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Comparative Study of Fractional Heun's Method and Fractional q-Heun's Method for Solving Fractional Differential Equations

Obaida Aljabari[1], *Shaher Momani*[1,2,*], *Iqbal M. Batiha*[2,3] *and Mohammad Odeh*[4]

[1]Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan
[2]Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE
[3]Department of Mathematics, Al Zaytoonah University of Jordan, Amman 11733, Jordan
[4]Department of Mathematics, Philadelphia University, Amman 19392, Jordan

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Abstract: This paper aims to study the Fractional Heun's Method (FHM) and the Fractional q-Heun's Method (FqHM). It provides a detailed comparison of their numerical performance, accuracy, and error behavior when solving fractional initial value problems (FIVPs). Numerical simulations are presented to validate the theoretical results and highlight the advantages and limitations of both methods.

Keywords: Fractional Heun's Method, Fractional q-Heun's Method, Fractional Calculus, Numerical Analysis, q-Calculus, Initial Value Problems.

1 Introduction

Fractional calculus is a branch of mathematical analysis that extends the concepts of differentiation and integration to non-integer orders [1,2,3,4,5]. The origins of this field date back to the late seventeenth century when Newton and Leibniz laid the foundations of differential and integral calculus. In particular, Leibniz introduced the symbol

$$\frac{d^n}{dx^n} f(x)$$

to define the nth derivative, where $n \in \mathbb{N}$ [6].

Many problems in various fields of science and technology cannot be solved analytically using known rules and techniques to obtain exact solutions [7,8,9,10]. In such cases, numerical methods are used to provide approximate solutions while accounting for an error associated with each method [11].

The subject of quantum calculus, also known as q-difference calculus, was initiated in 1910 when F.H. Jackson introduced the concepts of q-difference calculus. The fundamental definitions and characteristics of q-difference calculus were later refined by P.M. Rajkovic *et al.* (2007). Quantum calculus has proven to be a powerful tool in explaining various physical phenomena. For instance, the physicist V. Fock used q-difference equations to verify the symmetry of hydrogen atoms. Later, the foundations of this field were solidified in the late 1960s through the works of W. Al-Salam and R. Agarwal [12,13].

q-Calculus has been applied in diverse scientific domains such as physics, chemistry, engineering, and medicine, where differential equations are often used to model real-world problems [14,15]. However, many of these problems cannot be solved analytically, necessitating the use of numerical techniques to solve the corresponding equations [16].

* Corresponding author e-mail: s.momani@ju.edu.jo

Obtaining an exact solution for a differential equation is often difficult or even impossible. However, the use of computational techniques has facilitated the numerical solution of such equations, making these methods attractive to researchers. These methods provide an effective way to obtain approximate solutions for differential equations that were previously intractable analytically. One such method is Heun's method [17]. The present paper is devoted to studying the Fractional Heun's Method (FHM) and the Fractional q-Heun's Method (FqHM), followed by a detailed comparison between them.

2 Preliminaries

In this section, we review some basic notions and fundamental results associated with fractional calculus and fractional q-calculus, which are necessary for this study.

2.1 Fractional Calculus

This subsection introduces fundamental concepts of fractional calculus, including the Riemann-Liouville fractional integral and its essential properties [18,19,20,21].

Definition 1.*[22] Let α be a real non-negative number. The operator J_a^α is defined on $L_1[a,b]$, where $L_1[a,b]$ is the set of all functions whose absolute values are integrable on $[a,b]$, by*

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad a \le t \le b, \tag{1}$$

and is called the Riemann-Liouville fractional-order integral operator of order α.

2.1.1 Properties of the Riemann-Liouville Integral

The Riemann-Liouville fractional integral satisfies the following important properties [23,24]:

1.For $f(t) = K$ and $a = 0$, we have

$$J_0^\alpha K = \frac{K}{\Gamma(\alpha+1)} t^\alpha. \tag{2}$$

2.If $f(t) = (t-a)^p$, then

$$J_a^\alpha (t-a)^p = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} (\tau-a)^p d\tau. \tag{3}$$

3.Let $m,n \ge 0$ and $f \in L_1[a,b]$, then

$$J_a^m J_a^n f = J_a^{m+n} f. \tag{4}$$

4.For $m,n \ge 0$, we have

$$J_a^m J_a^n f = J_a^n J_a^m f. \tag{5}$$

Proposition 1.*Let $\alpha \in \mathbb{R}$ and $m = \lceil \alpha \rceil$. The operator D_a^α defined by*

$${}^{RL}D_a^\alpha f = D^m J_a^{m-\alpha} f, \tag{6}$$

is called the Riemann-Liouville fractional-order differential operator of order α.

Definition 2.*[22] Let α be a real non-negative number. For a positive integer m such that $m-1 < \alpha \le m$, the Riemann-Liouville fractional-order differential operator of a function f of order α is defined by*

$${}^{RL}D_a^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau. \tag{7}$$

Definition 3.*[22] Let $\alpha \in \mathbb{R}^+$ and $m = \lceil \alpha \rceil$ such that $m-1 < \alpha \le m$. Then, the Caputo fractional-order derivative operator of order α is given by*

$${}^{c}D_a^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \quad t > a. \tag{8}$$

2.1.2 Properties of the Caputo Fractional-Order Operator

Several useful results arise from the use of the Caputo fractional-order operator, including

1. If $\alpha = k$ for some $k \in \mathbb{N}$, then
$$^{C}D^{\alpha} f(t) = f^{(k)}(t), \quad \text{for } t \in \mathbb{R}^{+}. \tag{9}$$
2. For $m-1 < \alpha \leq m$ such that $m \in \mathbb{N}$, we have
$$^{c}D_a^{\alpha}(\lambda f(t) + \mu g(t)) = \lambda {}^{c}D_a^{\alpha} f(t) + \mu {}^{c}D_a^{\alpha} g(t),$$
where λ, μ are two scalars.
3. Let $m-1 < \alpha \leq m$ such that $m, \beta \in \mathbb{N}$ and $\alpha \in \mathbb{R}^{+}$. If $f(x) = (x-a)^{\beta}$, where $\beta \geq 0$, then
$$^{c}D_a^{\alpha}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(x-a)^{\beta-\alpha}. \tag{10}$$

Theorem 1. *(Generalized Taylor's formula).*
Suppose $D^{k\alpha} f(x) \in C((0,b])$, for $k = 0, 1, 2, ..., n+1$, where $0 < \alpha \leq 1$. Then
$$f(x) = \sum_{i=0}^{n} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)}(D_*^{i\alpha} f)(0_+) + \frac{(D^{(n+1)\alpha} f)(\zeta)}{\Gamma((n+1)\alpha+1)} x^{(n+1)\alpha}, \tag{11}$$
where $0 \leq \zeta \leq x, \ \forall x \in (0,b]$.

2.2 Fractional q-Calculus

If $q \in (0,1)$ and $a \in \mathbb{R}$, we define [26]
$$[a]_q = \frac{1-q^a}{1-q}. \tag{12}$$
As a result, the q-factorial of a positive integer is defined as
$$[1]_q = 1,$$
$$[n]_q = \frac{1-q^n}{1-q}.$$

The q-analogue of the power $(a-b)^{(\alpha)}$ is expressed by
$$(a-b)^{(\alpha)} = a^{\alpha} \prod_{i=0}^{n-1} \left(\frac{a-bq^i}{a-bq^{i+\alpha}} \right), \quad a, b \in \mathbb{R}, \ \alpha \in \mathbb{R}, \tag{13}$$
with the special case
$$(a-b)^{(0)} = 1.$$

Definition 4. *Consider an arbitrary function $f(x)$. Its q-differential is defined by*
$$d_q f(x) = f(qx) - f(x). \tag{14}$$

Definition 5. *The q-derivative of order $n \in \mathbb{N}$ of a function $f : [0,x] \to \mathbb{R}$ is defined by*
$$(D_q f)(t) = (D_q^1 f)(t) = \frac{(d_q f)(t)}{t-qt} = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \neq 0, \tag{15}$$
$$(D_q f)(0) = \lim_{t \to 0} (D_q f)(t),$$
and
$$(D_q^n f)(t) = (D_q^1 D_q^{n-1} f), \quad t \in \mathbb{J}, \ n \in \{1, 2, \ldots\},$$
where $\mathbb{J}$ is a given interval. Note that $(D_q^0) f(t) = f(t)$.

2.2.1 Properties of the q-Derivative

The following properties hold [26]:

1. For $n \in \mathbb{N}$ and $f(t) = t^n$, we have

$$D_q t^n = [n]_q t^{n-1}. \tag{16}$$

2. For all $n \in \mathbb{N}^*$, we have

$$D_q (x-a)_q^n = [n]_q (x-a)_q^{n-1}, \tag{17}$$

and

$$D_q (a-x)_q^n = -[n]_q (a-qx)_q^{n-1}. \tag{18}$$

Definition 6. *The q-integral (Jackson's Integral) of the function $f : [0,b] \to \mathbb{R}$ is defined as*

$$(I_q f)(t) = \int_0^t f(s) d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n),$$

provided that the series converges.

2.2.2 Properties of the q-Integral

The following properties hold:

1. For $n \in \mathbb{N}$, we have

$$I_q x^n = \frac{x^{n+1}}{[n+1]_q}. \tag{19}$$

2. For $n \in \mathbb{N}$, we have

$$(I_{q,a}^n D_q^n f)(x) = f(x) - \sum_{i=0}^{n-1} \frac{(D_q^i f)(a)}{[i]_q!}(x-a)^{(i)}. \tag{20}$$

Definition 7. *The fractional q-integral of order $\alpha \in \mathbb{R}^+$ in the Riemann-Liouville sense for a function $f : [0,b] \to \mathbb{R}$ is defined by*

$$(I_q^\alpha f)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) d_q s, \quad t \in [0,b]. \tag{21}$$

Note that when $\alpha = 1$ and $\alpha = 0$, we have

$$(I_q^1 f)(t) = (I_q f)(t) \quad \text{and} \quad (I_q^0 f)(t) = f(t),$$

respectively.

2.2.3 Properties of the Fractional q-Integral

The following properties hold:

1. Let $f, g : [0,b] \to \mathbb{R}$. For $\gamma, \lambda \in \mathbb{R}$ and for any $\alpha \in \mathbb{R}^+$, we have

$$I_q^\alpha (\gamma f(x) + \lambda g(x)) = \gamma (I_q^\alpha f)(x) + \lambda (I_q^\alpha g)(x). \tag{22}$$

2. Let $f : [0,b] \to \mathbb{R}$. For $\alpha, \beta \in \mathbb{R}^+$, we have

$$\begin{aligned}(I_q^\alpha I_q^\beta f)(x) &= (I_q^{\alpha+\beta} f)(x) \\ &= (I_q^\beta I_q^\alpha f)(x).\end{aligned} \tag{23}$$

Definition 8. *The fractional q-derivative of Riemann-Liouville of order $\alpha \in \mathbb{R}^+$ of a function $f : [0,b] \to \mathbb{R}$ is defined by*

$$({}^{RL}D_q^{\alpha} f)(t) = \left(D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f\right)(t), \quad t \in \mathbb{J},$$

where $[\alpha]$ is the integer part of α and $\mathbb{J}$ is a given interval. For $\alpha = 0$, we have

$$({}^{RL}D_q^{0} f)(t) = f(t).$$

Definition 9. *The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}^+$ of a function $f : [0,b] \to \mathbb{R}$ is defined by*

$$({}^{C}D_q^{\alpha} f)(t) = \left(I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f\right)(t), \quad t \in \mathbb{J},$$

where $[\alpha]$ is the integer part of α, and $\mathbb{J}$ is a given interval. Note that if $\alpha = 0$, we have

$$({}^{C}D_q^{0} f)(t) = f(t).$$

2.2.4 Properties of Fractional q-Derivatives

The following properties hold:

1. For $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ and $0 < a < x < b$, we have

$$({}^{C}D_q^{\alpha} D_q f)(x) = ({}^{C}D_q^{\alpha+1} f)(x). \tag{24}$$

$$(D_q^{C} D_q^{\alpha} f)(x) = ({}^{C}D_q^{\alpha+1} f)(x) + \frac{(D_q^{[\alpha]} f)(a)}{\Gamma_q([\alpha]-\alpha)}(x-a)^{([\alpha]-\alpha-1)}. \tag{25}$$

2. For $0 < \alpha < 1$, $n-1 < \beta < n$, $n-1 < \alpha+\beta \le n$, and $n \ge 1$, we have

$${}^{C}D_q^{\alpha}\, {}^{C}D_q^{\beta} f(t) = {}^{C}D_q^{\alpha+\beta} f(t). \tag{26}$$

Theorem 2. *(q-Taylor Theorem). Suppose that $(D_q^{\alpha} f)(t) \in C[a,b]$, $\forall k = 0,1,2,\ldots,n+1$, where $n \in \mathbb{N}$. For $\alpha \in (0,1]$, there exists $c \in (0,1)$ and $q^* \in (0,1)$ such that*

$$f(t) = \sum_{i=0}^{n} \frac{({}^{c}D_q^{\alpha})^i f(a)}{\Gamma_q(i\alpha+1)}(t-a)^{(i\alpha)} + \frac{({}^{c}D_q^{\alpha})^{n+1} f(c)}{\Gamma_q((n+1)\alpha+1)}(t-a)^{(n+1)\alpha}, \tag{27}$$

for all $q \in (q^*,1)$, where

$$({}^{c}D_q^{\alpha})^i = {}^{c}D_q^{\alpha}\, {}^{c}D_q^{\alpha} \ldots {}^{c}D_q^{\alpha} \quad (i \text{ times}).$$

3 Main Results

We will now review two lemmas that are useful for ensuring the accuracy and stability of the method. These lemmas ensure that the estimates produced by the method do not exceed the calculated limits, thereby enhancing confidence in the numerical results. The first lemma provides an upper bound for the function $(1+x)^m$, which frequently appears in numerical analysis, especially for small or negative values of x. This bound helps prove the convergence and error confinement of numerical calculations. The second lemma analyzes the behavior of the numerical sequence $\{a_i\}$, which follows a recurrence relationship. Such sequences frequently arise in solving differential equations using numerical methods such as Heun's method. By using this method, it can be shown that the recurring values in the method are bounded exponentially, thereby providing a guarantee of the stability of the method and the number of iterations.

Lemma 1. *For $x \ge -1$ and a positive integer m, we have*

$$0 \le (1+x)^m \le e^{mx}.$$

*Proof.*Suppose we want to expand $f(x)=e^x$ about $x=0$ using Taylor's theorem. Then we get

$$f(x)=f(0)+f'(0)(x-0)+f''(\zeta)\frac{(x-0)^2}{2}, \quad \text{for some } \zeta.$$

Since $f(x)=e^x$, we have

$$e^x=e^0+e^0x+e^\zeta\frac{x^2}{2}.$$

Simplifying, we obtain

$$e^x=1+x+e^\zeta\frac{x^2}{2}.$$

This implies

$$0\le(1+x)\le e^x \implies 0\le(1+x)^m\le e^{mx}.$$

Lemma 2.[28] *Suppose that t and s are two positive real numbers, and* $\{a_i\}_{i=0}^k$ *is a sequence satisfying*

$$a_i\ge\frac{-t}{s} \quad \textit{and} \quad a_{i+1}\le(1+s)a_i+t, \quad \forall i=0,1,2,\dots,k.$$

Then,

$$a_{i+1}\le e^{(i+1)s}\left(a_0+\frac{t}{s}\right)-\frac{t}{s}.$$

*Proof.*Suppose $a_{i+1}\le(1+s)a_i+t,\ \forall i=0,1,2,\dots,k$, then

$$a_{i+1}\le(1+s)a_i+t\le(1+s)^2a_{i-1}+(1+s)t+t,$$

$$a_{i+1}\le(1+s)^2a_{i-1}+(1+s)t+t,$$

or equivalently,

$$a_{i+1}\le(1+s)^2a_{i-1}+t\left[1+(1+s)\right]$$

Continuing this process, we get

$$a_{i+1}\le(1+s)^3a_{i-2}+(1+s)^2t+(1+s)t+t.$$

By repeating this pattern, we obtain

$$a_{i+1}\le(1+s)^{i+1}a_0+\left[1+(1+s)+(1+s)^2+\cdots+(1+s)^i\right]t. \tag{28}$$

Simplifying, this gives

$$a_{i+1}\le(1+s)^{i+1}a_0+\sum_{j=0}^{i}(1+s)^jt.$$

Since

$$\begin{aligned}\sum_{j=0}^{i}(1+s)^j&=\sum_{j=0}^{\infty}(1+s)^j-\sum_{j=i+1}^{\infty}(1+s)^j\\&=\frac{1}{1-(1+s)}-\sum_{k=0}^{\infty}(1+s)^{k+i+1}\\&=\frac{-1}{s}-(1+s)^{i+1}\sum_{k=0}^{\infty}(1+s)^k\\&=\frac{-1}{s}-(1+s)^{i+1}\frac{1}{1-(1+s)}\\\sum_{j=0}^{i}(1+s)^j&=\frac{(1+s)^{i+1}-1}{s},\end{aligned}$$

we can write (28) as

$$a_{i+1}\le(1+s)^{i+1}a_0+\left(\frac{(1+s)^{i+1}-1}{s}\right)t.$$

This simplifies to

$$a_{i+1} \leq (1+s)^{i+1}\left[a_0 + \frac{t}{s}\right] - \frac{t}{s}. \tag{29}$$

Using Lemma1, we Know that $(1+s)^{i+1} \leq e^{(i+1)s}$, so inequality (29) becomes

$$a_{i+1} \leq e^{(i+1)s}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}$$

3.1 Fractional Heun's Method

The Fractional Heun's Method (FHM) is a variant of the classical Heun's method, which is essentially a form of the Runge-Kutta method used to solve ordinary differential equations (ODEs), with a fractional step modification. It is designed to handle problems involving derivatives and fractional systems [22]. The method is based on two steps: the Fractional Predictor and the Fractional Corrector, which are adapted to the framework of fractional calculus.
Consider we have the following FIVP:

$$D^{\alpha}z(t) = f(t,z(t)), \quad 0 < \alpha \leq 1, \tag{30}$$

with the initial condition

$$z(0) = z_0. \tag{31}$$

With the use of fractional Taylor method, we can have

$$z_{i+1} = z_i + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f(t_i, z_i).$$

Herein, we can assume that z_{i+1} is an a predictor, i.e. $z_{i+1} = z_i^*$. This gives

$$z^* = z_i + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f(t_i, z_i). \tag{32}$$

Actually, z^* can be refined more by considering the corrector step that allows us to re-evaluate the function once again at the new point (t_{i+1}, z^*). This would give

$$z_{i+1} = z_i + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f(t_i, z^*), \tag{33}$$

or

$$z_{i+1} = z_i + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f\left(t_i,\ z_i + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f(t_i, z_i)\right), \tag{34}$$

which represents the so-called **Fractional Heun's method** (FHM).

Theorem 3.[28] *Suppose f is a continuous function satisfying the Lipschitz condition with constant L. Assume that the 2α-derivative of $z(t)$ is bounded by M. Then, the error bound of the fractional Heun's method is given by*

$$|z(t_i) - z_i| \leq \frac{\eta}{\psi}\ \left[e^{(i+1)\psi} - 1\right], \tag{35}$$

where

$$\psi = \frac{h^{\alpha}L}{\Gamma(\alpha+1)} + \frac{h^{2\alpha}L^2}{(\Gamma(\alpha+1))^2}, \quad \eta = \frac{h^{2\alpha}M}{\Gamma(2\alpha+1)} \ \textit{for}\ i = 0,1,2,\dots,n.$$

*Proof.*To prove this result, we consider again the FIVP

$$D^{\alpha}z(t) = f(t,z(t)), \quad 0 < \alpha \leq 1, \tag{36}$$

with the initial condition

$$z(0) = z_0. \tag{37}$$

Since f satisfies the Lipschitz condition, we have

$$|f(t,z_1) - f(t,z_2)| \leq L|z_1 - z_2|,$$

for some constant $L > 0$. Now, we assume that

$$|D^{2\alpha}z(t)| \leq M, \quad \forall t \in [a,b].$$

With the use of the fractional Taylor method (11), we can write the exact solution and the approximate solution of the FIVP (36)-(37), respectively, as follows:

$$z(t_{i+1}) = z(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f\left(t_i,\ z(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_i, z(t_i))\right) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} D^{2\alpha} z(\zeta_i), \tag{38}$$

and

$$z_{i+1} = z_i + \frac{h^\alpha}{\Gamma(\alpha+1)} f\left(t_i,\ z_i + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_i, z_i)\right), \tag{39}$$

Now, we subtract (39) from (38) to obtain

$$\begin{aligned} z(t_{i+1}) - z_{i+1} = z(t_i) - z_i + \frac{h^\alpha}{\Gamma(\alpha+1)} \Bigg(& f\left(t_i, z(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_i, z(t_i))\right) \\ & - f\left(t_i, z_i + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_i, z_i)\right)\Bigg) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} D^{2\alpha} z(\zeta_i) \end{aligned} \tag{40}$$

With the use of triangle inequality, we obtain

$$\begin{aligned} |z(t_{i+1}) - z_{i+1}| \leq |z(t_i) - z_i| + \frac{h^\alpha}{\Gamma(\alpha+1)} \Bigg| & f\left(t_i,\ z(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_i, z(t_i))\right) \\ & - f\left(t_i,\ z_i + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_i, z_i)\right)\Bigg| + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} |D^{2\alpha} z(\zeta_i)|. \end{aligned} \tag{41}$$

Now, by using Lipschitz condition, we get

$$\begin{aligned} |z(t_{i+1}) - z_{i+1}| \leq |z(t_i) - z_i| + \frac{h^\alpha L}{\Gamma(\alpha+1)} \Bigg| & \left(z(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_i, z(t_i))\right) \\ & - \left(z_i + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_i, z_i)\right)\Bigg| + \frac{h^{2\alpha} M}{\Gamma(2\alpha+1)}. \end{aligned} \tag{42}$$

This implies

$$|z(t_{i+1}) - z_{i+1}| \leq |z(t_i) - z_i| + \frac{h^\alpha L}{\Gamma(\alpha+1)} \left(|z(t_i) - z_i| + \frac{h^\alpha}{\Gamma(\alpha+1)} (f(t_i, z(t_i) - f(t_i, z_i))\right) + \frac{h^{2\alpha} M}{\Gamma(2\alpha+1)}.$$

By using Lipschitz condition once again, we get

$$|z(t_{i+1}) - z_{i+1}| \leq |z(t_i) - z_i| \left(1 + \frac{h^\alpha L}{\Gamma(\alpha+1)}\right) + \frac{h^{2\alpha} L^2}{(\Gamma(\alpha+1))^2} |z(t_i) - z_i| + \frac{h^{2\alpha} M}{\Gamma(2\alpha+1)},$$

or

$$|z(t_{i+1}) - z_{i+1}| \leq \left(1 + \frac{h^\alpha L}{\Gamma(\alpha+1)} + \frac{h^{2\alpha} L^2}{(\Gamma(\alpha+1))^2}\right) |z(t_i) - z_i| + \frac{h^{2\alpha} M}{\Gamma(2\alpha+1)}, \tag{43}$$

Now, assume that

$$\psi = \frac{h^\alpha L}{\Gamma(\alpha+1)} + \frac{h^{2\alpha} L^2}{(\Gamma(\alpha+1))^2}, \text{ and } \eta = \frac{h^{2\alpha} M}{\Gamma(2\alpha+1)},$$

$$a_i = |z(t_i) - z_i|, \text{ and } a_{i+1} = |z(t_{i+1}) - z_{i+1}|.$$

Then (43) becomes

$$a_{i+1} \leq (1+\psi) a_i + \eta, \text{ for } i = 0, 1, 2, \ldots, n.$$

This means that the condition of Lemma (2) is satisfied, and so the result of this Lemma holdes, i.e.,

$$a_{i+1} \leq e^{(i+1)\psi}(a_0 + \frac{\eta}{\psi}) - \frac{\eta}{\psi}.$$

This means that

$$|z(t_{i+1}) - z_{i+1}| \leq e^{(i+1)\psi}\left(|z(t_i) - z_i| + \frac{\eta}{\psi}\right) - \frac{\eta}{\psi},$$

or

$$|z(t_{i+1}) - z_{i+1}| \leq \frac{\eta}{\psi}\left(e^{(i+1)\psi} - 1\right).$$

3.2 Fractional q-Heun's Method

The fractional q-Heun's method is an extension of the fractional Heun's method. It is a numerical method for solving fractional q-differential equations (FqDEs), combining the predictor-corrector approach of Heun's method with the concepts of q-calculus and the q-Taylor series expansion (27). The method preserves stability for a range of step sizes. Consider we have the following FqIVP:

$$D_q^{\alpha} z(t) = f(t, z(t)), \quad 0 < \alpha \leq 1, \tag{44}$$

with the initial condition

$$z(0) = z_0. \tag{45}$$

With the use of fractional q-Taylor method (27), we can have

$$z_{i+1} = z_i + \frac{h^{\alpha}}{\Gamma_q(\alpha+1)} f(t_i, z_i)$$

Herein, we can assume that z_i is an a predictor, i.e. $z_i = z_q^*$. This gives

$$z_q^* = z_i + \frac{h^{\alpha}}{\Gamma_q(\alpha+1)} f(t_i, z_q).$$

As we said before, z_q^* can be refined more by considering the corrector step that allows us to re-evaluate the function once again at the new point (t_{i+1}, z_q^*). This would give

$$z_{i+1} = z_i + \frac{h^{\alpha}}{\Gamma_q(\alpha+1)} f\left(t_i, z_i + \frac{h^{\alpha}}{\Gamma_q(\alpha+1)} f(t_i, z_i)\right), \tag{46}$$

which represents the so-called **Fractional q-Heun's method** (FqHM).

Theorem 4.*Suppose f is a continuous function satisfying a q-Lipschitz condition with constant L_q. Assume that the 2α-derivative of the exact solution $z_q(t)$ is bounded by M_q. Then, the error bound of the fractional q-Heun's method is given by*

$$|z(t_i) - z_i| \leq \frac{\eta}{\psi} \left[e^{(i+1)\psi} - 1\right], \tag{47}$$

where

$$\psi = \frac{h^{\alpha} L}{\Gamma_q(\alpha+1)} + \frac{h^{2\alpha} L^2}{(\Gamma_q(\alpha+1))^2}, \quad \eta = \frac{h^{2\alpha} M}{\Gamma_q(2\alpha+1)} \text{ and } i = 0, 1, 2, \ldots, n.$$

*Proof.*With the use of the q-fractional Taylor method (27), we can write the exact solution and the approximate solution of the FqIVP (44)-(45), respectively, as follows:

$$z(t_{i+1}) = z(t_i) + \frac{h^{\alpha}}{\Gamma_q(\alpha+1)} f\left(t_i, \ z(t_i) + \frac{h^{\alpha}}{\Gamma_q(\alpha+1)} f(t_i, z(t_i))\right) + \frac{h^{2\alpha}}{\Gamma_q(2\alpha+1)} D^{2\alpha} z(\zeta_i), \tag{48}$$

and

$$z_{i+1} = z_i + \frac{h^{\alpha}}{\Gamma_q(\alpha+1)} f\left(t_i, \ z_i + \frac{h^{\alpha}}{\Gamma_q(\alpha+1)} f(t_i, z_i)\right), \tag{49}$$

Now, we subtract (49) from (48) to obtain

$$z(t_{i+1}) - z_{i+1} = z(t_i) - z_i + \frac{h^\alpha}{\Gamma_q(\alpha+1)} \left(f\left(t_i, z(t_i) + \frac{h^\alpha}{\Gamma_q(\alpha+1)} f(t_i, z(t_i))\right) - f\left(t_i, z_i + \frac{h^\alpha}{\Gamma_q(\alpha+1)} f(t_i, z_i)\right)\right) + \frac{h^{2\alpha}}{\Gamma_q(2\alpha+1)} D^{2\alpha} z(\zeta_i) \tag{50}$$

Appling triangle inequality, we obtain

$$|z(t_{i+1}) - z_{i+1}| \leq |z(t_i) - z_i| + \frac{h^\alpha}{\Gamma_q(\alpha+1)} \left| f\left(t_i,\ z(t_i) + \frac{h^\alpha}{\Gamma_q(\alpha+1)} f(t_i, z(t_i))\right) - f\left(t_i,\ z_i + \frac{h^\alpha}{\Gamma_q(\alpha+1)} f(t_i, z_i)\right)\right| + \frac{h^{2\alpha}}{\Gamma_q(2\alpha+1)} |D^{2\alpha} z(\zeta_i)|. \tag{51}$$

Now, by using Lipschitz condition, we get

$$|z(t_{i+1}) - z_{i+1}| \leq |z(t_i) - z_i| + \frac{h^\alpha L}{\Gamma_q(\alpha+1)} \left| \left(z(t_i) + \frac{h^\alpha}{\Gamma_q(\alpha+1)} f(t_i, z(t_i))\right) - \left(z_i + \frac{h^\alpha}{\Gamma_q(\alpha+1)} f(t_i, z_i)\right)\right| + \frac{h^{2\alpha} M}{\Gamma_q(2\alpha+1)}. \tag{52}$$

This implies

$$|z(t_{i+1}) - z_{i+1}| \leq |z(t_i) - z_i| + \frac{h^\alpha L}{\Gamma_q(\alpha+1)} \left(|z(t_i) - z_i| + \frac{h^\alpha}{\Gamma_q(\alpha+1)} (f(t_i, z(t_i) - f(t_i, z_i)) \right) + \frac{h^{2\alpha} M}{\Gamma_q(2\alpha+1)}.$$

By using Lipschitz condition once again, we get

$$|z(t_{i+1}) - z_{i+1}| \leq |z(t_i) - z_i| \left(1 + \frac{h^\alpha L}{\Gamma_q(\alpha+1)}\right) + \frac{h^{2\alpha} L^2}{(\Gamma_q(\alpha+1))^2} |z(t_i) - z_i| + \frac{h^{2\alpha} M}{\Gamma_q(2\alpha+1)},$$

or

$$|z(t_{i+1}) - z_{i+1}| \leq \left(1 + \frac{h^\alpha L}{\Gamma_q(\alpha+1)} + \frac{h^{2\alpha} L^2}{(\Gamma_q(\alpha+1))^2}\right) |z(t_i) - z_i| + \frac{h^{2\alpha} M}{\Gamma_q(2\alpha+1)}, \tag{53}$$

Now, assume that

$$\psi = \frac{h^\alpha L}{\Gamma_q(\alpha+1)} + \frac{h^{2\alpha} L^2}{(\Gamma_q(\alpha+1))^2}, \quad \text{and} \quad \eta = \frac{h^{2\alpha} M}{\Gamma_q(2\alpha+1)},$$

$$a_i = |z(t_i) - z_i|, \quad \text{and} \quad a_{i+1} = |z(t_{i+1}) - z_{i+1}|.$$

Then (53) becomes

$$a_{i+1} \leq (1+\psi) a_i + \eta, \quad \text{for} \quad i = 0, 1, 2, \ldots, n.$$

This means that the condition of Lemma (2) is satisfied, and so the result of this Lemma holdes, i.e.,

$$a_{i+1} \leq e^{(i+1)\psi} (a_0 + \frac{\eta}{\psi}) - \frac{\eta}{\psi}.$$

This means that

$$|z(t_{i+1}) - z_{i+1}| \leq e^{(i+1)\psi} \left(|z(t_i) - z_i| + \frac{\eta}{\psi}\right) - \frac{\eta}{\psi},$$

or

$$|z(t_{i+1}) - z_{i+1}| \leq \frac{\eta}{\psi} \left(e^{(i+1)\psi} - 1\right).$$

4 Numerical Simulation

To study the comparison between the two methods, FHM and FqHM, let us consider the subsequent two examples.

*Example 1.*Consider the following FIVP:

$$D^{\alpha}u(t) = -u(t)^2 + u(t), \tag{54}$$

with the initial condition

$$u(0) = \frac{1}{2}, \quad 0 < \alpha \leq 1. \tag{55}$$

The exact solution is given by

$$u(t) = \frac{1}{1 + e^{\frac{-t^{\alpha}}{\Gamma(1+\alpha)}}}.$$

Let

$$D^{\alpha}u(t) = f(t, u(t)),$$

where

$$f(t, u(t)) = -u(t)^2 + u(t).$$

Now, applying the Fractional Heun's Method (FHM) yields

$$z^* = z_i + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f_i(t_i, z_i),$$

where

$$f(t_i, z_i) = -z_i^2 + z_i.$$

Then, compute

$$f(t_{i+1}, z^*) = -(z^*)^2 + z^*.$$

This allows us to compute z_{i+1} as

$$z_{i+1} = z_i + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f\left(t_i, \;\; z_i + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f(t_i, z_i)\right). \tag{56}$$

We will perform these steps for $i = 0$ to $i = 0.9$.. Now, consider the FqIVP

$$D_q^{\alpha}u(t) = -u(t)^2 + u(t), \tag{57}$$

with the initial condition

$$u(0) = \frac{1}{2}, \quad 0 < \alpha \leq 1. \tag{58}$$

Let

$$f(t, u) = -u^2 + u, \quad \text{for } \alpha = 0.5, \; h = 0.1, \text{ and } z_0 = \frac{1}{2}.$$

Consequently, applying the Fractional q-Heun's Method (FqHM) yields

$$z^* = z_i + \frac{h^{\alpha}}{\Gamma_q(\alpha+1)} f(t_i, z_i),$$

where

$$f(t_i, z_i) = -z_i^2 + z_i.$$

Then, compute

$$f(t_i, z^*) = -(z^*)^2 + z^*,$$

where

$$z^* = z_i + \frac{h^{\alpha}}{\Gamma_q(\alpha+1)} f(t_i, z_i).$$

Now, compute

$$f(t_i, z^*) = -(z^*)^2 + z^*,$$

and find z_{i+1} as

$$z_{i+1} = z_i + \frac{h^{\alpha}}{\Gamma_q(\alpha+1)} f\left(t_i, \;\; z_i + \frac{h^{\alpha}}{\Gamma_q(\alpha+1)} f(t_i, z_i)\right). \tag{59}$$

For example if one computes z_1 for $\alpha = 0.5$, $q = 0.9$, and $h = 0.1$, the following steps should be performed:

–Compute $f(t_0, z_0)$, where $z_0 = \frac{1}{2}$:

$$f(t_0, z_0) = -(0.5)^2 + 0.5 = 0.25.$$

–Compute the predictor:

$$z^* = 0.5 + \frac{0.1^{0.5}}{\Gamma_q(1.5)}(0.25) \approx 0.5223.$$

–Compute z_1:

$$z_1 = 0.5 + \frac{0.1^{0.5}}{\Gamma_q(1.5)}\left(-(0.5223)^2 + 0.5223\right).$$

Simplifying, we get

$$z_1 \approx 0.5890.$$

To verify our theoretical work, we will display numerical results. In particular, Figure 1 illustrates a comparison of FHM, FqHM, and the exact solution for $\alpha = 1$ and $q = 0.99$. Both numerical methods, FHM and FqHM, provide an accurate approximation to the exact solution. However, the Fractional Heun's Method (FHM) may be more accurate than the Fractional q-Heun's Method (FqHM), depending on how close it is to the exact solution. Figure 2 presents an absolute error comparison between FHM and FqHM. The FHM demonstrates a very high level of accuracy, with the resulting error being extremely small compared to FqHM. Consequently, the error for FHM is so minimal that it does not appear clearly in the plot. On the other hand, FqHM shows a gradual increase in error over time. Therefore, in this context, FHM performs better than FqHM in minimizing the absolute error.

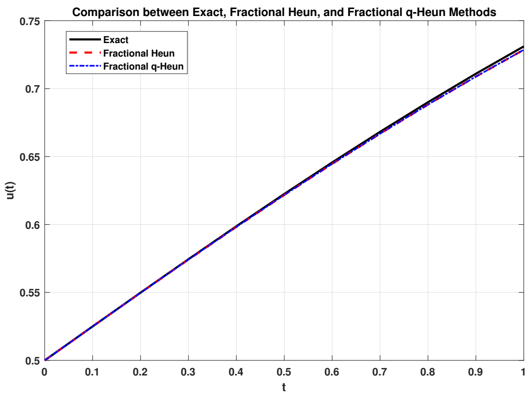

Fig. 1: Comparison of numerical results between the exact solution, FHM, and FqHM for $\alpha = 1$ and $q = 0.99$.

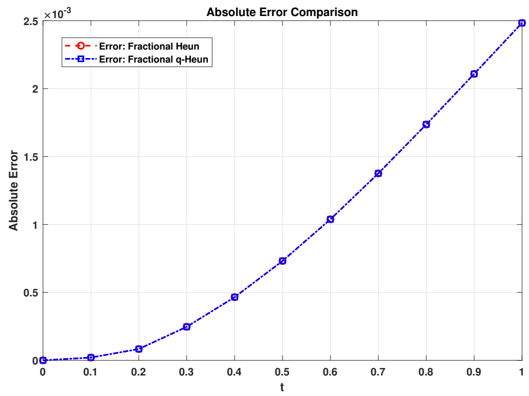

Fig. 2: Absolute error comparison between FHM and FqHM.

In Figure 3, we observe that FqHM provides more accurate solutions at higher values of q, especially when $q = 1$. The difference between the exact solution and the numerical solutions is more noticeable at smaller values of t (near the beginning) and gradually reduces as t approaches 1.

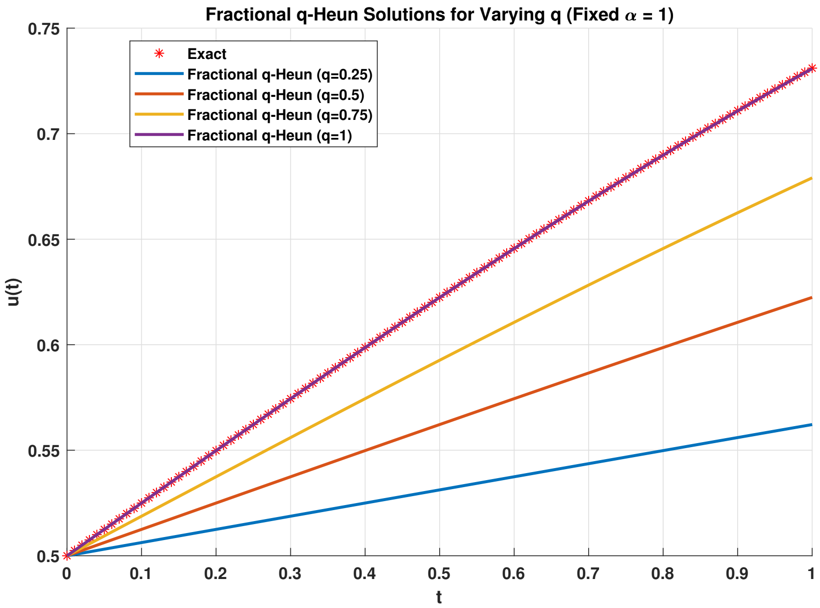

Fig. 3: Comparison of FqHM and the exact solution for different values of q with fixed $\alpha = 1$.

In Figure 4, we observe that the performance of FqHM depends on α. The solutions decrease monotonically as the independent variable increases. The q-Heun's method converges more slowly to the exact solution at smaller values of α. Solutions with α values close to 1 are consistent with the exact solution, while smaller α values produce significant deviations.

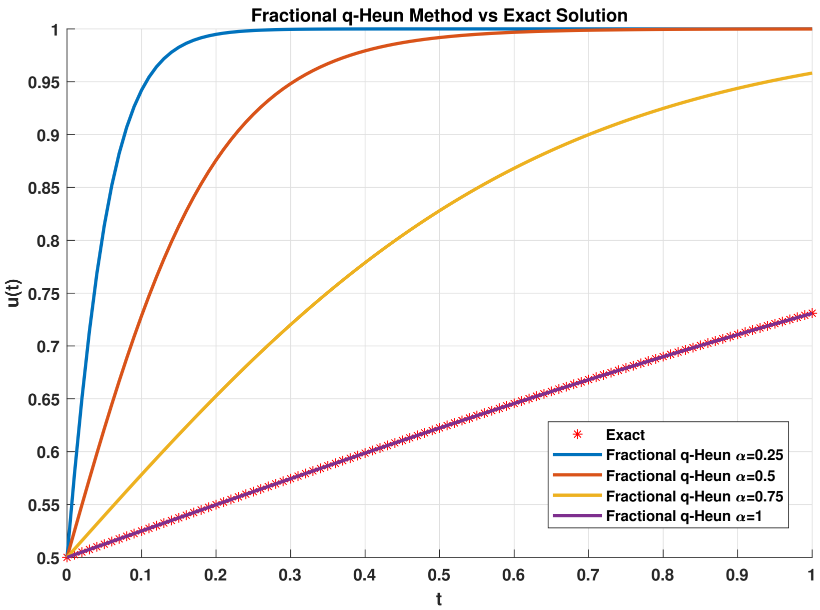

Fig. 4: Comparison of FqHM and the exact solution for fixed $q = 1$ and varying α.

Table 1: Total Comparison of Exact Solution, Heun, and q-Heun Methods

t	Exact Solution	Heun Solution	Heun Error	q-Heun Solution	q-Heun Error
0	0.5	0.5	0	0.5	0
0.1	0.52498	0.525	2.0813E-05	0.525	2.0813E-05
0.2	0.54983	0.54975	8.34E-05	0.54975	8.3373E-05
0.3	0.57444	0.5742	0.00024696	0.5742	0.00024696
0.4	0.59869	0.59822	0.00046519	0.59822	0.00046519
0.5	0.62246	0.62173	0.00073156	0.62173	0.00073156
0.6	0.64566	0.64462	0.0010382	0.64462	0.0010382
0.7	0.66819	0.66681	0.0013762	0.66681	0.0013762
0.8	0.68997	0.68824	0.0017361	0.68824	0.0017361
0.9	0.71095	0.70884	0.0021083	0.70884	0.0021083
1.0	0.73106	0.72858	0.0024835	0.72858	0.0024835

In Figure 5, when $\alpha = 1.00$, the approximate solution is nearly identical to the exact solution, indicating the accuracy of the method at this value. Conversely, for lower values of α (such as 0.25, 0.50, and 0.75), the deviation between the approximate solution and the exact solution becomes more noticeable, highlighting the effect of the α value. The lower the α value, the less accurate the method becomes.

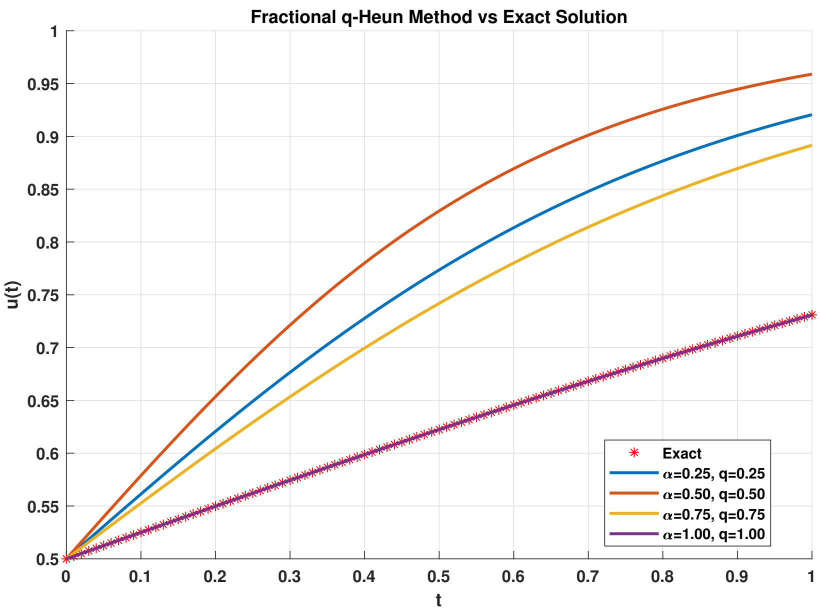

Fig. 5: Comparison of FqHM and the exact solution for varying values of α and q.

In Table 1, we observe that the solution values for the exact solution, FHM, and FqHM range from 0.5 to 0.73106, while the error ranges from 0 to 0.0024835. Both FHM and FqHM yield exactly the same values and errors for each value of t, suggesting that the q adjustment in this case made no difference in terms of accuracy or performance. The errors increase almost linearly as t increases from 0 to 1, with the error approximately doubling every 0.1 unit of time. Despite this, the overall accuracy remains very high, with errors remaining below 0.25% at all points. The highest error occurs at $t = 1$. If we take $h = 0.01$, we can see in Figure 6 that the two methods produce similar results. The differences between them are not large but are noticeable. Both methods approach the exact solution, although to varying degrees. In Figure 7, the errors range from 0 to 1.5, indicating significant variation in solution accuracy. Peak errors occur in the mid-range values ($0.4 \leq t \leq 0.6$). Both methods approach zero error at parameter extremes, close to 0.1 and 0.9.

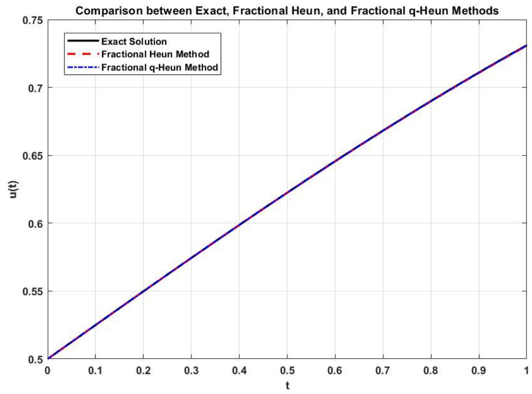

Fig. 6: Comparison of numerical results between the exact solution, FHM, and FqHM for smaller step size ($h = 0.01$).

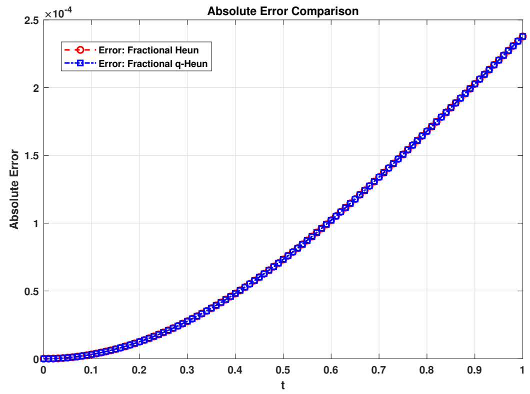

Fig. 7: Absolute error comparison between FHM and FqHM for smaller step size ($h = 0.01$).

*Example 2.*Consider the following FIVP:

$$D^{\alpha}u(t) = u(t)^2, \tag{60}$$

with the initial condition

$$u(0) = 1, \quad 0 < \alpha \leq 1. \tag{61}$$

The exact solution is given by

$$u(t) = \frac{1}{(1-t^{\alpha})\Gamma(1+\alpha)}.$$

Let

$$D^{\alpha}u(t) = f(t, u(t)),$$

where

$$f(t, u(t)) = u(t)^2.$$

Now, applying the Fractional Heun's Method (FHM) gives

$$z^* = z_i + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f_i(t_i, z_i),$$

where

$$f(t_i, z_i) = z_i^2.$$

Then, compute

$$f(t_i, z^*) = (z^*)^2,$$

which allows us to compute z_{i+1} as

$$z_{i+1} = z_i + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f(t_i, z_i^*), \quad \text{for } h = 0.1.$$

We will perform these steps for $i = 0$ to $i = 0.9$.. Now, consider the FqIVP

$$D_q^{\alpha}u(t) = u(t)^2, \tag{62}$$

with the initial condition

$$u(0) = 1, \quad 0 < \alpha \leq 1. \tag{63}$$

Let

$$f(t, u) = u^2, \quad \text{for } h = 0.1, \quad z_0 = 1.$$

Consequently, applying the Fractional q-Heun's Method (FqHM) gives

$$z^* = z_i + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f_i(t_i, z_i),$$

where

$$f(t_i, z_i) = z_i^2.$$

Then, compute

$$f(t_i, z^*) = (z^*)^2,$$

where

$$z^* = z_i + \frac{h^\alpha}{\Gamma_q(\alpha+1)} f(t_i, z_i).$$

Now, compute

$$f_q(t_i, z^*) = (z^*)^2,$$

and find z_{i+1} as

$$z_{i+1} = z_i + \frac{h^\alpha}{\Gamma_q(\alpha+1)} f\left(t_i,\ z_i + \frac{h^\alpha}{\Gamma_q(\alpha+1)} f(t_i, z_i)\right).$$

For example, if one computes z_1 for $\alpha = 0.5$, $q = 0.9$, and $h = 0.1$, the following steps should be then performed:

– Compute $f(t_0, z_0)$, where $z_0 = 1$:

$$f(t_0, z_0) = (1)^2 = 1.$$

– Compute the predictor:

$$z^* = 1 + \frac{0.1^{0.5}}{\Gamma_q(1.5)} (1)^2 \approx 1.356826.$$

– Compute z_1:

$$z_1 = 1 + \frac{0.1^{0.5}}{\Gamma_q(1.5)} (1.356826)^2.$$

Simplifying yields

$$z_1 \approx 1.657.$$

In Figure 8, we observe that both numerical methods, FHM and FqHM, produce results that are very close to the exact solution when $\alpha = 1$. Meanwhile, Figure 9 suggests that FqHM appears to outperform FHM in some areas, while their performance is comparable in others. Overall, there is no significant difference in performance between the two methods in most regions. The error for both methods remains low at the beginning and stays stable for a period at point F, which is approximately equal to the speed of convergence. However, the first method (FHM) appears to be more stable.

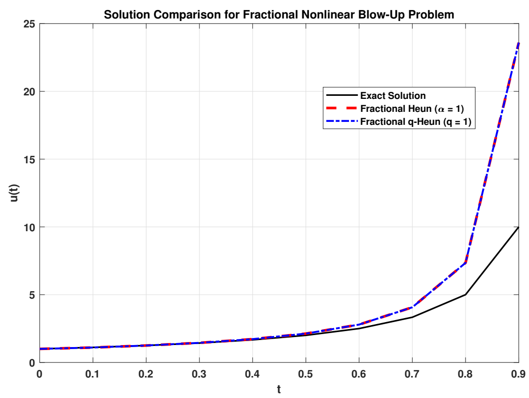

Fig. 8: Comparison of numerical results between the exact solution, FHM, and FqHM for the fractional nonlinear blow-up problem.

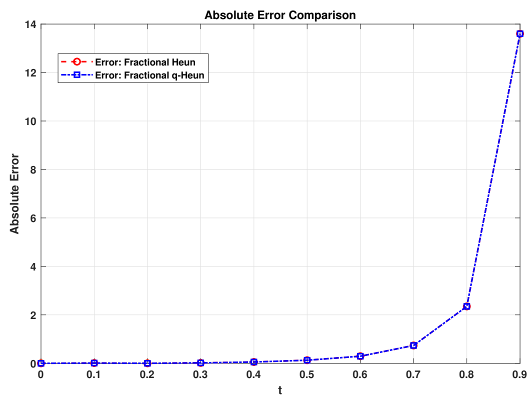

Fig. 9: Absolute error comparison between FHM and FqHM for the fractional nonlinear blow-up problem.

Figure 10 shows that both numerical solutions are identical to the exact solution when $q = 1$, and the growth is rapid. As the value of q decreases, the growth slows down.

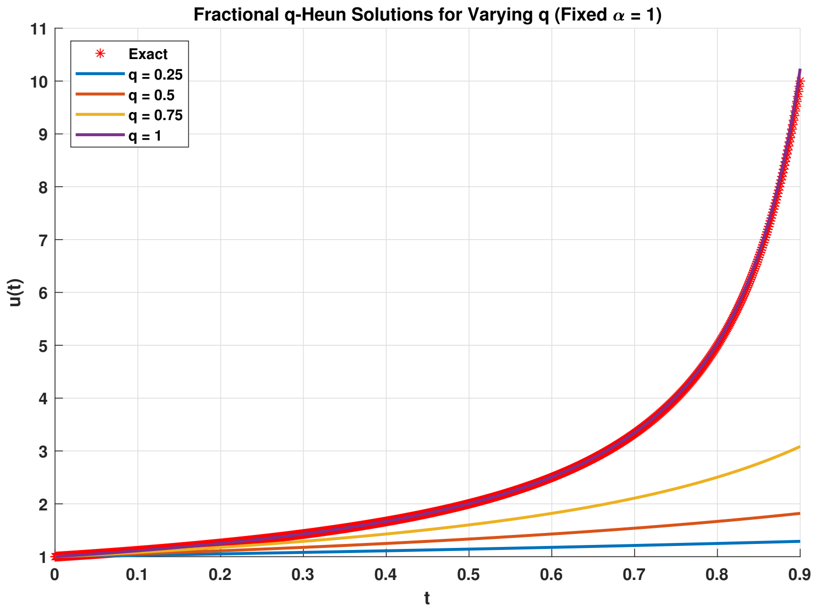

Fig. 10: Comparison of FqHM and the exact solution for varying q with fixed $\alpha = 1$.

In Figure 11, the numerical solution matches the exact solution closely. However, as α decreases (0.75, 0.5, and 0.25), the numerical solution deviates more from the exact solution. The largest deviation occurs at $\alpha = 0.50$ and $\alpha = 0.25$, with significant deviations increasing over time.

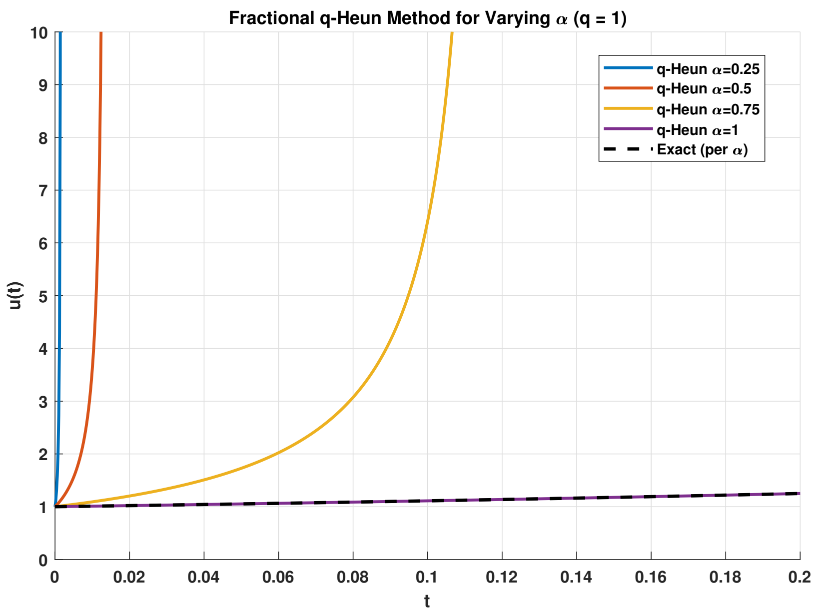

Fig. 11: Comparison of FqHM and the exact solution for varying α with fixed $q = 1$.

Table 2: Total Comparison of Exact Solution, Heun, and q-Heun Methods

t	Exact Solution	Heun Solution	Heun Error	q-Heun Solution	q-Heun Error
0	1.0000	1.0000	0.0000	1.0000	0.0000
0.1	1.1111	1.1000	0.0111	1.1000	0.0111
0.2	1.2500	1.2491	0.0009	1.2491	0.0009
0.3	1.4286	1.4465	0.0179	1.4465	0.0179
0.4	1.6667	1.7207	0.0540	1.7207	0.0540
0.5	2.0000	2.1274	0.1274	2.1274	0.1274
0.6	2.5000	2.7930	0.2930	2.7930	0.2930
0.7	3.3333	4.0697	0.7364	4.0697	0.7364
0.8	5.0000	7.3485	2.3485	7.3485	2.3485
0.9	10.0000	23.6010	13.6010	23.6010	13.6010

In Figure 12, we observe that the closer α and q are to the value 1, the closer the numerical solution becomes to the exact solution. Solutions at smaller values, such as 0.25, differ significantly from the exact solution. This indicates that to improve the accuracy of the solution, larger values of these parameters should be chosen.

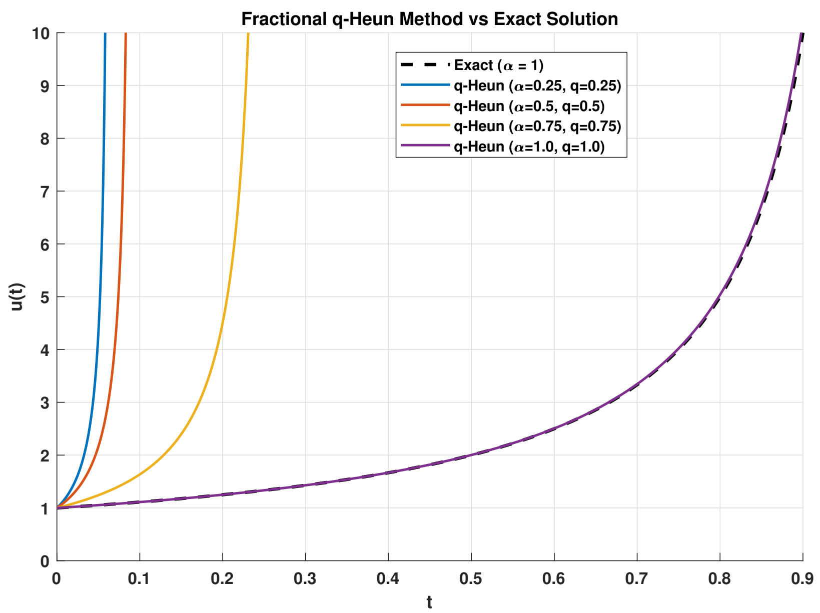

Fig. 12: Comparison of FqHM and the exact solution for varying α and q.

In Table 2, we observe that at larger values of t (such as 0.8 and 0.9), the differences between the values (both first and second differences) increase significantly. Conversely, at smaller values of t (0, 0.1, and 0.2), the differences between the values are very small, indicating a high degree of agreement between the methods. In Figure 13, we see that at $\alpha = 1$, the two methods are expected to reduce to the classical case. "Blow-up" problems typically refer to solutions that tend to infinity in finite time. On the other hand, Figure 14 shows that the error reaches its maximum values, which differ between the two methods.

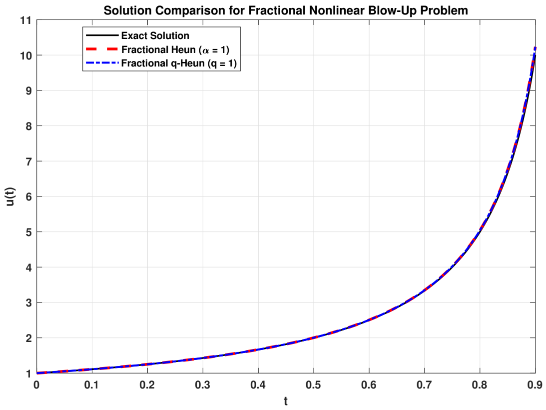

Fig. 13: Comparison of numerical results between the exact solution, FHM, and FqHM for $\alpha = 1$ and $q = 0.99$ with a smaller step size ($h = 0.01$).

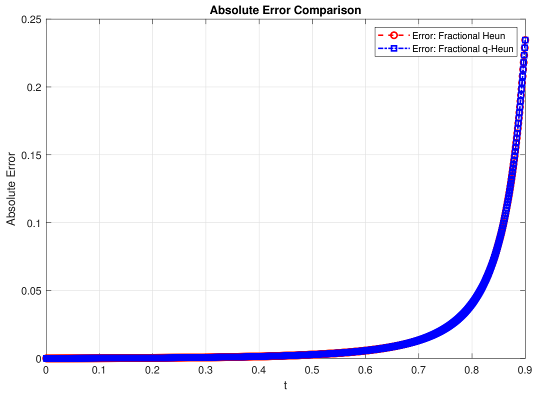

Fig. 14: Absolute error comparison between FHM and FqHM for $\alpha = 1$ and $q = 0.99$ with a smaller step size ($h = 0.01$).

5 Conclusion

In conclusion, there are significant differences and deviations in error between the two methods, with the error generally being larger in the FqHM method. This suggests that the algorithms used should be reviewed to improve the convergence of the results. Increasing differences with large inputs may indicate an approximation error, which can be mitigated by using very small values of h or modifying the method to enhance stability. The increasing differences in FqHM also suggest potential numerical instability or approximation errors, which require further analysis. To improve the performance of FqHM, more efficient numerical stabilization techniques should be explored or modifications to the method should be considered.

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