

Conformable Double Sumudu-Elzaki Transform

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Abstract: In this paper, we define the conformable fractional Sumudu-Elzaki transform (CSET) and establish some essential properties and theorems. By employing CSET and its associated properties, integral and partial differential equations were transformed into algebraic forms. The findings demonstrated that the utilization of the CSET was more effective and practical for addressing equations of this nature.

Keywords: Conformable, Sumudu-transform, Elzaki-transform.

1 Introduction

Over the last few decades, the influence of fractional calculus on various realms of pure and applied science as well as engineering has seen a significant upswing. This surge has become particularly pronounced over the past twenty years.

The fundamental concept underpinning the establishment of fractional calculus can be merged into two principal approaches. The first approach, known as the Riemann–Liouville method, involves iteratively applying the integral operator n -times and subsequently condensing it into a single integral through the renowned Cauchy formula. This transformation entails the substitution of $n!$ with the Gamma function, leading to the formulation of fractional integrals for non-integer orders. This, in turn, paves the way for defining Riemann and Caputo fractional operators [9, 10].

The second approach, termed the Grünwald–Letnikov approach, centers on iteratively applying the operator n times, followed by fractionalization using the Gamma function within binomial coefficients [9].

The fractional operators derived within this calculus exhibited complexity and deviated from some fundamental properties typically associated with standard operators, such as the product rule and the chain rule. Nevertheless, the fractional operators demonstrated favorable semigroup characteristics under specific conditions. A recent advancement by Khalil, R., and coworkers [8], introduces a novel and well-behaved fractional operator termed the "conformable fractional operator." This operator is constructed solely based on the fundamental limit definition of an operator.

To elaborate, for a function $f : [0, \infty) \rightarrow \mathbb{R}$, the conformable fractional operator of order $0 < \beta \leq 1$ for f at a given point $t > 0$ is defined as follows:

$$(T_\beta f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\beta}) - f(t)}{\varepsilon}$$

Additionally, the fractional operator at the point 0 is define as:

$$(T_\beta f)(0) = \lim_{t \rightarrow 0^+} (T_\beta f)(t).$$

Subsequently, they proceed to introduce the concept of higher-order fractional operators (specifically, those with an order $\beta > 1$). Their efforts extend to proving essential principles, including the product rule and the fractional mean value theorem. They established the notion of the fractional integral for cases where the order falls within the range of $0 < \beta \leq 1$. They also addressed specific (conformable) fractional differential equations, where the fractional exponential function $e^{\frac{t^\beta}{\beta}}$ plays a pivotal role.

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Lately, a range of potent techniques has emerged for acquiring precise solutions to conformable fractional partial differential equations. These methods encompass reliable approaches mentioned in [1, 5, 6], the utilization of single and double Laplace transforms detailed in [1, 15], as well as the application of the double Shehu transform as outlined in [13].

Notably, in [13], Ozkan and colleagues introduced a novel concept, namely the definition of the conformable fractional double Laplace transform along with several associated properties. These concepts proved valuable in solving certain conformable fractional partial differential equations and hold particular significance for the subsequent discussions in this research.

Through the passage of time, transform techniques, such as the Elzaki transform, have demonstrated their effectiveness in resolving a multitude of practical, both symmetric and asymmetric, real-world challenges in the domains of applied sciences and engineering. In [6], the authors introduced a solitary Elzaki transformation, and in reference [14], a double Elzaki transform tailored to conformably tackle partial differential equations of the conformable fractional variety was introduced. Authors of [7] employed the double Elzaki transform to derive solutions for a Volterra Integro equation. On the other hand, authors of [2] utilized the conformable double Elzaki transform to derive solutions for nonlinear fractional Burger's equations. Al-Sook and coworkers [4] employed the double Laplace-Elzaki transform to derive solutions for nonhomogeneous partial differential equations. In reference [12], Mohamed et al. provided numerical solutions for the conformable fractional coupled Burger's equation utilizing a double Sumudu transform.

This paper introduces a revised definition of the double Sumudu-Elzaki transform and explores additional properties, notably its convolution characteristics previously unverified in existing literature. We establish its existence under specific conditions. Leveraging this modification, we offer exact solutions for significant conformable fractional differential and integral equations such as the Klein-Gordon equation, wave equation, and other integral equations. Our adaptation builds upon the application of the conformable fractional integral, pioneered by Khalil and colleagues [8]. It's crucial to highlight that our methodology diverges from conventional approaches found in current literature.

2 Conformable double Sumudu-Elzaki transform

Definition 1.[11] Consider $g : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$, a function of two variables τ, t , which can be expressed as an infinite convergent series. The (SET) of the function $g(\tau, t)$ is denoted by:

$$\mathcal{L}g[g(\tau, t)] = \frac{v}{\omega} \int_0^\infty \int_0^\infty e^{-\frac{\tau}{\omega} - \frac{t}{v}} g(\tau, t) d\tau dt, \quad \omega, v \neq 0.$$

Definition 2.[3] A bivariate function g is said to be conformable exponentially order bounded if $\exists N, p, q, \alpha, \beta > 0$, with $0 < \alpha, \beta \leq 1$ such that

$$|g(\tau, t)| \leq Ne^{p\frac{t^\beta}{\beta} + q\frac{\tau^\alpha}{\alpha}},$$

for every sufficiently large t and τ .

Definition 3. Let $g : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ be a piecewise continuous function of exponential order bounded. Then the double conformable Laplace-Elzaki transform of g is defined as follows:

$$\mathcal{L}_\alpha \mathcal{L}_\beta [g(\tau, t)] = \frac{v}{\omega} \int_0^\infty \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\beta v}} g(\tau, t) d_\alpha \tau d_\beta t,$$

where $d_\alpha \tau = \tau^{\alpha-1} d\tau$, $d_\beta t = t^{\beta-1} dt$, $\omega, v \in \mathbb{C} - \{0\}$ and $\alpha, \beta \in (0, 1]$.

The inverse conformable double Sumudu Elzaki transform $(\mathcal{L}_\alpha \mathcal{L}_\beta)^{-1}[\mathcal{G}(\omega, v)] = g(\tau, t)$ is defined by the complex double integral formula

$$\begin{aligned} (\mathcal{L}_\alpha \mathcal{L}_\beta)^{-1}[\mathcal{G}(\omega, v)] &= g(\tau, t) \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{a-i\infty}^{a+i\infty} \int_{b-i\infty}^{b+i\infty} e^{\frac{\tau^\alpha}{\alpha\omega} + \frac{t^\beta}{\beta v}} \mathcal{G}(\omega, v) d_\alpha \tau d_\beta t \\ &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\frac{\tau^\alpha}{\alpha\omega}} d_\alpha \tau \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} e^{\frac{t^\beta}{\beta v}} \mathcal{G}(\omega, v) d_\beta t, \end{aligned}$$

where $\mathcal{G}(\omega, v)$ must be an analytic function for all ω and v in the region defined by the inequalities $\text{Re } \omega \geq a$ and $\text{Re } v \geq b$, where a and b are real constants to be chosen suitably.

Theorem 1. (Existence of the (CSET)) Let g be an exponentially order bounded piecewise continuous function on $[0, \infty) \times [0, \infty)$ of conformable type. Then the (CSET) $\mathcal{S}_{\alpha\beta} [g(\tau, t)]_{(\omega, \nu)}$ exists, for $p_1, p_2 > 0$ and for $\text{Re}(\frac{1}{\omega}) > \frac{1}{p_2}$ and $\text{Re}(\frac{1}{\nu}) > \frac{1}{p_1}$, and converges absolutely.

Proof. Given that g possesses the attributes of conformability and boundedness within an exponential order for $p_1, p_2, \frac{1}{p_1} > 0$ and for $\text{Re}(\frac{1}{\omega}) > \frac{1}{p_2}$ and $\text{Re}(\frac{1}{\nu}) > \frac{1}{p_1}$. Then $\exists N_1, p_1, p_2, r_0, t_0 > 0$ such that,

$$|g(\tau, t)| \leq N_1 e^{\frac{t^\beta}{p_1\beta} + \frac{\tau^\alpha}{\alpha p_2}}, \forall t > t_0, \forall \tau > r_0.$$

Also g is piecewise continuous on $[0, \infty) \times [0, \infty)$, which means,

$$|g(\tau, t)| \leq N_2 e^{\frac{t^\beta}{p_1\beta} + \frac{\tau^\alpha}{\alpha p_2}},$$

$$\forall (\tau, t) \in [0, r_0] \times [0, t_0].$$

Now $e^{\frac{t^\beta}{p_1\beta} + \frac{\tau^\alpha}{\alpha p_2}}$ exhibits a positive minimum within $[0, \infty) \times [0, \infty)$, it becomes possible to select a suitably large N value in a manner that,

$$|g(\tau, t)| \leq N \left| e^{\frac{t^\beta}{p_1\beta} + \frac{\tau^\alpha}{\alpha p_2}} \right|, \forall \tau, t > 0.$$

Hence, if $\frac{1}{\omega} = \frac{1}{a_1} + \frac{1}{a_2}i$ and $\frac{1}{\nu} = \frac{1}{b_1} + \frac{1}{b_2}i$, then

$$\begin{aligned} & \left| \frac{\nu}{\omega} \int_0^\lambda \int_0^\eta e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\nu\beta}} g(\tau, t) d_\alpha \tau d_\beta t \right| \leq \left| \frac{\nu}{\omega} \int_0^\lambda \int_0^\eta e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\nu\beta}} |g(\tau, t)| d_\alpha \tau d_\beta t \right| \\ & \leq \left| \frac{\nu}{\omega} \int_0^\lambda \int_0^\eta N \left| e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\nu\beta}} e^{\frac{\tau^\alpha}{\alpha p_2} + \frac{t^\beta}{p_1\beta}} \right| d_\alpha \tau d_\beta t \right| \\ & = \left| \frac{\nu}{\omega} \int_0^\lambda \int_0^\eta N \left| e^{-\left(\left(\frac{1}{a_1} + \frac{1}{a_2}i\right)\frac{\tau^\alpha}{\alpha} + \left(\frac{1}{b_1} + \frac{1}{b_2}i\right)\frac{t^\beta}{\beta}\right)} e^{\frac{\tau^\alpha}{\alpha p_2} + \frac{t^\beta}{p_1\beta}} t^{\beta-1} \tau^{\alpha-1} \right| d\tau dt \right| \\ & = \left| \frac{\nu}{\omega} \int_0^\lambda \int_0^\eta N \left| e^{\left(\frac{1}{p_2} - \frac{1}{a_1}\right)\frac{\tau^\alpha}{\alpha}} \right| \left| e^{-i\frac{\tau^\alpha}{\alpha a_2}} \right| \left| e^{\left(\frac{1}{p_1} - \frac{1}{b_1}\right)\frac{t^\beta}{\beta}} \right| \left| e^{-i\frac{t^\beta}{\beta}} \right| t^{\beta-1} \tau^{\alpha-1} d\tau dt \right| \\ & = \left| \frac{\nu}{\omega} \int_0^\lambda \int_0^\eta N \left| e^{\left(\frac{1}{p_2} - \frac{1}{a_1}\right)\frac{\tau^\alpha}{\alpha}} \right| \left| e^{\left(\frac{1}{p_1} - \frac{1}{b_1}\right)\frac{t^\beta}{\beta}} \right| t^{\beta-1} \tau^{\alpha-1} d\tau dt \right| \\ & = \left| \frac{\nu}{\omega} N \int_0^\lambda \left| e^{\left(\frac{1}{p_2} - \frac{1}{a_1}\right)\frac{\tau^\alpha}{\alpha}} \right| \tau^{\alpha-1} d\tau \int_0^\eta \left| e^{\left(\frac{1}{p_1} - \frac{1}{b_1}\right)\frac{t^\beta}{\beta}} \right| t^{\beta-1} dt \right| \end{aligned}$$

Let $x = \frac{\tau^\alpha}{\alpha}, y = \frac{t^\beta}{\beta}$. Then

$$\begin{aligned} & \left| \frac{\nu}{\omega} N \int_0^\lambda \left| e^{\left(\frac{1}{p_2} - \frac{1}{a_1}\right)\frac{\tau^\alpha}{\alpha}} \right| \tau^{\alpha-1} d\tau \int_0^\eta \left| e^{\left(\frac{1}{p_1} - \frac{1}{b_1}\right)\frac{t^\beta}{\beta}} \right| t^{\beta-1} dt \right| \\ & = \left| \frac{\nu}{\omega} N \int_0^{\frac{\lambda^\alpha}{\alpha}} \left| e^{\left(\frac{1}{p_2} - \frac{1}{a_1}\right)x} \right| dx \int_0^{\frac{\eta^\beta}{\beta}} \left| e^{\left(\frac{1}{p_1} - \frac{1}{b_1}\right)y} \right| dy \right| \\ & = \left| \frac{\nu}{\omega} N \left| \frac{e^{\left(\frac{1}{p_2} - \frac{1}{a_1}\right)x}}{\frac{1}{p_2} - \frac{1}{a_1}} \right|_0^{\frac{\lambda^\alpha}{\alpha}} \left| \frac{e^{\left(\frac{1}{p_1} - \frac{1}{b_1}\right)y}}{\frac{1}{p_1} - \frac{1}{b_1}} \right|_0^{\frac{\eta^\beta}{\beta}} \right| \\ & = \left| \frac{\nu}{\omega} N \left(\frac{e^{\left(\frac{1}{p_2} - \frac{1}{a_1}\right)\frac{\lambda^\alpha}{\alpha}}}{\frac{1}{p_2} - \frac{1}{a_1}} - \frac{1}{\frac{1}{p_2} - \frac{1}{a_1}} \right) \left(\frac{e^{\left(\frac{1}{p_1} - \frac{1}{b_1}\right)\frac{\eta^\beta}{\beta}}}{\frac{1}{p_1} - \frac{1}{b_1}} - \frac{1}{\frac{1}{p_1} - \frac{1}{b_1}} \right) \right|. \end{aligned}$$

Now since $\text{Re}(\frac{1}{\omega}) > \frac{1}{p_2}$, $\text{Re}(\frac{1}{\nu}) > \frac{1}{p_1}$ and as $\lambda, \eta \rightarrow \infty$, then,

$$\left| \frac{\nu}{\omega} \int_0^\infty \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\beta\nu}} \mathbf{g}(\tau, t) d_\alpha \tau d_\beta t \right| \leq \left| \frac{\nu}{\omega} \right| N \left(\frac{1}{\frac{1}{\rho_2} - \frac{1}{a_1}} \right) \left(\frac{1}{\frac{1}{\rho_1} - \frac{1}{b_1}} \right).$$

Theorem 2.(Linearity property). Let $\mathbf{h}, \mathbf{g} : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ be piecewise continuous functions of exponential order. Then

$$\mathcal{S}_\alpha \mathcal{E}_\beta [a\mathbf{g}(\tau, t) + b\mathbf{h}(\tau, t)] = a\mathcal{S}_\alpha \mathcal{E}_\beta [\mathbf{g}(\tau, t)] + b\mathcal{S}_\alpha \mathcal{E}_\beta [\mathbf{h}(\tau, t)],$$

where a, b are constants.

Proof.

$$\begin{aligned} & \mathcal{S}_\alpha \mathcal{E}_\beta [a\mathbf{g}(\tau, t) + b\mathbf{h}(\tau, t)] \\ &= \frac{\nu}{\omega} \int_0^\infty \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\beta\nu}} [a\mathbf{g}(\tau, t) + b\mathbf{h}(\tau, t)] t^{\beta-1} \tau^{\alpha-1} d\tau dt \\ &= \frac{\nu}{\omega} \int_0^\infty \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\beta\nu}} a\mathbf{g}(\tau, t) t^{\beta-1} \tau^{\alpha-1} d\tau dt + \frac{\nu}{\omega} \int_0^\infty \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\beta\nu}} b\mathbf{h}(\tau, t) t^{\beta-1} \tau^{\alpha-1} d\tau dt \\ &= a \frac{\nu}{\omega} \int_0^\infty \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\beta\nu}} \mathbf{g}(\tau, t) t^{\beta-1} \tau^{\alpha-1} d\tau dt + b \frac{\nu}{\omega} \int_0^\infty \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\beta\nu}} \mathbf{h}(\tau, t) t^{\beta-1} \tau^{\alpha-1} d\tau dt \\ &= a\mathcal{S}_\alpha \mathcal{E}_\beta [\mathbf{g}(\tau, t)] + b\mathcal{S}_\alpha \mathcal{E}_\beta [\mathbf{h}(\tau, t)]. \end{aligned}$$

Theorem 3. If $\mathbf{g}(\tau, t) = \mathbf{h}(\tau)k(t)$, then

$$\mathcal{S}_\alpha \mathcal{E}_\beta [\mathbf{g}(\tau, t)] = \mathcal{S}_\alpha \mathcal{E}_\beta [\mathbf{h}(\tau)k(t)] = \mathcal{S}_\alpha [\mathbf{h}(\tau)] \mathcal{E}_\beta [k(t)].$$

Proof.

$$\begin{aligned} \mathcal{S}_\alpha \mathcal{E}_\beta [\mathbf{h}(\tau)k(t)] &= \frac{\nu}{\omega} \int_0^\infty \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\beta\nu}} \mathbf{h}(\tau)k(t) \tau^{\alpha-1} t^{\beta-1} d\tau dt \\ &= \frac{1}{\omega} \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega}} \mathbf{h}(\tau) \tau^{\alpha-1} d\tau \int_0^\infty \nu e^{-\frac{t^\beta}{\beta\nu}} k(t) t^{\beta-1} dt \\ &= \mathcal{S}_\alpha [\mathbf{h}(\tau)] \mathcal{E}_\beta [k(t)]. \end{aligned}$$

Theorem 4. Let $\mathbf{g} : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ be a real valued continuous function and $\omega, \nu \in \mathbb{C}$ such that $\frac{\nu}{\omega} > 0$. Then

$$|\mathcal{S}_\alpha \mathcal{E}_\beta [\mathbf{g}(\tau, t)]| \leq \mathcal{S}_\alpha \mathcal{E}_\beta [|\mathbf{g}(\tau, t)|]$$

for any $\beta, \alpha \in (0, 1]$

Proof.

$$\begin{aligned} |\mathcal{S}_\alpha \mathcal{E}_\beta [\mathbf{g}(\tau, t)]| &= \left| \frac{\nu}{\omega} \int_0^\infty \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\beta\nu}} \mathbf{g}(\tau, t) \tau^{\alpha-1} t^{\beta-1} d\tau dt \right| \\ &\leq \left| \frac{\nu}{\omega} \right| \int_0^\infty \int_0^\infty \left| e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\beta\nu}} \mathbf{g}(\tau, t) \tau^{\alpha-1} t^{\beta-1} d\tau dt \right| \\ &= \frac{\nu}{\omega} \int_0^\infty \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\beta\nu}} |\mathbf{g}(\tau, t)| \tau^{\alpha-1} t^{\beta-1} d\tau dt \\ &= \mathcal{S}_\alpha \mathcal{E}_\beta [|\mathbf{g}(\tau, t)|]. \end{aligned}$$

Theorem 5.(Shifting property)

$$\mathcal{S}_\alpha \mathcal{E}_\beta \left[e^{\left(q \frac{\tau^\alpha}{\alpha} + r \frac{t^\beta}{\beta} \right)} \mathbf{g}(\tau, t) \right] = \frac{(1-p\nu)}{(1-q\omega)} \mathcal{S}_\alpha \mathcal{E}_\beta [\mathbf{g}(\tau, t)] \left(\frac{\omega}{1-q\omega}, \frac{\nu}{1-p\nu} \right).$$

Proof.

$$\begin{aligned} \mathcal{S}_\alpha \mathcal{E}_\beta \left[e^{\left(q \frac{\tau^\alpha}{\alpha} + p \frac{t^\beta}{\beta} \right)} \mathbf{g}(\tau, t) \right] &= \frac{\mathbf{v}}{\omega} \int_0^\infty \int_0^\infty e^{-\left(\frac{\tau^\alpha}{\alpha\omega} + \frac{t^\beta}{\beta\mathbf{v}} \right)} e^{\left(q \frac{\tau^\alpha}{\alpha} + p \frac{t^\beta}{\beta} \right)} \mathbf{g}(\tau, t) t^{\beta-1} \tau^{\alpha-1} d\tau dt \\ &= \frac{\mathbf{v}}{\omega} \int_0^\infty \int_0^\infty e^{-\left(\frac{1}{\omega} - q \right) \frac{\tau^\alpha}{\alpha}} e^{-\left(\frac{1}{\mathbf{v}} - p \right) \frac{t^\beta}{\beta}} \mathbf{g}(\tau, t) t^{\beta-1} \tau^{\alpha-1} d\tau dt. \end{aligned}$$

Let $u = \frac{\mathbf{v}}{(1-p\mathbf{v})}$ or equivalently $\frac{1}{u} = \frac{(1-p\mathbf{v})}{\mathbf{v}}$, and $\frac{1}{\mathfrak{z}} = \frac{(1-q\omega)}{\omega}$. Then

$$\begin{aligned} \mathcal{S}_\alpha \mathcal{E}_\beta \left[e^{\left(q \frac{\tau^\alpha}{\alpha} + p \frac{t^\beta}{\beta} \right)} \mathbf{g}(\tau, t) \right] &= \frac{(1-p\mathbf{v})}{(1-q\omega)} \frac{u}{\mathfrak{z}} \int_0^\infty \int_0^\infty e^{-\frac{t^\beta}{u\beta}} e^{-\frac{\tau^\alpha}{\mathfrak{z}\alpha}} \mathbf{g}(\tau, t) t^{\beta-1} \tau^{\alpha-1} d\tau dt \\ &= \frac{(1-p\mathbf{v})}{(1-q\omega)} \mathcal{S}_\alpha \mathcal{E}_\beta [\mathbf{g}(\tau, t)]_{(\mathfrak{z}, u)} \\ &= \frac{(1-p\mathbf{v})}{(1-q\omega)} \mathcal{S}_\alpha \mathcal{E}_\beta [\mathbf{g}(\tau, t)]_{\left(\frac{\omega}{1-q\omega}, \frac{\mathbf{v}}{1-p\mathbf{v}} \right)}. \end{aligned}$$

Theorem 6. (Conformable fractional operator property) Let $\mathcal{G}(\omega, \mathbf{v}) = \mathcal{S}_\alpha \mathcal{E}_\beta [\mathbf{g}(\tau, t)]$. Then

Proposition 1. (1) $\mathcal{S}_\alpha \mathcal{E}_\beta \left[\frac{\partial^\alpha \mathbf{g}(\tau, t)}{\partial \tau^\alpha} \right] = \frac{1}{\omega} \mathcal{G}(\omega, \mathbf{v}) - \frac{1}{\omega} \mathcal{E}_\beta [\mathbf{g}(0, t)]$.

(2) $\mathcal{S}_\alpha \mathcal{E}_\beta \left[\frac{\partial^\beta \mathbf{g}(\tau, t)}{\partial t^\beta} \right] = \frac{1}{\mathbf{v}} \mathcal{G}(\omega, \mathbf{v}) - \mathbf{v} \mathcal{S}_\alpha [\mathbf{g}(\tau, 0)]$.

(3) $\mathcal{S}_\alpha \mathcal{E}_\beta \left[\frac{\partial^{\alpha+\beta} \mathbf{g}(\tau, t)}{\partial \beta \partial \tau^\alpha} \right] = \frac{1}{\mathbf{v}\omega} \mathcal{G}(\omega, \mathbf{v}) - \frac{\mathbf{v}}{\omega} \mathcal{S}_\alpha [\mathbf{g}(\tau, 0)] - \mathcal{E}_\beta \left[\frac{\partial^\beta \mathbf{g}(0, t)}{\partial t^\beta} \right]$
 $= \frac{1}{\mathbf{v}\omega} \mathcal{G}(\omega, \mathbf{v}) - \frac{1}{\mathbf{v}\omega} \mathcal{E}_\beta [\mathbf{g}(\tau, 0)] - \mathbf{v} \mathcal{S}_\alpha \left[\frac{\partial^\beta \mathbf{g}(0, t)}{\partial t^\beta} \right]$

(4) $\mathcal{S}_\alpha \mathcal{E}_\beta \left[\frac{\partial^{2\alpha} \mathbf{g}(\tau, t)}{\partial \tau^{2\alpha}} \right] = \frac{1}{\omega^2} \mathcal{G}(\omega, \mathbf{v}) - \frac{1}{\omega^2} \mathcal{E}_\beta [\mathbf{g}(0, t)] - \frac{1}{\omega} \mathcal{E}_\beta \left[\frac{\partial^\alpha \mathbf{g}(0, t)}{\partial \tau^\alpha} \right]$.

(5) $\mathcal{S}_\alpha \mathcal{E}_\beta \left[\frac{\partial^{2\beta} \mathbf{g}(\tau, t)}{\partial t^{2\beta}} \right] = \frac{1}{\mathbf{v}^2} \mathcal{G}(\omega, \mathbf{v}) - \mathcal{S}_\alpha [\mathbf{g}(\tau, 0)] - \mathbf{v} \mathcal{S}_\alpha \left[\frac{\partial^\beta \mathbf{g}(\tau, 0)}{\partial t^\beta} \right]$.

Proof. (1) By the definition of the conformable fractional double Laplace-Elzaki transform, we have,

$$\begin{aligned} \mathcal{S}_\alpha \mathcal{E}_\beta \left[\frac{\partial^\alpha \mathbf{g}(\tau, t)}{\partial \tau^\alpha} \right] &= \frac{\mathbf{v}}{\omega} \int_0^\infty \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\beta\mathbf{v}}} \left[\frac{\partial^\alpha \mathbf{g}(\tau, t)}{\partial \tau^\alpha} \right] t^{\beta-1} \tau^{\alpha-1} d\tau dt \\ &= \frac{\mathbf{v}}{\omega} \int_0^\infty \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\beta\mathbf{v}}} \left[\frac{\partial \mathbf{g}(\tau, t)}{\partial \tau} \right] t^{\beta-1} d\tau dt \\ &= \frac{\mathbf{v}}{\omega} \int_0^\infty e^{-\frac{t^\beta}{\beta\mathbf{v}}} t^{\beta-1} dt \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega}} \frac{\partial \mathbf{g}(\tau, t)}{\partial \tau} d\tau. \end{aligned}$$

Using integration by parts, we get

$$\begin{aligned} &\mathcal{S}_\alpha \mathcal{E}_\beta \left[\frac{\partial^\alpha \mathbf{g}(\tau, t)}{\partial \tau^\alpha} \right] \\ &= \frac{\mathbf{v}}{\omega} \int_0^\infty e^{-\frac{t^\beta}{\beta\mathbf{v}}} t^{\beta-1} \left[e^{-\frac{\tau^\alpha}{\alpha\omega}} \mathbf{g}(\tau, t) \Big|_{\tau=0} + \int_0^\infty \mathbf{g}(\tau, t) e^{-\frac{\tau^\alpha}{\alpha\omega}} \tau^{\alpha-1} d\tau \right] dt \\ &= \frac{\mathbf{v}}{\omega} \int_0^\infty e^{-\frac{t^\beta}{\beta\mathbf{v}}} t^{\beta-1} (-\mathbf{g}(0, t)) dt + \frac{1}{\omega} \int_0^\infty \int_0^\infty \mathbf{g}(\tau, t) e^{-\frac{t^\beta}{\beta\mathbf{v}} - \frac{\tau^\alpha}{\alpha\omega}} t^{\beta-1} \tau^{\alpha-1} d\tau dt \\ &= -\frac{1}{\omega} \mathcal{E}_\beta [\mathbf{g}(0, t)] + \frac{1}{\omega} \mathcal{S}_\alpha \mathcal{E}_\beta [\mathbf{g}(\tau, t)] \\ &= \frac{1}{\omega} \mathcal{G}(\omega, \mathbf{v}) - \frac{1}{\omega} \mathcal{E}_\beta [\mathbf{g}(0, t)]. \end{aligned}$$

(2) To prove the second property,

$$\begin{aligned} \mathcal{S}_\alpha \mathcal{E}_\beta \left[\frac{\partial^\beta \mathbf{g}(\tau, t)}{\partial t^\beta} \right] &= \frac{\mathbf{v}}{\omega} \int_0^\infty \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\beta\mathbf{v}}} \left[\frac{\partial^\beta \mathbf{g}(\tau, t)}{\partial t^\beta} \right] t^{\beta-1} \tau^{\alpha-1} d\tau dt \\ &= \frac{\mathbf{v}}{\omega} \int_0^\infty \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\beta\mathbf{v}}} \left[\frac{\partial \mathbf{g}(\tau, t)}{\partial t} \right] \tau^{\alpha-1} d\tau dt \\ &= \frac{\mathbf{v}}{\omega} \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega}} \tau^{\alpha-1} d\tau \int_0^\infty \frac{\partial \mathbf{g}(\tau, t)}{\partial t} e^{-\frac{t^\beta}{\beta\mathbf{v}}} dt. \end{aligned}$$

Using integration by parts, we get

$$\begin{aligned} &\int_0^\infty \frac{\mathbf{v}}{\omega} e^{-\frac{\tau^\alpha}{\alpha\omega}} \tau^{\alpha-1} d\tau \int_0^\infty \frac{\partial \mathbf{g}(\tau, t)}{\partial t} e^{-\frac{t^\beta}{\beta\mathbf{v}}} dt \\ &= \frac{\mathbf{v}}{\omega} \int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega}} \tau^{\alpha-1} d\tau \left[e^{-\frac{t^\beta}{\beta\mathbf{v}}} \mathbf{g}(\tau, t) \Big|_{t=0} + \frac{1}{\mathbf{v}} \int_0^\infty t^{\beta-1} e^{-\frac{t^\beta}{\beta\mathbf{v}}} \mathbf{g}(\tau, t) dt \right] \\ &= \frac{\mathbf{v}}{\omega} \left[\int_0^\infty e^{-\frac{\tau^\alpha}{\alpha\omega}} (-\mathbf{g}(\tau, 0)) \tau^{\alpha-1} d\tau + \frac{1}{\mathbf{v}} \int_0^\infty \int_0^\infty \tau^{\alpha-1} t^{\beta-1} e^{-\frac{\tau^\alpha}{\alpha\omega} - \frac{t^\beta}{\beta\mathbf{v}}} \mathbf{g}(\tau, t) d\tau dt \right] \\ &= -\mathbf{v} \mathcal{S}_\alpha [\mathbf{g}(\tau, 0)] + \frac{1}{\mathbf{v}} \mathcal{S}_\alpha \mathcal{E}_\beta [\mathbf{g}(\tau, t)] \\ &= \frac{1}{\mathbf{v}} \mathcal{G}(\mathbf{s}, \mathbf{v}) - \mathbf{v} \mathcal{S}_\alpha [\mathbf{g}(\tau, 0)]. \end{aligned}$$

Likewise we can use part (1) and (2) to obtain part (3). Also part (4) and (5) can be obtained by applying part (1) and (2) respectively twice.

3 Applications

Problem 1. Use the double conformable fractional Sumudu-Elzaki transform to solve the following conformable Klein-Gordon equation

$$u_{tt}^{2\beta} \left(\frac{\tau^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) - u_{\tau\tau}^{2\alpha} \left(\frac{\tau^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) - 2u \left(\frac{\tau^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) = -2 \sin \frac{\tau^\alpha}{\alpha} \sin \frac{t^\beta}{\beta}, \tag{3.1}$$

where:

$$\begin{aligned} u \left(\frac{\tau^\alpha}{\alpha}, 0 \right) &= 0 \\ u_t^\beta \left(\frac{\tau^\alpha}{\alpha}, 0 \right) &= \sin \frac{\tau^\alpha}{\alpha} \\ u \left(0, \frac{t^\beta}{\beta} \right) &= 0 \\ u_\tau^\alpha \left(0, \frac{t^\beta}{\beta} \right) &= \sin \frac{t^\beta}{\beta}, \end{aligned}$$

$\beta, \alpha \in (0, 1]$, $\tau, t > 0$ and u_t^β, u_τ^α denote the β th- and α th-order partial conformable fractional derivative of $u \left(\frac{\tau^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right)$ with respect to t and τ , respectively.

Solution. Let $\mathcal{U}(\mathbf{w}, \mathbf{v}) = \mathcal{S}_\alpha \mathcal{E}_\beta \left[u \left(\frac{\tau^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) \right]$. Then,

$$\begin{aligned} &\frac{1}{\mathbf{v}^2} \mathcal{U}(\mathbf{w}, \mathbf{v}) - \mathcal{S}_\alpha \left[u \left(\frac{\tau^\alpha}{\alpha}, 0 \right) \right] - \mathbf{v} \mathcal{S}_\alpha \left[u_t^\beta \left(\frac{\tau^\alpha}{\alpha}, 0 \right) \right] \\ &- \frac{1}{\mathbf{w}^2} \mathcal{U}(\mathbf{w}, \mathbf{v}) + \frac{1}{\mathbf{w}^2} \mathcal{E}_\beta \left[u \left(0, \frac{t^\beta}{\beta} \right) \right] + \frac{1}{\mathbf{w}} \mathcal{E}_\beta \left[u_\tau^\alpha \left(0, \frac{t^\beta}{\beta} \right) \right] - 2\mathcal{U}(\mathbf{w}, \mathbf{v}) \\ &= -2 \frac{\mathbf{v}^3}{1 + \mathbf{v}^2} \frac{\mathbf{w}}{1 + \mathbf{w}^2} \end{aligned}$$

Substituting the initial and boundary conditions:

$$\begin{aligned} & \left(\frac{1}{v^2} - \frac{1}{w^2} - 2 \right) \mathcal{U}(w, v) - \frac{wv}{1+w^2} + \frac{1}{w} \frac{v^3}{1+v^2} \\ &= -2 \frac{v^3}{1+v^2} \frac{w}{1+w^2} \end{aligned}$$

$$\mathcal{U}(w, v) = \frac{-2 \frac{v^3}{1+v^2} \frac{w}{1+w^2} + \frac{wv}{1+w^2} - \frac{1}{w} \frac{v^3}{1+v^2}}{\left(\frac{1}{v^2} - \frac{1}{w^2} - 2 \right)}$$

$$\begin{aligned} \mathcal{U}(w, v) &= \\ \mathcal{U}(w, v) &= (w^2 v^2) \frac{-2w^2 v^3 + w^2 v^3 + w^2 v - v^3 - v^3 w^2}{(1+v^2)(1+w^2)(w^2 - v^2 - 2w^2 v^2)} \\ &= \frac{wv^2(-v(2w^2 v^2 + v^2 - w^2))}{(1+v^2)(1+w^2)(w^2 - v^2 - 2w^2 v^2)} \\ &= \frac{v^3 w (w^2 - v^2 - 2w^2 v^2)}{(1+v^2)(1+w^2)(w^2 - v^2 - 2w^2 v^2)} \\ &= \frac{v^3}{(1+v^2)} \frac{w}{(1+w^2)}. \end{aligned}$$

Then

$$\begin{aligned} u \left(\frac{\tau^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) &= [\mathcal{L}_\alpha \mathcal{L}_\beta]^{-1} \left[\frac{v^3}{(1+v^2)} \frac{w}{(1+w^2)} \right] \\ &= \sin \frac{\tau^\alpha}{\alpha} \sin \frac{t^\beta}{\beta}. \end{aligned}$$

The following set of figures shows exact solutions for different values of β and α of equation (3.1).

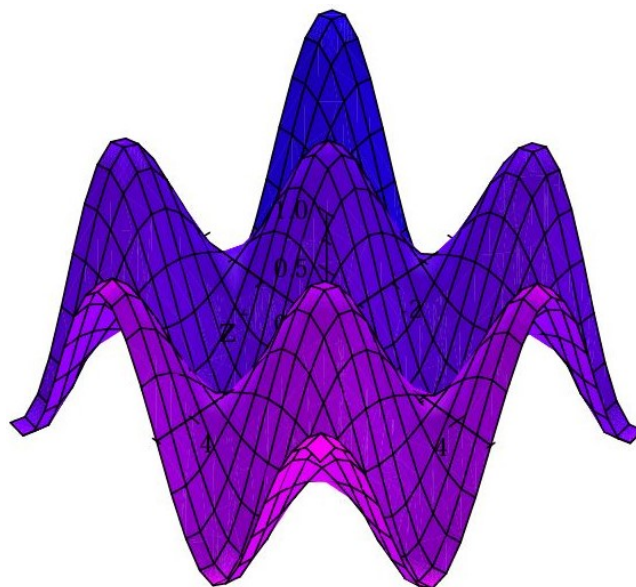


Fig. 1: Figure 3.1-a: ($\beta = \alpha = 1$)

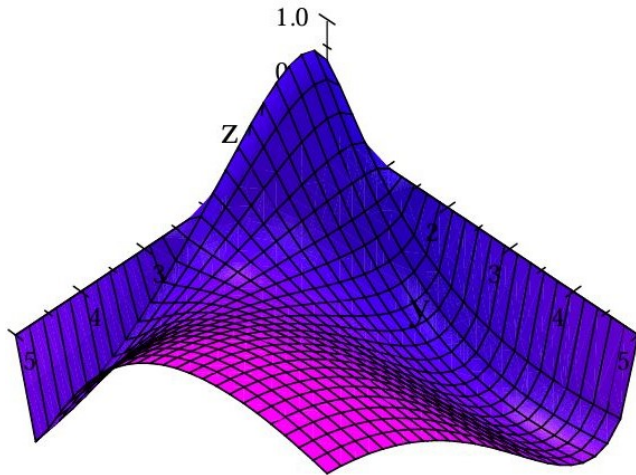


Fig. 2: Figure 3.1-b: ($\beta = \frac{1}{3}$ and $\alpha = \frac{1}{2}$)

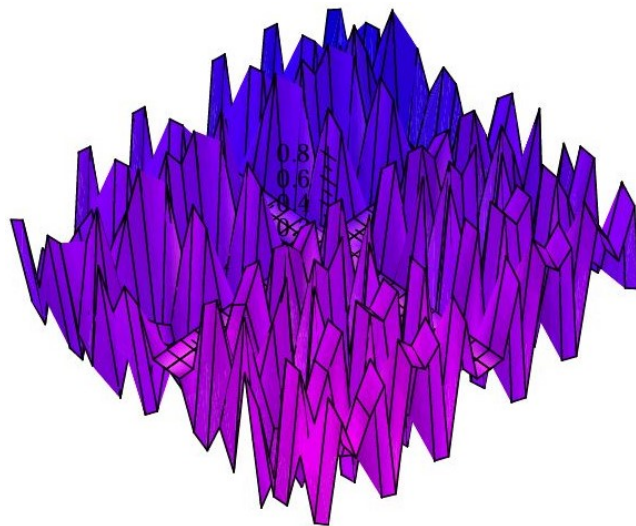


Fig. 3: Figure 3.1-c: ($\beta = \frac{1}{17}$ and $\alpha = \frac{1}{10}$)

Problem 2. Use the double conformable fractional Laplace-Elzaki transform to solve the following homogeneous fractional wave equation

$$u_t^{2\beta}(\tau, t) = u_{\tau\tau}^{2\alpha}(\tau, t), \quad (3.2)$$

Substituting the initial and boundary conditions:

$$\begin{aligned} & \left(\frac{1}{v^2} - \frac{1}{w^2} - 2 \right) \mathcal{U}(w, v) - \frac{wv}{1+w^2} + \frac{1}{w} \frac{v^3}{1+v^2} \\ &= -2 \frac{v^3}{1+v^2} \frac{w}{1+w^2} \end{aligned}$$

$$\mathcal{U}(w, v) = \frac{-2 \frac{v^3}{1+v^2} \frac{w}{1+w^2} + \frac{wv}{1+w^2} - \frac{1}{w} \frac{v^3}{1+v^2}}{\left(\frac{1}{v^2} - \frac{1}{w^2} - 2 \right)}$$

$$\begin{aligned} \mathcal{U}(\mathfrak{w}, \mathfrak{v}) &= \\ \mathcal{U}(\mathfrak{w}, \mathfrak{v}) &= (\mathfrak{w}^2 \mathfrak{v}^2) \frac{-2\mathfrak{w}^2 \mathfrak{v}^3 + \mathfrak{w}^2 \mathfrak{v}^3 + \mathfrak{w}^2 \mathfrak{v} - \mathfrak{v}^3 - \mathfrak{v}^3 \mathfrak{w}^2}{(1 + \mathfrak{v}^2)(1 + \mathfrak{w}^2)(\mathfrak{w}^2 - \mathfrak{v}^2 - 2\mathfrak{w}^2 \mathfrak{v}^2)} \\ &= \frac{\mathfrak{w} \mathfrak{v}^2 (-\mathfrak{v}(2\mathfrak{w}^2 \mathfrak{v}^2 + \mathfrak{v}^2 - \mathfrak{w}^2))}{(1 + \mathfrak{v}^2)(1 + \mathfrak{w}^2)(\mathfrak{w}^2 - \mathfrak{v}^2 - 2\mathfrak{w}^2 \mathfrak{v}^2)} \\ &= \frac{\mathfrak{v}^3 \mathfrak{w} (\mathfrak{w}^2 - \mathfrak{v}^2 - 2\mathfrak{w}^2 \mathfrak{v}^2)}{(1 + \mathfrak{v}^2)(1 + \mathfrak{w}^2)(\mathfrak{w}^2 - \mathfrak{v}^2 - 2\mathfrak{w}^2 \mathfrak{v}^2)} \\ &= \frac{\mathfrak{v}^3}{(1 + \mathfrak{v}^2)} \frac{\mathfrak{w}}{(1 + \mathfrak{w}^2)}. \end{aligned}$$

Then

$$\begin{aligned} u\left(\frac{\tau^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) &= [\mathcal{S}_\alpha \mathcal{E}_\beta]^{-1} \left[\frac{\mathfrak{v}^3}{(1 + \mathfrak{v}^2)} \frac{\mathfrak{w}}{(1 + \mathfrak{w}^2)} \right] \\ &= \sin \frac{\tau^\alpha}{\alpha} \sin \frac{t^\beta}{\beta}. \end{aligned}$$

The following set of figures shows exact solutions for different values of β and α of equation (3.1). with conditions:

$$u(0, t) = e^{\frac{t^\beta}{\beta}}$$

$$u(\tau, 0) = e^{\frac{\tau^\alpha}{\alpha}}$$

$$u_\tau^\alpha(0, t) = e^{\frac{t^\beta}{\beta}}$$

$$u_t^\beta(\tau, 0) = e^{\frac{\tau^\alpha}{\alpha}},$$

where $\beta, \alpha \in (0, 1]$, $\tau, t > 0$ and u_t^β, u_τ^α denote the β th- and α th-order partial conformable fractional derivative of $u(\tau, t)$ with respect to t and τ , respectively.

Solution. Let $\mathcal{U}(\mathfrak{w}, \mathfrak{v}) = \mathcal{S}_\alpha \mathcal{E}_\beta [u(\tau, t)]$. Then,

$$\begin{aligned} &\frac{1}{\mathfrak{v}^2} \mathcal{U}(\mathfrak{w}, \mathfrak{v}) - \mathcal{S}_\alpha [u(\tau, 0)] - \mathfrak{v} \mathcal{S}_\alpha [u_t^\beta(\tau, 0)] \\ &= \frac{1}{\mathfrak{w}^2} \mathcal{U}(\mathfrak{w}, \mathfrak{v}) - \frac{1}{\mathfrak{w}^2} \mathcal{E}_\beta [u(0, t)] - \frac{1}{\mathfrak{w}} \mathcal{E}_\beta [u_\tau^\alpha(0, t)] \end{aligned}$$

Substituting the initial and boundary conditions:

$$\begin{aligned} &\left(\frac{1}{\mathfrak{w}^2} - \frac{1}{\mathfrak{v}^2}\right) \mathcal{U}(\mathfrak{w}, \mathfrak{v}) \\ &= \frac{1}{\mathfrak{w}^2} \mathcal{E}_\beta \left[e^{\frac{t^\beta}{\beta}} \right] + \frac{1}{\mathfrak{w}} \mathcal{E}_\beta \left[e^{\frac{t^\beta}{\beta}} \right] - \mathcal{S}_\alpha \left[e^{\frac{\tau^\alpha}{\alpha}} \right] - \mathfrak{v} \mathcal{S}_\alpha \left[e^{\frac{\tau^\alpha}{\alpha}} \right] \end{aligned}$$

$$\begin{aligned}
 \mathcal{U}(\mathfrak{w}, \mathfrak{v}) &= \frac{1}{\mathfrak{w}^2} \frac{\mathfrak{v}^2}{1-\mathfrak{v}} + \frac{1}{\mathfrak{w}} \frac{\mathfrak{v}^2}{1-\mathfrak{v}} - \frac{1}{1-\mathfrak{w}} - \frac{\mathfrak{v}}{1-\mathfrak{w}} \\
 &= \frac{\frac{1}{\mathfrak{w}} \left(\frac{1}{\mathfrak{w}} + 1 \right) \frac{\mathfrak{v}^2}{1-\mathfrak{v}} - \frac{(1+\mathfrak{v})}{1-\mathfrak{w}}}{\left(\frac{1}{\mathfrak{w}^2} - \frac{1}{\mathfrak{v}^2} \right)} \\
 &= \frac{\frac{1}{\mathfrak{w}} \left(\frac{1}{\mathfrak{w}} + 1 \right) \mathfrak{v}^2}{(1-\mathfrak{v}) \left(\frac{1}{\mathfrak{w}^2} - \frac{1}{\mathfrak{v}^2} \right)} - \frac{(1+\mathfrak{v})}{(1-\mathfrak{w}) \left(\frac{1}{\mathfrak{w}^2} - \frac{1}{\mathfrak{v}^2} \right)} \\
 &= \frac{(1-\mathfrak{w})(1+\mathfrak{w})\mathfrak{v}^2 - \mathfrak{w}^2(1+\mathfrak{v})(1-\mathfrak{v})}{\mathfrak{w}^2(1-\mathfrak{w})(1-\mathfrak{v}) \left(\frac{1}{\mathfrak{w}^2} - \frac{1}{\mathfrak{v}^2} \right)} \\
 &= \frac{\mathfrak{v}^2(\mathfrak{v}^2(1-\mathfrak{w}^2) - \mathfrak{w}^2(1-\mathfrak{v}^2))}{(1-\mathfrak{w})(1-\mathfrak{v})(\mathfrak{v}^2 - \mathfrak{w}^2)} \\
 &= \frac{\mathfrak{v}^4 - \mathfrak{w}^2\mathfrak{v}^4 - \mathfrak{w}^2\mathfrak{v}^2 + \mathfrak{w}^2\mathfrak{v}^4}{(1-\mathfrak{w})(1-\mathfrak{v})(\mathfrak{v}^2 - \mathfrak{w}^2)} \\
 &= \frac{\mathfrak{v}^4 - \mathfrak{v}^2\mathfrak{w}^2}{(1-\mathfrak{w})(1-\mathfrak{v})(\mathfrak{v}^2 - \mathfrak{w}^2)} \\
 &= \frac{\mathfrak{v}^2(\mathfrak{v}^2 - \mathfrak{w}^2)}{(1-\mathfrak{w})(1-\mathfrak{v})(\mathfrak{v}^2 - \mathfrak{w}^2)} \\
 &= \frac{\mathfrak{v}^2}{(1-\mathfrak{w})(1-\mathfrak{v})} \\
 &= \frac{1}{(1-\mathfrak{w})} \frac{\mathfrak{v}^2}{(1-\mathfrak{v})}.
 \end{aligned}$$

$$\begin{aligned}
 u(\mathfrak{r}, \mathfrak{t}) &= [\mathcal{S}_\alpha \mathcal{E}_\beta]^{-1} \left[\frac{\mathfrak{w}}{(1-\mathfrak{w})} \frac{\mathfrak{v}^2}{(1-\mathfrak{v})} \right] \\
 &= e^{\frac{\mathfrak{r}^\alpha}{\alpha} + \frac{\mathfrak{t}^\alpha}{\alpha}}.
 \end{aligned}$$

Figure (3.2) shows the exact solution of equation (3.2) when $\alpha = \beta = 1$.

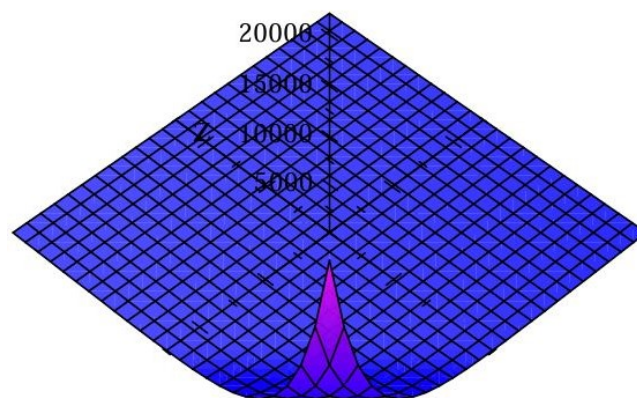


Fig. 4: Figure 3.2: ($\beta = \alpha = 1$)

Problem 3. Consider the following conformable integral equation

$$u_{\mathfrak{t}\mathfrak{t}}^{2\beta}(\mathfrak{r}, \mathfrak{t}) - u_{\mathfrak{r}\mathfrak{r}}^{2\alpha}(\mathfrak{r}, \mathfrak{t}) + u(\mathfrak{r}, \mathfrak{t}) + \int_0^{\mathfrak{r}} \int_0^{\mathfrak{t}} e^{\frac{(\mathfrak{r}-\delta)^\alpha}{\alpha}} e^{\frac{(\mathfrak{t}-\gamma)^\beta}{\beta}} u(\delta, \gamma) d\delta d\gamma = \frac{\mathfrak{r}^\alpha}{\alpha} e^{\frac{\mathfrak{r}^\alpha}{\alpha} + \frac{\mathfrak{t}^\beta}{\beta}} + \frac{\mathfrak{t}^\beta}{\beta} e^{\frac{\mathfrak{r}^\alpha}{\alpha} + \frac{\mathfrak{t}^\beta}{\beta}} \quad (3.3)$$

with I.B.Cs

$$\begin{aligned}
 u(0, t) &= e^{\frac{t^\beta}{\beta}} \\
 u(\tau, 0) &= e^{\frac{\tau^\alpha}{\alpha}} \\
 u_\tau^\alpha(0, t) &= e^{\frac{t^\beta}{\beta}} \\
 u_t^\beta(\tau, 0) &= e^{\frac{\tau^\alpha}{\alpha}}.
 \end{aligned}$$

Solution. Applying the CSET and using the convolution theorem

$$\begin{aligned}
 &\frac{1}{v^2} \mathcal{U}(w, v) - \mathcal{S}_\alpha[u(\tau, 0)] - v \mathcal{S}_\alpha[u_t^\beta(\tau, 0)] \\
 &- \left(\frac{1}{w^2} \mathcal{U}(w, v) - \frac{1}{w^2} \mathcal{E}_\beta[u(0, t)] - \frac{1}{w} \mathcal{E}_\beta[u_\tau^\alpha(0, t)] \right) \\
 &+ \mathcal{U}(w, v) + \mathcal{S}_\alpha \mathcal{E}_\beta \left[\int_0^\tau \int_0^t e^{\frac{(\tau-\delta)^\alpha}{\alpha}} e^{\frac{(t-\gamma)^\beta}{\beta}} u(\delta, \gamma) d\delta d\gamma \right] \\
 &= \frac{v^2}{1-v} \frac{1}{1-w} + \frac{w}{(1-w)^2} \frac{v^3}{(1-v)^2}
 \end{aligned}$$

Substituting the I.B.Cs implies

$$\begin{aligned}
 &\frac{1}{v^2} \mathcal{U}(w, v) - \mathcal{S}_\alpha \left[e^{\frac{\tau^\alpha}{\alpha}} \right] - v \mathcal{S}_\alpha \left[e^{\frac{\tau^\alpha}{\alpha}} \right] \\
 &- \left(\frac{1}{w^2} \mathcal{U}(w, v) - \frac{1}{w^2} \mathcal{E}_\beta \left[e^{\frac{t^\beta}{\beta}} \right] - \frac{1}{w} \mathcal{E}_\beta \left[e^{\frac{t^\beta}{\beta}} \right] \right) \\
 &+ \mathcal{U}(w, v) + \mathcal{U}(w, v) \frac{w}{1-w} \frac{v}{1-v} \\
 &= \frac{v^2}{1-v} \frac{1}{1-w} + \frac{w}{(1-w)^2} \frac{v^3}{(1-v)^2}
 \end{aligned}$$

$$\begin{aligned}
 &\left(\frac{1}{v^2} - \frac{1}{w^2} + 1 + \frac{v}{1-v} \frac{w}{1-w} \right) \mathcal{U}(w, v) \\
 &= \frac{1}{1-w} + \frac{v}{1-w} - \frac{1}{w^2} \frac{v^2}{1-v} - \frac{1}{w} \frac{v^2}{1-v} + \frac{v^2}{1-v} \frac{1}{1-w} + \frac{w}{(1-w)^2} \frac{v^3}{(1-v)^2} \\
 \mathcal{U}(w, v) &= \frac{\frac{1}{1-w} + \frac{v}{1-w} - \frac{1}{w^2} \frac{v^2}{1-v} - \frac{1}{w} \frac{v^2}{1-v} + \frac{v^2}{1-v} \frac{1}{1-w} + \frac{w}{(1-w)^2} \frac{v^3}{(1-v)^2}}{\left(\frac{1}{v^2} - \frac{1}{w^2} + 1 + \frac{v}{1-v} \frac{w}{1-w} \right)}
 \end{aligned}$$

After simplifying the above equation,

$$\begin{aligned}
 \mathcal{U}(w, v) &= \frac{v^2}{(1-v)(1-w)} \\
 &= \frac{v^2}{1-v} \frac{1}{1-w}.
 \end{aligned}$$

And,

$$\begin{aligned}
 u(\tau, t) &= (\mathcal{S}_\alpha \mathcal{E}_\beta)^{-1} \left[\frac{v^2}{1-v} \frac{1}{1-w} \right] \\
 &= e^{\frac{\tau^\alpha}{\alpha} + \frac{t^\beta}{\beta}}.
 \end{aligned}$$

Problem 4. Consider the following conformable integral equation

$$\int_0^{\tau} \int_0^t e^{\delta-\gamma} u(\tau-\delta, t-\gamma) d\delta d\gamma = te^{\frac{\tau^\alpha}{\alpha} - \frac{t^\beta}{\beta}} - te^{\frac{t^\beta}{\beta}}. \quad (3.4)$$

Solution. Applying the convolution theorem

$$\begin{aligned} \int_0^{\tau} \int_0^t e^{\delta-\gamma} u(\tau-\delta, t-\gamma) d\delta d\gamma &= e^{\tau-t} * u(\tau, t) \\ e^{\tau-t} * u(\tau, t) &= te^{\frac{\tau^\alpha}{\alpha} - \frac{t^\beta}{\beta}} - te^{\frac{t^\beta}{\beta}} \end{aligned} \quad (3.4-a)$$

Applying the CSET on both sides of equation (3.4 - a)

$$\begin{aligned} \frac{w}{v} \frac{v^2}{(1+v)} \frac{1}{(1-w)} \mathcal{U}(w, v) &= \frac{w}{(1-w)^2} \frac{v^2}{(1+v)} - \frac{wv^2}{(1-w)^2} \\ \mathcal{U}(w, v) &= \frac{\frac{w}{(1-w)^2} \frac{v^2}{(1+v)} - \frac{wv^2}{(1-w)^2}}{\frac{w}{v} \frac{v^2}{(1+v)} \frac{1}{(1-w)}}. \end{aligned}$$

Simplify the equation we get,

$$\mathcal{U}(w, v) = \frac{v^2}{w-1}.$$

Therefore,

$$u(\tau, t) = (\mathcal{S}_\alpha \mathcal{E}_\beta)^{-1} \left(\frac{v^2}{w-1} \right) = -e^{\frac{\tau^\alpha}{\alpha}}.$$

Problem 5. Consider the following conformable integral equation

$$c^2 t = \int_0^{\tau} \int_0^t u(\delta, \gamma) u(\tau-\delta, t-\gamma) d\delta d\gamma, \quad (3.5)$$

where c is a constant.

Solution. Applying the convolution theorem

$$c^2 t = \int_0^{\tau} \int_0^t u(\delta, \gamma) u(\tau-\delta, t-\gamma) d\delta d\gamma = u(\tau, t) * u(\tau, t)$$

Now performing the CSET on both sides of the equation

$$c^2 v^3 = \frac{w}{v} \mathcal{U}(w, v) \mathcal{U}(w, v)$$

$$\begin{aligned} (\mathcal{U}(w, v))^2 &= \frac{c^2 v^3}{\frac{w}{v}} \\ &= \frac{c^2 v^4}{w} \end{aligned}$$

$$\begin{aligned} \mathcal{U}(w, v) &= \frac{cv^2}{\sqrt{w}} \\ &= w^{-\frac{1}{2}} cv^2. \end{aligned}$$

Then,

$$u(\tau, t) = \frac{c\sqrt{\pi}}{\sqrt{\tau}}.$$

The following figures shows the exact solution of equation (3.5) at different values of c .

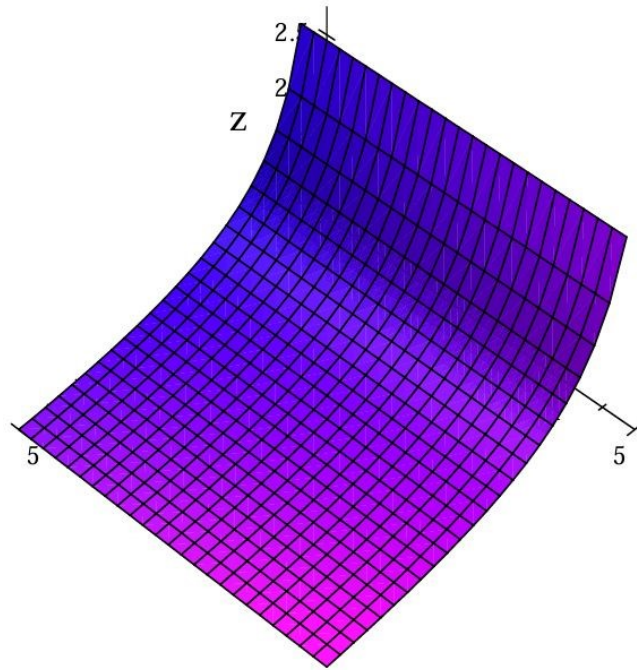


Fig. 5: Figure 3.5-a: ($c=1$)

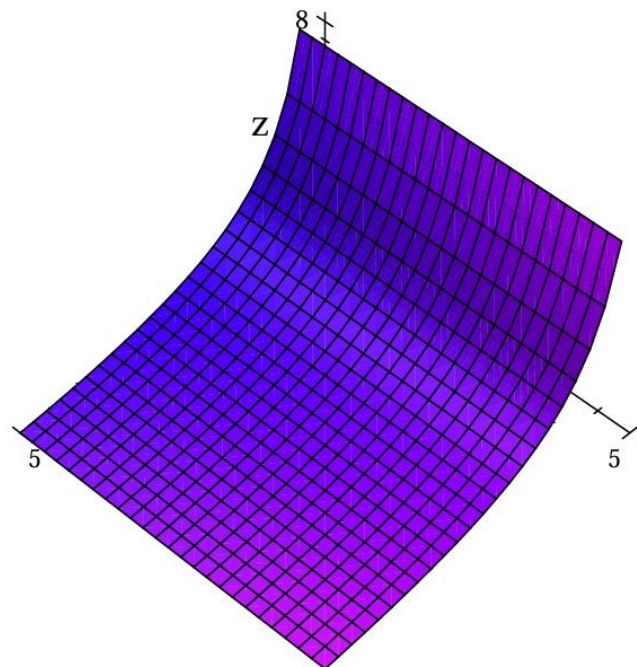


Fig. 6: Figure 3.5-b: ($c=3$)

4 Conclusions

In conclusion, this paper introduced the conformable fractional Sumudu-Elzaki transform (CSET) and presented several fundamental properties and theorems associated with it. The proposed transform was successfully applied to integral and partial differential equations. The obtained results indicate that the CSET provides an efficient, reliable, and practical framework for solving equations of this type. Therefore, the conformable fractional Sumudu-Elzaki transform can be regarded as a promising mathematical technique for future research in fractional calculus and its applications across mathematics, physics, and engineering sciences.

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