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On Essential Ideals of a Ternary Semiring

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Abstract: This paper is based on the idea of essential ideals in ternary semiring. The concepts of the essential ideals, semi-essential ideals and weak essential ideals of a semiring have been generalized to ternary semiring and study some of its properties. We also generalize the Andrunakievich lemma for ternary semiring. We proved that the intersection of any two ideals of a ternary semiring is weak essential if one is essential and other is weak essential. Also proved that the intersection of any two weak essential ideals of a ternary semiring is weak essential if one of them is does not contained in any semi-prime ideal of a ternary semiring.

Keywords: Ternary Semiring, Essential Ideal, Semi-Essential Ideal, Weak Essential Ideal, etc.

1 Introduction

In 1934, Vandiver [17] introduced a new type of algebraic system which is commonly known as Semiring. More information about the concepts of semiring can be found in [4]-[8]. The idea of a ternary algebraic system was first invented in 1924 by Prufer in [16]. Then Lehmer [10] introduced certain ternary algebraic systems called triplexes which turn out to be commutative ternary groups in 1932. In 1971, Lister in [11] investigated the notion of ternary ring and studied some properties of a ternary ring.

In 2003, Dutta and Kar [2] introduced the notion of a Ternary Semiring which generalizes the notion of ternary ring. More specifically, in an ordered ring the positive cone forms a semiring where as its negative cone forms a ternary semiring. Thus a ternary semiring may be considered as a counterpart of semiring in an ordered ring. They also define the notion of right ternary semimodule and right ternary subsemimodule over a ternary semiring.

In 2012, Pawar and Deore [13] proved some results of an essential ideals for semiring. The concepts of essential subsemimodules and semi-essential subsemimodules of semiring given in [1] and [12] has been extended to the weak essential ideals for semirings in [14]. In this paper we have generalized the notions of an essential ideals, semi-essential ideals and weak essential ideals of a semiring, so far introduced in [1], [12]-[15] to a ternary semiring.

2 Preliminary Definitions

Definition 1.[2] *A non-empty set S together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if S is an additive commutative semigroup satisfying the following conditions:*

(i) $(abc)de = a(bcd)e = ab(cde)$ *(Associative Law)*
(ii) $(a+b)cd = acd + bcd$ *(Right Distributive Law)*
(iii) $a(b+c)d = abd + acd$ *(Lateral Distributive Law)*
(iv) $ab(c+d) = abc + abd$ *(Left Distributive Law) for all* $a,b,c,d,e \in T$.

Example 1.[2] Let $\mathbb{Z}_0^-$ be the set of all negative integers with zero. Then with the usual binary addition and ternary multiplication, $\mathbb{Z}_0^-$ forms a ternary semiring.

Definition 2.[2] *An additive subsemigroup I of a ternary semiring S is called ideal of S if $SSI \subseteq I, SIS \subseteq I$ and $ISS \subseteq I$. An ideal I of a ternary semiring S is called k-ideal (subtractive) if for $a \in I, a+b \in I, b \in S$ imply $b \in I$. We denote $I \triangleleft S$, a ternary semiring ideal in S.*

Definition 3.[3] *A proper ideal P of a ternary semiring S is called a prime ideal of S if for any three ideals A,B,C of S; $ABC \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$.*

Definition 4.[9] *A ternary semiring S is called a prime ternary semiring if the zero ideal 0 is a prime ideal of S.*

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Definition 5.[3] *A proper ideal Q of a ternary semiring S is called a semi-prime ideal of S if $I^3 \subseteq Q$ implies $I \subseteq Q$ for any ideal I of S.*

Definition 6.[3] *A proper ideal I of a ternary semiring S is said to be irreducible if for ideals H and K of S, $H \cap K = I$ implies that $I = H$ or $I = K$.*

3 Essential Ideal

Definition 7. *An ideal I of a ternary semiring S is said to be an essential ideal of S if $I \cap K \neq 0$, for every non-zero ideal K of S. We denote an essential ideal I of a ternary semiring S by $I \triangleleft \cdot S$.*

Proposition 1. *If $0 \neq K \triangleleft \cdot I \triangleleft \cdot S$ and $\overline{K}$ is k-closure of K. Then $\overline{K}$ is essential ideal in S.*

Proof. Let L be any non-zero ideal of S. Since I is essential ideal in S, we have $I \cap L \neq 0$. But K is essential ideal in I. Thus, we have $0 \neq K \cap (I \cap L) \subseteq K \cap L \subseteq \overline{K} \cap L$. Hence $\overline{K}$ is an essential ideal in S.

Lemma 1. Andrunakievich Lemma: *If $0 \neq K \triangleleft I \triangleleft S$ and $\overline{K}$ is k-closure of K. then $\overline{K}^3 \subseteq K$.*

Proof. Since $\overline{K}$ is k-closure of K and $0 \neq K \triangleleft I \triangleleft S$, we have $\overline{K}^3 \subseteq I(K + SSK + KSS + SSKSS)I \subseteq IKI + ISSKI + IKSSI + ISSKSSI \subseteq K$.

Proposition 2. *If $I \triangleleft S$ and S is prime ternary semiring. Then I is also a prime ternary semiring.*

Proof. Let J, K, L be ideals of I such that $JKL = 0$. Let $\overline{J}$, $\overline{K}$, $\overline{L}$ be k-closures of J, K, L respectively. Thus by lemma 1, $\overline{J}^3 \overline{K}^3 \overline{L}^3 \subseteq JKL = 0$. But S is prime ternary semiring. Therefore either $\overline{J}^3 = 0$ or $\overline{K}^3 = 0$ or $\overline{L}^3 = 0$ implies either $J = 0$ or $K = 0$ or $L = 0$. Thus I is a prime ternary semiring.

Proposition 3. *If $I \triangleleft \cdot S$ and I is prime ternary semiring. Then S itself is a prime ternary semiring.*

Proof. Let J, K, L be ideals of S such that $JKL = 0$. Suppose $J \neq 0$ and $K \neq 0$. Then $I \cap J \neq 0$ and $I \cap K \neq 0$. Since I is prime ternary semiring and $I \triangleleft \cdot S$. We have $(I \cap J)(I \cap K)(I \cap L) \subseteq JKL = 0$ implies $I \cap L = 0$ implies $L = 0$. Therefore S is a prime ternary semiring.

Lemma 2. *Let $I \triangleleft \cdot J \triangleleft \cdot S$. Then $I \triangleleft \cdot S$.*

Proof. Let L be any non-zero ideal of S such that L is also ideal of J. Since $I \triangleleft \cdot J$ and $J \triangleleft \cdot S$, we have $I \cap L \subseteq J \cap L \neq 0$. Thus $I \triangleleft \cdot S$.

Proposition 4. *If $I_1, I_2, I_3, \ldots, I_k$ be ideals in ternary semiring S. Then $\bigcap_{i=1}^{k} I_i \triangleleft \cdot S$ if and only if each $I_i \triangleleft \cdot S$, for all i.*

Definition 8. *The heart $H(S)$ of a ternary semiring S is defined as $H(S) = \cap \{I \triangleleft S : I \neq 0\}$.*

Definition 9. *A ternary semiring S is said to be subdirectly irreducible, if $H(S) \neq 0$, i.e. S has a unique minimal ideal.*

Proposition 5. *In a subdirectly irreducible ternary semiring S, every non-zero ideal, in particular its heart is an essential ideal.*

4 Semi-Essential Ideal

Definition 10. *An ideal I of a ternary semiring S is said to be a semi-essential ideal of S if $I \cap K \neq 0$, for every non-zero prime ideal K of S. We denote an semi-essential ideal I of a ternary semiring S by $I \triangleleft_s \cdot S$.*

Note 1. Every essential ideal in ternary semiring S is semi-essential. But the converse is not true in general.

Example 2. In $\mathbb{Z}_{12}$ as a ternary semiring, the ideal $(\overline{6})$ is semi-essential ideal of $\mathbb{Z}_{12}$, but $(\overline{6})$ is not essential ideal of $\mathbb{Z}_{12}$ as $(\overline{6}) \cap (\overline{4}) = (\overline{0})$.

The following proposition gives a necessary and sufficient condition for an ideal to be semi-essential.

Proposition 6. *Let S be a ternary semiring. A non-zero ideal I of S is semi-essential if and only if for each non-zero prime ideal K of S there exists $x \in K$ and $s_1, s_2 \in S$ such that $0 \neq s_1 s_2 x,\ x s_1 s_2,\ s_1 x s_2 \in I$.*

Proof. Suppose that for each non-zero prime ideal K of S, there exists $x \in K$ and $s_1, s_2 \in S$ such that $0 \neq s_1 s_2 x,\ x s_1 s_2,\ s_1 x s_2 \in I$. But $s_1 s_2 x,\ x s_1 s_2,\ s_1 x s_2 \in K$. Therefore $0 \neq s_1 s_2 x,\ x s_1 s_2,\ s_1 x s_2 \in I \cap K$. Thus $I \cap K \neq 0$, for each non-zero prime ideal K of S. Hence I is a semi-essential ideal of S.

Conversely, suppose that I is a semi-essential ideal of S. Then $I \cap K \neq 0$, for each non-zero prime ideal K of S. Thus $0 \neq s_1 s_2 x,\ x s_1 s_2,\ s_1 x s_2 \in I \cap K$, for each $x \in K$ and $s_1, s_2 \in S$ implies that $0 \neq s_1 s_2 x,\ x s_1 s_2,\ s_1 x s_2 \in I$.

Lemma 3. *Let H and K be any two ideals of a ternary semiring S such that H is ideal of K. If H is semi-essential ideal of S, then K is semi-essential ideal of S.*

Note 2. The converse of lemma 3 is not true in general.

Example 3. In $\mathbb{Z}_{12}$ as a ternary semiring, the ideal $(\overline{4})$ is semi-essential ideal of $(\overline{2})$ and $(\overline{2})$ is semi-essential ideal of $\mathbb{Z}_{12}$. But $(\overline{4})$ is not semi-essential ideal of $\mathbb{Z}_{12}$ since $(\overline{4}) \cap (\overline{3}) = (\overline{0})$ and $(\overline{3})$ is prime ideal of $\mathbb{Z}_{12}$.

Lemma 4. *Let H and K be any two ideals of a ternary semiring S. If $H \cap K$ is semi-essential ideal of S, then both H and K are semi-essential ideals of S.*

Note 3. The converse of lemma 4 is not true in general.

*Example 4.*In $\mathbb{Z}_{36}$ as a ternary semiring, the ideals $(\overline{2})$ and $(\overline{3})$ are prime ideals of $\mathbb{Z}_{36}$. Now $(\overline{12}) \cap (\overline{2}) = (\overline{12})$, $(\overline{12}) \cap (\overline{3}) = (\overline{12})$, $(\overline{18}) \cap (\overline{2}) = (\overline{18})$ and $(\overline{18}) \cap (\overline{3}) = (\overline{18})$. Thus $(\overline{12})$ and $(\overline{18})$ are semi-essential ideals of $\mathbb{Z}_{36}$. But $(\overline{12}) \cap (\overline{18}) = (\overline{0})$. This shows that $(\overline{12}) \cap (\overline{18})$ is not semi-essential ideal of $\mathbb{Z}_{36}$.

The following proposition gives a condition under which the converse of lemma 4 is true.

Proposition 7.*Let H and K be any two ideals of a ternary semiring S such that H is essential ideal and K is semi-essential ideal of S. Then $H \cap K$ is a semi-essential ideal of S.*

*Proof.*Since K is semi-essential ideal of S, we have $K \cap J \neq 0$, for every non-zero prime ideal J of S. But H is an essential ideal of S. Therefore $H \cap (K \cap J) \neq 0$ implies that $(H \cap K) \cap J \neq 0$. Hence $H \cap K$ is a semi-essential ideal of S.

5 Weak Essential Ideal

Definition 11.*An ideal I of a ternary semiring S is said to be a weak essential ideal of S if $I \cap K \neq 0$, for every non-zero semi-prime ideal K of S. We denote an weak essential ideal I of a ternary semiring S by $I \lhd_w \cdot S$.*

*Note 4.*Every essential ideal in ternary semiring S is weak essential. But the converse is not true in general.

*Example 5.*In $\mathbb{Z}_{36}$ as a ternary semiring, the ideal $(\overline{9})$ of $\mathbb{Z}_{36}$ is weak essential but not essential. Since $(\overline{9}) \cap (\overline{2}) \neq (\overline{0})$, $(\overline{9}) \cap (\overline{3}) \neq (\overline{0})$ and $(\overline{9}) \cap (\overline{6}) \neq (\overline{0})$, where $(\overline{2})$, $(\overline{3})$ and $(\overline{6})$ are the only non-zero semi-prime ideals of $\mathbb{Z}_{36}$. But $(\overline{9}) \cap (\overline{12}) = (\overline{0})$.

*Note 5.*Also every weak essential ideal in ternary semiring S is semi-essential. But the converse is not true in general.

*Example 6.*In $\mathbb{Z} \oplus \mathbb{Z}$ as a ternary semiring, the only prime ideals are of the form $\mathbb{Z} \oplus p\mathbb{Z}$ and $p\mathbb{Z} \oplus \mathbb{Z}$, where p is the prime number. The ideal $I = (\overline{0}) \oplus \mathbb{Z}$ of $\mathbb{Z} \oplus \mathbb{Z}$ is semi-essential but not weak essential, since $I \cap 2\mathbb{Z} \oplus (\overline{0}) = (\overline{0})$, where $2\mathbb{Z} \oplus (\overline{0})$ is semi-prime ideal of S but not prime ideal of S.

The following proposition gives a necessary and sufficient condition for an ideal to be weak essential.

Proposition 8.*Let S be a ternary semiring. A non-zero ideal I of S is weak essential if and only if for each non-zero semi-prime ideal K of S there exists $x \in K$ and $s_1, s_2 \in S$ such that $0 \neq s_1 s_2 x$, $x s_1 s_2$, $s_1 x s_2 \in I$.*

Lemma 5.*Let I be an irreducible ideal of a ternary semiring S. Then I is semi-prime if and only if I is prime ideal of S.*

Proposition 9.*Let S be a ternary semiring such that every semi-prime ideal of S is irreducible. If an ideal I of S is semi-essential then I is a weak essential ideal of S.*

*Proof.*Let K be any non-zero semi-prime ideal of ternary semiring S. Then K is irreducible ideal of S. Therefore by lemma 5, K is prime ideal. Now I is semi-essential ideal of S. Therefore $I \cap K \neq 0$, for every non-zero semi-prime ideal K of S. Hence I is a weak essential ideal of S.

Lemma 6.*Let S be a ternary semiring and let H and K be any two ideals of S such that $H \subseteq K$. If H is a weak essential ideal of S then K is also a weak essential ideal of S.*

Lemma 7.*Let S be a ternary semiring and let H and K be any two ideals of S such that $H \cap K$ is a weak essential ideal of S. Then both H and K are weak essential ideals of S.*

*Note 6.*The converse of lemma 7 is not true in general.

*Example 7.*In $\mathbb{Z}_{36}$ as a ternary semiring, the ideals $(\overline{2})$, $(\overline{3})$ and $(\overline{6})$ are the only non-zero semi-prime ideals of $\mathbb{Z}_{36}$. Now $(\overline{12}) \cap (\overline{2}) \neq (\overline{0})$, $(\overline{12}) \cap (\overline{3}) \neq (\overline{0})$, $(\overline{12}) \cap (\overline{6}) \neq (\overline{0})$, $(\overline{18}) \cap (\overline{2}) \neq (\overline{0})$, $(\overline{18}) \cap (\overline{3}) \neq (\overline{0})$ and $(\overline{18}) \cap (\overline{6}) \neq (\overline{0})$. Thus $(\overline{12})$ and $(\overline{18})$ are weak essential ideals of $\mathbb{Z}_{36}$. But $(\overline{12}) \cap (\overline{18}) = (\overline{0})$. This shows that $(\overline{12}) \cap (\overline{18})$ is not weak essential ideal of $\mathbb{Z}_{36}$.

The following proposition gives a condition under which the converse of lemma 7 is true.

Proposition 10.*Let H and K be any two ideals of a ternary semiring S such that H is essential ideal and K is weak essential ideal of S. Then $H \cap K$ is a weak essential ideal of S.*

*Proof.*Since K is weak essential ideal of S, we have $K \cap J \neq 0$, for every non-zero semi-prime ideal J of S. But H is an essential ideal of S. Therefore $H \cap (K \cap J) \neq 0$ implies that $(H \cap K) \cap J \neq 0$. Hence $H \cap K$ is a weak essential ideal of S.

Proposition 11.*Let H and K be any two ideals of a ternary semiring S such that one of them does not contained in any semi-prime ideal of S. If both H and K are weak essential ideals of S. Then $H \cap K$ is a weak essential ideal of S.*

*Proof.*Let J is non-zero semi-prime ideal of S such that $(H \cap K) \cap J = 0$. Then $H \cap (K \cap J) = 0$ implies that $K \cap J = 0$ is semi-prime ideal of H. But H is weak essential ideal of S, so $K \cap J = 0$. Also K is weak essential ideal of S, therefore $J = 0$. Hence $H \cap K$ is a weak essential ideal of S.

6 Conclusion

In this paper we have generalized the Andrunakievich lemma for ternary semiring. Also we investigated certain types of ideals such as essential ideal, semi-essential ideal, weak essential ideal for ternary semiring and given inter-relation between them with examples.

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