\left(\vec{\overline{y}}_{k-1}\right). \end{cases} \tag{49}$$

For $k \geq 1$, we apply the fractional integral operator J^{β} from Definition 2.4 on both sides of Eq. (49) and obtain

$$\begin{cases} \underline{y}_k(x;\alpha) = \chi_k \underline{y}_{k-1}(x;\alpha) + \hbar J^{\beta} \mathcal{R}_k\left(\vec{\underline{y}}_{k-1}\right), \\ \overline{y}_k(x;\alpha) = \chi_k \overline{y}_{k-1}(x;\alpha) + \hbar J^{\beta} \mathcal{R}_k\left(\vec{\overline{y}}_{k-1}\right). \end{cases} \tag{50}$$

The fuzzy fractional HAM series solution for the lower bound of Eq. (44) can be expressed as follows

$$\underline{y}(x;\alpha) = \underline{y}_0(x;\alpha) + \underline{y}_1(x;\alpha) + \underline{y}_2(x;\alpha) + \underline{y}_3(x;\alpha) + \cdots. \tag{51}$$

Some of the first lower bound terms of the proposed fuzzy fractional series solution using HAM of Eq. (44), and $\beta = 0.9$ starting with the lower initial guess are listed as follows:

$\underline{y}_0 = (\alpha - 1),$
$\underline{y}_1 = c_0 x^{\beta},$
$\underline{y}_2 = (h+1)c_0 x^{\beta} + h c_1 x^{2\beta},$
$\underline{y}_3 = (h+1)^2 c_0 x^{\beta} + (h(h+1)c_1 + c_3)x^{2\beta} + (c_2 + c_4)x^{3\beta},$
$\underline{y}_4 = (h+1)^3 c_0 x^{\beta} + (h(h+1)^2 c_1 + (h+1)c_3 + c_5)x^{2\beta} + \left((h+1)(c_2+c_4) + c_6\right)x^{3\beta} + c_7 x^{4\beta},$

and the first few coefficients of c_k are:

$c_0 = \frac{(\alpha^2 - 2\alpha)}{\Gamma(1.7)} h,$
$c_1 = \frac{\Gamma(1.7)}{\Gamma(2.4)}(2\alpha - 2)c_0,$
$c_2 = \frac{\Gamma(2.4)}{\Gamma(3.1)} h {c_0}^2,$
$c_3 = \frac{\Gamma(1.7)}{\Gamma(2.4)}(2\alpha - 2)h(h+1)c_0,$
$c_4 = \frac{\Gamma(2.4)}{\Gamma(3.1)}(2\alpha - 2)h^2 c_1,$
$c_5 = \frac{\Gamma(1.7)}{\Gamma(2.4)}(2\alpha - 2)\, h(h+1)^2 c_0,$
$c_6 = \frac{\Gamma(2.4)}{\Gamma(3.1)}\left((2\alpha - 2)\left(h(h(h+1)c_1 + c_3)\right) + 2h(h+1){c_0}^2\right),$
$c_7 = \frac{\Gamma(3.1)}{\Gamma(3.8)}\left((2\alpha - 2)(c_2 + c_4)h + 2(c_0 c_1)h\right).$

On the other hand, the fuzzy fractional HAM series solution for the upper bound of Eq. (44) can be expressed as below

$$\overline{y}(x;\alpha) = \overline{y}_0(x;\alpha) + \overline{y}_1(x;\alpha) + \overline{y}_2(x;\alpha) + \overline{y}_3(x;\alpha) + \cdots. \tag{52}$$

Several of the first upper bound terms of the proposed fuzzy fractional series solution via HAM of Eq. (44) starting with upper initial guess are listed as follows:

$$\overline{y}_0 = (1-\alpha),$$
$$\overline{y}_1 = r_0 x^{\beta},$$
$$\overline{y}_2 = r_1 x^{\beta} + r_2 x^{2\beta},$$
$$\overline{y}_3 = (r_1 + hr_1)x^{\beta} + (r_2 + r_3 + r_5)x^{2\beta} + (r_4 + r_6)x^{3\beta},$$
$$\overline{y}_4 = (h+1)^2 x^{\beta} + \big((h+1)(r_2+r_3+r_5)+r_7\big)x^{2\beta} + \big((h+1)(r_4+r_6)+r_8\big)x^{3\beta} + r_9 x^{4\beta},$$

while some of the first coefficients for the upper bound are:

$$r_0 = c_0,$$
$$r_1 = r_0 + \frac{(\alpha^2-2\alpha)}{\Gamma(1.7)}h^2,$$
$$r_2 = \frac{\Gamma(1.7)}{\Gamma(2.4)}(2-2\alpha)hr_0,$$
$$r_3 = \frac{\Gamma(1.7)}{\Gamma(2.4)}(2-2\alpha)h^2 r_0,$$
$$r_4 = \frac{\Gamma(2.4)}{\Gamma(3.1)}h{r_0}^2,$$
$$r_5 = \frac{\Gamma(1.7)}{\Gamma(2.4)}(2-2\alpha)hr_1,$$
$$r_6 = \frac{\Gamma(2.4)}{\Gamma(3.1)}(2-2\alpha)hr_2,$$
$$r_7 = \frac{\Gamma(1.7)}{\Gamma(2.4)}(2-2\alpha)h(h+1)r_1,$$
$$r_8 = \frac{\Gamma(2.4)}{\Gamma(3.1)}\big((2-2\alpha)(r_2+r_3+r_5)+2r_0r_1\big)h,$$
$$r_9 = \frac{\Gamma(3.1)}{\Gamma(3.8)}\big((2-2\alpha)(r_4+r_6)+2r_0r_2\big)h.$$

Now, by setting the $\tilde{\alpha}$-cut as $[\underline{\alpha}, \overline{\alpha}]$ and depending on the Section 5, we can pick the optimal values of $\tilde{h}(\tilde{\alpha})$-curves of fuzzy fractional HAM for fuzzy solution of Eq. (44) as illustrated in Figure 9.

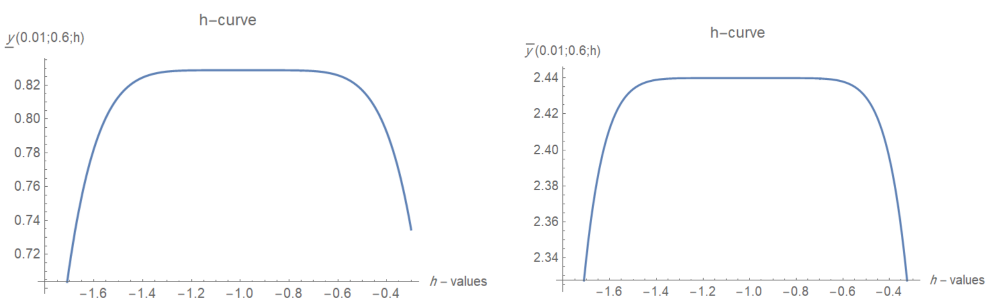


Fig. 9: Sixth order fuzzy fractional HAM solution $\tilde{h}(\alpha)$-curves of Eq. (44) at $\tilde{\alpha} = 0.6$ and $\beta = 0.9$.

According to Figure 9 which illustrates the valid region of the convergence control parameters $\tilde{h}$, we can conclude that the region of the valid $\tilde{h}$ values will be bounded by $-1.5 \le \tilde{h} \le -0.5$. To obtain the approximation for the lower analytical solution of Eq. (44) using the lower fuzzy fractional HAM series that showed in Eq. (51), we selected the best values of the convergence control parameters $\underline{h}$ as listed in Table 9.

Table 9: Optimal values of $\underline{h}$ (0.6) of the sixth order solution of fuzzy fractional HAM of Eq. (44) at $\underline{\alpha} = 0.6$ and $\beta = 0.9$.

$\underline{y}(x, \underline{h}(0.6); 0.6)$	$\underline{h}_1 \to -1.0827795143 72233$	$\underline{h}_2 \to -1.0170948620587954$
$\underline{h}_3 \to -1.0051037230460373$	$\underline{h}_4 \to -0.996267552349457$	$\underline{h}_5 \to -0.9721253852779959$

Figure 10 showed the accuracy of the sixth order fuzzy fractional HAM based on $\underline{h}(0.6)$ values that extracted from Table 9, and the corresponding residual error of Eq. (44) is as follows:

$$\widetilde{Er}\left(x;\alpha;\tilde{h}(\alpha)\right) = D^{\beta}\tilde{y}_6\left(x;\alpha;\tilde{h}(\alpha)\right) + \tilde{y}_6{}^2\left(x;\alpha;\tilde{h}(\alpha)\right) - 1. \quad (53)$$

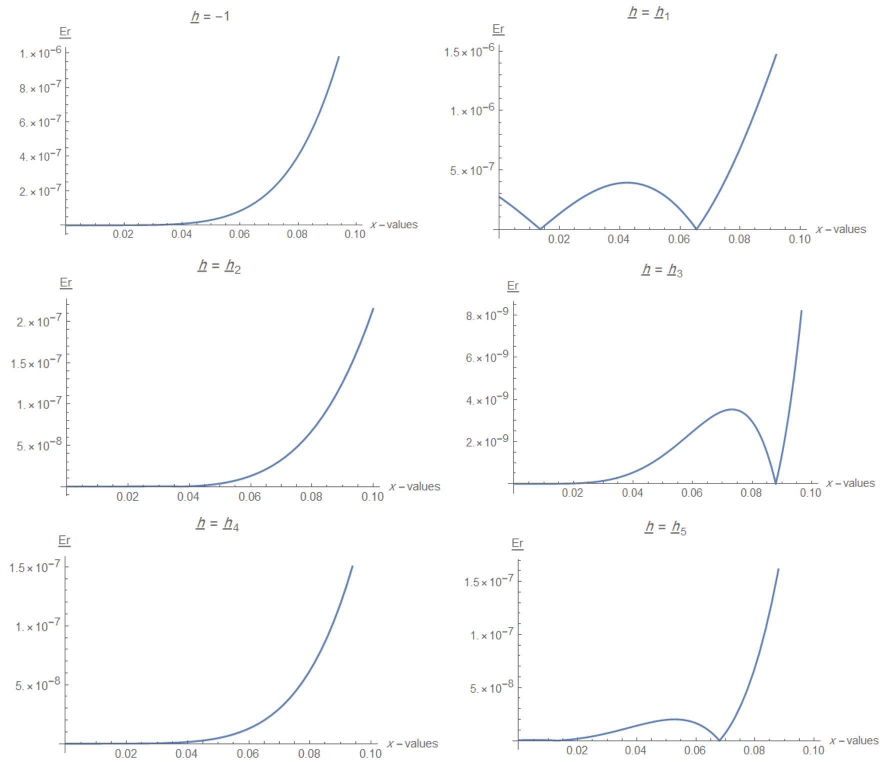


Fig. 10: Fuzzy fractional HAM of order six which linked with the optimal lower control parameters values $\underline{h}(0.6)$ in Table **9** for the lower solution of Eq. (44) $\forall\, x \in [0,0.1]$ and $\alpha = 0.6$ when $\beta = 0.9$.

Figure 10 showed the accuracy of Eq. (44) based on the convergence control parameters $\underline{h}$. Thus, the best value of $\underline{h}(0.6)$ that can conclude from Figure10 is $\underline{h}_3$ which gives the optimal lower approximation $\underline{y}(x;\alpha)$ of Eq. (51) at fractional order $\beta = 0.9$ as displayed in Table 10 and Table 11.

Table 10: Sixth order fuzzy fractional HAM lower solution and accuracy of Eq. (44) at $x = 0.1$, $h = -1$, and $\alpha = 0.6$ when $\beta = 0.9$.

α	$\underline{y}(x;\alpha)$	$\underline{Er}(x,\alpha;-1)$
0	-1	0
0.1	-0.87162099940462010	$2.612237467237600 \times 10^{-7}$
0.2	-0.74710477767498320	$2.600924798796820 \times 10^{-7}$
0.3	-0.62626495043126340	$-4.11566670160357 \times 10^{-7}$
0.4	-0.50892710057501160	$-1.26137889874138 \times 10^{-6}$
0.5	-0.39492784528668260	$-1.68741890619526 \times 10^{-6}$
0.6	-0.28411397973362110	$-1.36803578168178 \times 10^{-6}$
0.7	-0.17634169285377277	$-3.65107877533546 \times 10^{-7}$
0.8	-0.07147585058036897	$9.565783732798605 \times 10^{-7}$
0.9	0.030610658127133320	$2.104301653364130 \times 10^{-6}$
1	0.130037517080618300	$2.651795617455331 \times 10^{-6}$

Table 11: Sixth order fuzzy fractional HAM lower solution and accuracy of Eq. (44) at $x = 0.1$, $h = \underline{h}_3$ and $\alpha = 0.6$ when $\beta = 0.9$.

α	$\underline{y}(x;\alpha)$	$\underline{Er}(x,\alpha;\underline{h}_3)$
0	-1	0
0.1	-0.87162100705229100	$-6.302367075877674\times10^{-8}$
0.2	-0.74710478351909670	$-1.593312247008181\times10^{-7}$
0.3	-0.62626493067095680	$4.3809482112217140\times10^{-8}$
0.4	-0.50892704554733860	$6.9639497219715410\times10^{-7}$
0.5	-0.39492776725621970	$1.5838978413168510\times10^{-7}$
0.6	-0.28411390776417245	$2.2386011615660850\times10^{-7}$
0.7	-0.17634166124353670	$2.1729544806518840\times10^{-7}$
0.8	-0.07147588604323521	$1.1116407923027950\times10^{-7}$
0.9	0.030610546650189720	$-8.584790205343330\times10^{-7}$
1	0.130037343394749900	$-3.263567567884928\times10^{-7}$

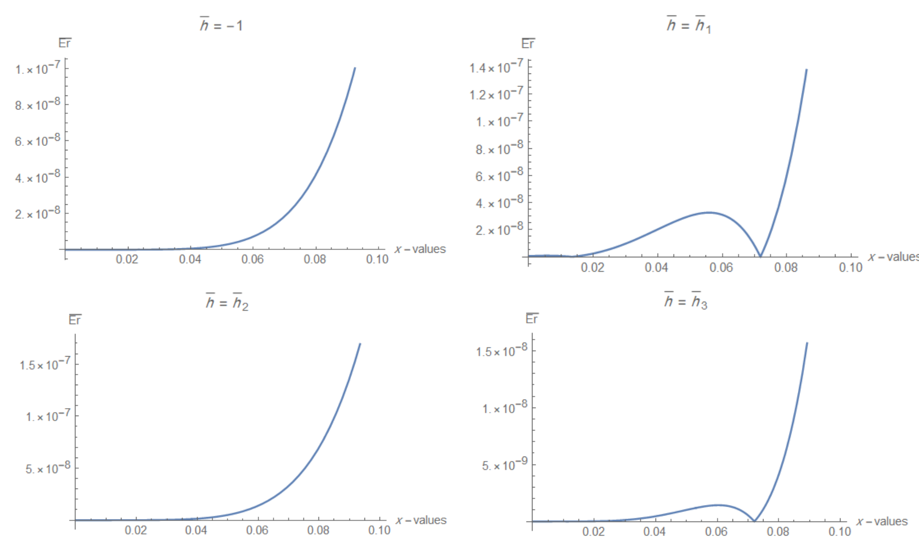
Tables 10 and 11 show the lower solutions at $h = -1$ and $h = \underline{h}_3$ respectively via fuzzy fractional HAM, which show that the best convergence control parameter $\underline{h}_3$ provides solution that is more accurate.

For the upper solution via upper fuzzy fractional HAM series described in Eq. (52), we list the best values of the upper control convergence parameters $\overline{h}(0.6)$ of fuzzy fractional HAM series solution of Eq. (44) at fuzzy level set $\alpha = 0.6$.

Table 12. Optimal values of $\overline{h}(0.6)$ of the sixth order solution of fuzzy fractional HAM of Eq. (44) at $\underline{\alpha} = 0.6$ and $\beta = 0.9$.

$\overline{y}(x,\underline{h}(0.6);0.6)$	$\overline{h}_1 \to -1.029520351079232$	$\overline{h}_2 \to -1.0037605248999117$
$\overline{h}_3 \to -0.994947838798828$	$\overline{h}_4 \to -0.9834702872804917$	$\overline{h}_5 \to -0.9289787025877265$

Fig. 11: illustrated the accuracy of the sixth order fuzzy fractional HAM based on $\overline{h}(0.6)$ values which gained from Table 12 and corresponds to the residual error described in Eq. (53).



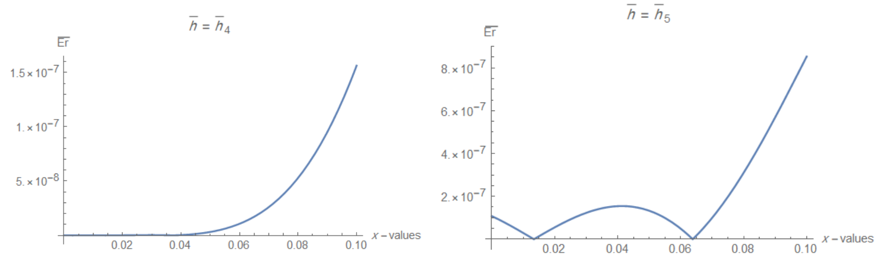


Fig. 11: Fuzzy fractional HAM of order six which linked with the upper control parameters values $\overline{h}(0.6)$ in Table 12 for the upper solution of Eq. (44) $\forall\, x \in [0,0.1]$ and $\alpha = 0.6$ when $\beta = 0.9$.

Figure 11 show the accuracy of the series solutions of Eq. (44). Therefore, we can conclude the best value of $\underline{h}(0.6) = \overline{h}_3$ gives the optimal upper approximation $\overline{y}(x;\alpha)$ of Eq. (52) at fractional order $\beta = 0.9$ as displayed in Table 13 and Table 14.

Table 13: Sixth order fuzzy fractional HAM upper solution and accuracy of Eq. (44) at $x = 0.1$, $h = -1$ and $\alpha = 0.6$ when $\beta = 0.9$.

α	$\overline{y}(x;\alpha)$	$\overline{Er}(x,\alpha;-1)$
0	1	0
0.1	0.92188828647286350	$2.6695756018885550 \times 10^{-7}$
0.2	0.84200402531672000	$1.839679286815950 \times 10^{-7}$
0.3	0.76027852003391700	$-5.681393667906749 \times 10^{-7}$
0.4	0.67663942508334010	$-1.373997012023458 \times 10^{-6}$
0.5	0.59101047688730290	$-1.617115451521994 \times 10^{-6}$
0.6	0.50331121301432500	$-1.063405441361245 \times 10^{-6}$
0.7	0.41345667490303306	$1.0176368112695670 \times 10^{-7}$
0.8	0.32135708949247765	$1.4235374454400170 \times 10^{-6}$
0.9	0.22691752512409650	$2.3943045318806640 \times 10^{-6}$
1	0.13003751708061830	$2.6517956174532120 \times 10^{-6}$

Table 14: Sixth order fuzzy fractional HAM upper solution and accuracy of Eq. (44) at $x = 0.1$, $h = \overline{h}_3$ and $\alpha = 0.6$ when $\beta = 0.9$.

α	$\overline{y}(x;\alpha)$	$\overline{Er}(x,\alpha;\overline{h}_3)$
0	1	0
0.1	0.92188827985384550	$-6.302367075877674 \times 10^{-8}$
0.2	0.84200401836617240	$-1.593312247008181 \times 10^{-7}$
0.3	0.76027853217293160	$4.3809482112217140 \times 10^{-8}$
0.4	0.67663946572204610	$6.9639497219715410 \times 10^{-7}$
0.5	0.59101053846665130	$1.5838978413168510 \times 10^{-7}$
0.6	0.50331127428126140	$2.2386011615660850 \times 10^{-7}$
0.7	0.41345670944502390	$2.1729544806518840 \times 10^{-7}$
0.8	0.32135707553621007	$1.1116407923027950 \times 10^{-7}$
0.9	0.22691745358211568	$-8.584790205343330 \times 10^{-7}$
1	0.13003739580852433	$-3.263567567884928 \times 10^{-7}$

Tables 13 and 14 show the upper solutions at $h = -1$ and $h = \overline{h}_3$ respectively that show that the best convergence control parameter $\overline{h}_3$ provides solution that is more accurate.

Clearly, from Tables 10, 11, 13 and 14, the solutions of Eq. (44) with fractional order $\beta = 0.9$ via fuzzy fractional HAM of order six at $x = 0.1$ fulfilled the properties of fuzzy solutions [39] which can be displyed in Figure 12.

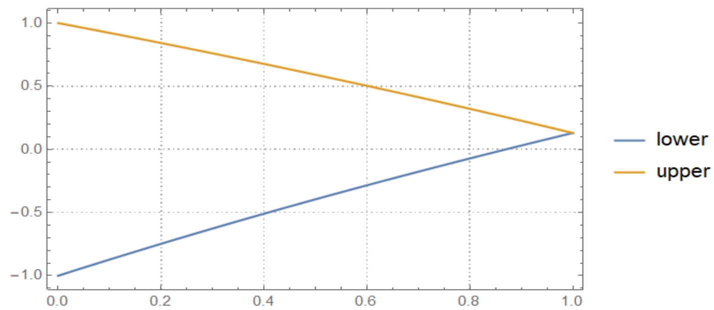


Fig. 12: Sixth order fuzzy fractional HAM corresponding with optimal value $\tilde{h}_3 = [\underline{h}_3, \overline{h}_3]$ of Eq. (44) at $x = 0.1$ and for all $\alpha \in [0,1]$.

Thus, the solutions of sixth order fuzzy fractional HAM overall $x \in [0,0.1]$ and $\alpha \in [0,1]$ corresponding with optimal values $\tilde{h}_3$ of Eq. (44) for $\beta = 0.9$ can display in the following three-dimensional figure as in Figure 13.

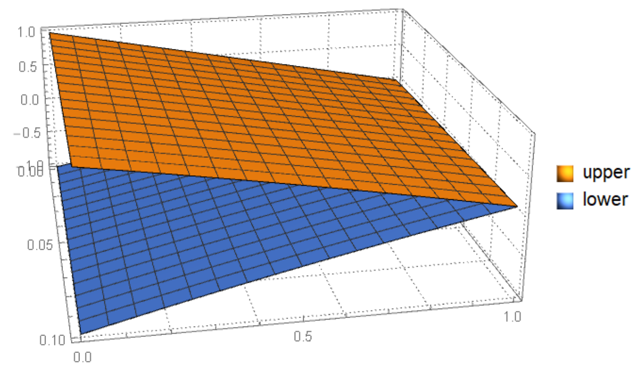


Fig. 13: Sixth order fuzzy fractional HAM corresponding with optimal value $\tilde{h}_3 = [\underline{h}_3, \overline{h}_3]$ of Eq. (44) with $\beta = 0.9$ for all $x \in [0,0.1]$ and $\alpha \in [0,1]$.

We conclude that the results in Figure 13 which obtained by sixth order fuzzy fractional HAM solutions of Eq. (44) with fractional order $\beta = 0.9$ fulfilled the fuzzy differential equation properties in the form of triangular fuzzy numbers for all $\alpha \in [0,1]$ and $x \in [0,0.1]$.

6 Conclusion

This study aims to develop a new approach based on HAM to approximate the analytical solution of fuzzy fractional differential equations. The HAM has the potential primarily to provide a certain way of guaranteeing a convergence series solution and the derivation of this convergence analysis is illustrated mathematically corresponding with first order FFIVPs under the sense of Caputo's derivative. A new HAM general fractional differential equation is introduced and analyzed in fuzzy domain and extended to provide a stable formulation of FFIVPs by using the properties of fuzzy sets theory. Two examples followed by full fuzzy analysis demonstrated the capability of the method. The results obtained showed that first order linear and nonlinear FFIVPs possessed convergence approximate solutions for different order of HAM with different order of fractional derivative that satisfy the properties of fuzzy numbers. Moreover, fuzzy fractional HAM provides more accurate solutions compared to other approaches such as FRPS method [47]. We also solved the fuzzified first order fractional Ricatti differential equation, which has potential in real-world application. So, it is highly recommended to use HAM in place of conventional approximate-analytical approaches because HAM converges faster to the solution and more abridged. From this study, we think that HAM is a powerful and efficient approach for obtaining approximate solutions for fuzzy fractional differential equations that have great significance in different fields of science and engineering.

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