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Refinements for Hermite-Hadamard Type Inequalities for Operator $h-$Convex Function

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Abstract: In the present paper we introduce the notion of *operator h-convex function* . Also, we obtain new Jensen and Hermite-Hadamard inequalities for these *operator h-convex functions* in Hilbert spaces.

Keywords: Self-adjoint operators, operator convex functions, operator *h*- convex functions, Jensen inequalities type, Hermite-Hadamard inequalities

1 Introduction

In recent years several extensions and generalizations have been considered for classical convexity, and the theory of inequalities has made essential contributions to many areas of Mathematics. In this paper we shall deal with an important and useful class of functions called *operator convex functions*. We introduce a new class of generalized convex functions, namely the class of *operator h-convex function*. The theory of operator/matrix monotone functions was initiated by the celebrated paper of C. Löwner [43], which was soon followed by F. Kraus [40] on operator/matrix convex functions. After further developments due to some authors (for instance, J. Bendat and S. Sherman [15], A. Korányi [39], and U. Franz [26]), in their seminal paper [32] F.Hansen and G.K. Pedersen established a modern treatment of operator monotone and convex functions. In [2,11,19,34] are found comprehensive expositions on the subject matter.

Inequalities are one of the most important instrument in many branches of Mathematics such as Functional Analysis, Theory of Differential and Integral Equations, Probability Theory, etc. They are also useful in mechanics, physics and other sciences. A systematic study of inequalities was started in the classical book [33] and continued in [8]. Nowadays the theory of inequalities is still being intensively developed. This fact is confirmed by a great number of recent published books [7,56]and a huge number of articles on inequalities [3,4,5,14,16,17,24,27,42,51,52,54]. Thus, the theory of inequalities may be regarded as an independent area of mathematics.

The convexity of functions plays a significant role in many fields, for example, in biological system, economy, optimization and so on [29,49]. And many important inequalities are established for the class of convex functions. The Hermite-Hadamard inequality (1) have been the subject of intensive research, and many applications, generalizations and improvements of them can be found in the literature (see, for instance [10,23,41,47,48] and the references therein).

From the results founded by Hadamard in [30], the Hermite-Hadamard (double) inequality for convex functions on an interval of the real line is usually stated as follows. This classical inequality provides estimates of the mean value of a continuous function $f:[a,b]\to\mathbb{R}$.

Theorem 1. *Hermite-Hadamard's Inequality [45]. Let f be a convex function on $[a,b]$, with $a<b$.If f is*

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integrable on $[a,b]$ *, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \quad (1)$$

The interested reader can find the history of the Hermite-Hadamard inequality in the historical note by D.S.Mitrinovic and I.B. Lackovic [45] and [44]. Both has been studied widely and in recent years they have found generalizations thereof using generalized convex functions. In particular, for operator functions of positive self-adjoint operators in a Hilbert space H.

In recent years many authors have been interested in giving some refinements and extensions of the Hermite-Hadamard inequality (1). For more about convex functions and the Hermite-Hadamard inequality, see . Zabandan in [55] presents the Hermite-Hadamard type inequality for convex functions by sequences. In this paper, a new refinement of the Hermite-Hadamard type inequality is presented for operator $h-$convex function. Bakac and Türkmen, in [5], gave a general form of the first of certain inequalities showed by Bacak in [6] and showed that the inequalities therein are satisfied for operator convex functions.

Inspired and motivate by the work of Dragomir [22], Ghazanfari in [27], Erdas et al. [24], Horváth et al. [36], T. Ando in [1], L. Horvath [36], I. Kim [38], S. Salas [50], in this paper, we define a novel class of convex functions called *operator h-convex function*. We establish some refinements of generalized Hermite-Hadamard inequalities for operator *h-convex functions*. This paper is organized as follows: In Section 2 we provide some notations, definitions and recall well known fundamental theorems. In section 3, we establish the main results of the article: refinements of generalized Hermite-Hadamard‘s inequality for *operator h-convex functions*.

2 Preliminaries

Our purpose in this section is to establish some basic terminology, we review briefly and without proofs some elementary results from the continuous functional calculus. The functional calculus is defined by the spectral theorem.

The notion of a convex function plays a fundamental role in modern mathematics. The theory of convex functions has been studied mostly due to its usefulness and applicability in Optimization. We recall some concepts of convexity that are well known in the literature.

Definition 1. *A function* $f: I \to \mathbb{R}$ *is said to be convex function over I if for any* $x,y \in I$ *and for any* $t \in [0,1]$ *we have the following inequality*

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y). \quad (2)$$

Definition 2. *[[28]] We shall say that a function* $f: I \subset \mathbb{R} \to \mathbb{R}$ *is a* ***Godunova-Levin*** *function or* $f \in Q(I)$ *if f is non negative and for each* $x,y \in I$ *and* $t \in (0,1)$ *we have*

$$f(tx+(1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

Definition 3. *[[21]]We say that* $f: I \to \mathbb{R}$ *is a* ***P-function****, or that f belongs to the class* $P(I)$*, if f is a non-negative function and for all* $x,y \in\in I$, $t \in [0,1]$ *we have*

$$f(tx+(1-t)y) \leq f(x)+f(y).$$

Definition 4. *[[14]] Let* $s \in (0,1]$*. A function* $f: [0,\infty) \to [0,\infty)$ *is named* ***s-convex*** *(in the second sense), or* $f \in K_s^2$ *if*

$$f(\lambda x+(1-\lambda)y) \leq \lambda^s f(x)+(1-\lambda)^s f(y)$$

for each $x,y \in (0,\infty)$ *and* $\lambda \in [0,1]$.

It can be easily seen that for $s = 1$, $s-$convexity reduces to ordinary convexity function.

A significant generalization of convex functions is that of *h-convex functions* introduced by S.Varosanec in [32].

Definition 5. *[[53]] Let* $h: J \to \mathbb{R}$ *be a non negative function and* $h \not\equiv 0$*, definided on an interval* $J \subset \mathbb{R}$*, with* $(0,1) \subset J$*. We shall say that a function* $f: I \to \mathbb{R}$ *, defined on an interval* $I \subset \mathbb{R}$*, is* ***h-convex*** *if f is non negative and the following inequality holds*

$$f(tx+(1-t)y) \leq h(t)f(x)+h(1-t)f(y)$$

for any $x,y \in I$ *and for all* $t \in (0,1)$ *.*

For some results concerning this class of functions see [12, 42, 51].

We can see, from this definition, that this class of functions contains the class of Godunova-Levin functions. It also contains the class of

1. If $h(t) = 1$ then an $h-$convex function f is a $P-$function.
2. If $h(t) = t^s, s \in (0,1]$ then an $h-$convex function f is an $s-$function.
3. If $h(t) = t^s$, with $s = -1$ then an $h-$convex function f is a Godunova-Levin function.

In order to achieve our results we need the following definitions and preliminary. With $B(H)$ we shall denote the C^*-algebra commutative of all bounded operators over a Hilbert space H with inner product $\langle,\rangle$. Let $\mathscr{A}$ be a subalgebra of $B(H)$. An operator $A \in \mathscr{A}$ is positive if $\langle Ax,x\rangle \geq 0$ for all $x \in H$. Over $\mathscr{A}$ there exists an order relation by means

$$A \leq B \text{ if } \langle Ax,x\rangle \leq \langle Bx,x\rangle$$

or

$$B \geq A \text{ if } \langle Bx,x\rangle \geq \langle Ax,x\rangle$$

for $A,B \in \mathscr{A}$ selfadjoint operators and for all $x \in H$.

The Gelfand map established a $*-$isometrically isomorphism Φ between the set $C(\sigma(A))$ of all continuous functions defined over the spectrum of A, denoted by $\sigma(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator $\mathbf{1}_H$ over H as follows:
For any $f,g \in C(\sigma(A))$ and $\alpha, \beta \in \mathbb{C}$ (Complex numbers) we have

1. $\Phi(\alpha f + \beta g) = \alpha\Phi(A) + \beta\Phi(B)$
2. $\Phi(fg) = \Phi(A)\Phi(B)$ and $\Phi(\overline{f}) = \Phi(f)^*$
3. $\|\Phi(f)\| = \|f\| := \sup_{t\in\sigma(A)} |f(t)|$
4. $\Phi(f_0) = \mathbf{1}_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ y $f_1(t) = t$ for all $t \in \sigma(A)$

with this notation we define

$$f(A) = \Phi(f)$$

and we call it the continuous functional calculus for a selfadjoint operator A.
If A is a selfadjoint operator and f is a continuous real valued function on $\sigma(A)$ then

$$f(t) \geq 0 \text{ for all } t \in \sigma(A) \Rightarrow f(A) \geq 0$$

that is to say $f(A)$ is a positive operator over H. Moreover, if both functions f,g are continuous real valued functions on $\sigma(A)$ then

$$f(t) \geq g(t) \text{ for all } t \in \sigma(A) \Rightarrow f(A) \geq g(A)$$

respect to the order in $B(H)$.

Definition 6. *Let H be a Hilbert space and $I \subseteq \mathbb{R}$ an interval. A continuous function $f : I \to \mathbb{R}$ is called operator convex with respect to H if*

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$$

for all $A, B \in B(H)^{sa}$ with $\sigma(A) \cup \sigma(B) \subset I$ and for all scalars $\lambda \in [0,1]$. f is called operator convex of order $n \in N$ if it is operator convex with respect to $H = C^n$. Finally, f is simply called operator convex if there is an infinite dimensional Hilbert space H such that f is operator convex with respect to H.

Here $B(H)^{sa}$ is the set of selfadjoint bounded operators on the Hilbert space H, $\sigma(A), \sigma(B)$, denotes the spectrum of A and B, and $f(A)$ and $f(B)$ are defined by the continuous functional calculus. We refer the reader to [46] for undefined notions on C^*-algebra theory.
As illustration below we state some classical theorems on operator inequalities.

Theorem 2. *[Bendat and Sherman [15]] f is operator convex if and only if it is operator convex of every order $n \in N$, and this last property holds if and only if it is operator convex with respect to the Hilbert space $\ell^2(C)$.*

Theorem 3. *[F. Hansen and G.K. Pedersen [32]] A continuous function f defined on an interval I is operator convex if and only if*

$$f\left(\sum_{j\in J} a_j^* x_j a_j\right) \leq \sum_{j\in J} a_j^* f(x_j) a_j$$

for every finite family $\{x_j : j \in J\}$ of bounded, self-adjoint operators on a separable Hilbert space H, with spectra contained in I, and every family of operators $\{a_j : j \in J\}$ in $B(H)$ with $\sum_{j\in J} a_j^* a_j = 1$, where $1 \in B(H)$ is the identity operator.

Theorem 4. *[D.R. Farenick and F. Zhou [25]] Let (Ω, Σ, μ) be a probability measure space, and suppose f is an operator convex function defined on an open interval $I \subseteq \mathbb{R}$. If $g : \Omega \to B(C^n)^{sa}$ is a measurable function for which $\sigma(g(\omega)) \subset [\alpha, \beta] \subset I$ for all $\omega \in \Omega$, then*

$$f\left(\int_\Omega g d\mu\right) \leq \int_\Omega f \circ g d\mu.$$

Some other references about this topic are in [34,35]. Dragomir in [22] has proved a Hermite-Hadamard type inequality for operator convex functions.

Theorem 5. *[[20],Theorem 1] Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality*

$$\left(f\left(\frac{A+B}{2}\right) \leq\right) \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right]$$
$$\leq \int_0^1 f((1-t)A + tB)dt$$
$$\leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2}\right]\left(\leq \frac{f(A)+f(B)}{2}\right)$$

The definition of operator $s-$convex function is proposed by Ghazanfari in [23].

Definition 7. *Let I be an interval in $[0,\infty)$ y K a convex subset of $B(H)^+$. A continuous function $f : I \to \mathbb{R}$ is said to be operatos $s-$convex on I for operators in K if*

$$f((1-\lambda)A + \lambda B) \leq (1-\lambda)^s f(A) + \lambda^s f(B)$$

in the operator order in $B(H)$, for all $\lambda \in [0,1]$ and for every positive operator A and B in K whose spectra are contained in I and for some fixed $s \in (0,1]$.

The following Hermite-Hadamard inequality for operator s-convex functions holds.

Theorem 6. *[[27],Theorem 6] Let $f : I \to \mathbb{R}$ be an operator $s-$convex function on the interval $I \subseteq [0,\infty)$ for operators in $K \subset B(H)^+$. Then for all positive operators A and B in K with spectra in I, we have the inequality*

$$2^{s-1} f\left(\frac{A+B}{2}\right) \leq \int_0^1 f((1-t)A + tB)dt \leq \frac{f(A)+f(B)}{s+1}$$

Dragomir in [52] introduced an even more general definition of *operator h-convex functions*.

Definition 8. *Let J be an interval include in $\mathbb{R}$ with $(0,1)\subset J$. Let $h : J\to\mathbb{R}$ be a non negative and identically nonzero function. We shall say that a continuous function $f : I\to\mathbb{R}$, defined on an interval $I\subset\mathbb{R}$, is an operator $h-$convex for operators in K if*

$$f(tA+(1-t)B)\leq h(t)f(A)+h(1-t)f(B)$$

for all $t\in(0,1)$ and $A,B\in K\subseteq B(H)^+$ such that $Sp(A)\subset I$ and $Sp(B)\subset I$.

With this concept Dragomir obtained some results involving operators *h-convex functions*. The first of them is located as Lemma 2.3 in [52] and it involves the associated function φ. The second is the Theorem 2.4 in [52], which establishes the Hermite-Hadamard type inequality for *operator h-convex functions*.

Lemma 1. *If f is an operator $h-$convex function then*

$$\varphi_{x,A,B}(t)=\langle(f(tA+(1-t)B)x,x)\rangle$$

for $x\in H$ with $\|x\|=1$ is an $h-$convex function over $(0,1)$.

Theorem 7. *Let f be an operator $h-$convex function. Then*

$$\frac{1}{2h(1/2)}f\left(\frac{A+B}{2}\right)\leq\int_0^1 f(tB+(1-t)A)dt$$
$$\leq(f(A)+f(B))\int_0^1 h(t)dt \tag{3}$$

3 Main Results

Theorem 8.*Let $f : I\to R$ be an operator $h-$convex function on some interval I. Then for any self-adjoint operators A and B with spectra in I, we have the inequality*

$$\left(f\left(\frac{A+B}{2}\right)\leq\right)\frac{1}{k}\sum_{i=0}^{k-1}f\left(\frac{(2k-2i-1)A+(2i+1)B}{2k}\right) \tag{4}$$

$$\leq\int_0^1 f((1-t)A+tB)dt$$

$$\leq\frac{1}{k}\left[\sum_{i=0}^{k-1}f\left(\frac{(k-i)A+iB}{k}\right)+h(1/2)(f(A)+f(B))\right]$$

$$(\leq h(1/2)(f(A)+f(B)))$$

where k is the numbers of steps.

*Proof.*The function f is continuous, $\int_0^1 f((1-t)A+tB)dt$ exists for any self-adjoint operators A and B with spectra in I.

We can give two proofs of the theorem. The first using the definition of operator $h-$convex functions and the second using the Hermite-Hadamard inequality for real-valued functions.

The first proof.

From the definition of operator $h-$convex functions, we have the inequalities

$$f\left(\frac{X+Y}{2}\right)=f\left(\frac{(1-t)X+tY}{2}+\frac{(1-t)Y+tX}{2}\right) \tag{5}$$

$$\leq h\left(\frac{1}{2}\right)(f((1-t)X+tY)+f((1-t)Y+tX))$$

$$\leq h\left(\frac{1}{2}\right)(f(X)+f(Y))$$

for any$t\in[0,1]$ and self-adjoint operators X and Y with spectra in I. If we integrate the inequality (5) over t and take into account that

$$\int_0^1 f((1-t)X+tY)dt=\int_0^1 f(tX+(1-t)Y)dt$$

then we conclude the Hermite-Hadamard inequality for operator $h-$convex functions

$$f\left(\frac{X+Y}{2}\right)\leq\int_0^1 f((1-t)X+tY)dt$$
$$\leq h\left(\frac{1}{2}\right)(f(X)+f(Y)) \tag{6}$$

that holds for any self-adjoint operators X and Y with spectra in I. Utilizing the change of variable $u=kt$, we have

$$\int_0^{1/k} f((1-t)A+tB)dt$$
$$=\frac{1}{k}\left(\int_0^1 f\left(\left(1-\frac{u}{k}\right)A+\frac{u}{k}B\right)du\right)$$
$$=\frac{1}{k}\left(\int_0^1 f\left(A-\frac{u}{k}A+\frac{u}{k}B\right)du\right)$$
$$=\frac{1}{k}\left(\int_0^1 f\left((1-u)A+u\frac{(k-1)A+B}{k}\right)du\right)$$

and by the change of variable $u=kt-1$, we have

$$\int_{1/k}^{2/k} f((1-t)A+tB)dt$$
$$=\frac{1}{k}\left(\int_0^1 f\left(\left(1-\frac{u+1}{k}\right)A+\frac{u+1}{k}B\right)du\right)$$
$$=\frac{1}{k}\left(\int_0^1 f\left(A-\frac{Au}{k}-\frac{A}{k}+\frac{Bu}{k}+\frac{B}{k}\right)du\right)$$

$$= \frac{1}{k}\left(\int_0^1 f\left((1-u)\frac{(k-1)A+B}{k}+u\frac{(k-2)A+2B}{k}\right)du\right)$$

We can change the variables until the variable $u = kt - (k-1)$ by using the same procedure above. By the change of variable $u = kt-(k-1)$, we get

$$\int_{(k-1)/k}^1 f((1-t)A+tB)\,dt$$

$$= \frac{1}{k}\left(\int_0^1 f\left(\left(1-\frac{u+k-1}{k}\right)A+\frac{u+k-1}{k}B\right)du\right)$$
$$= \frac{1}{k}\left(\int_0^1 f\left(A-\frac{Au}{k}-A+\frac{A}{k}+\frac{Bu}{k}+B-\frac{B}{k}\right)du\right)$$
$$= \frac{1}{k}\left(\int_0^1 f\left((1-u)\frac{A+(k-1)B}{k}+uB\right)du\right).$$

Using the Hermite-Hadamard inequality in (6), we have

$$f\left(\frac{A+\frac{(k-1)A+B}{k}}{2}\right) = f\left(\frac{(2k-1)A+B}{2k}\right) \tag{7}$$

$$\leq \int_0^1 f\left((1-u)A+u\frac{(k-1)A+B}{k}\right)du$$
$$\leq h\left(\frac{1}{2}\right)\left(f(A)+f\left(\frac{(k-1)A+B}{k}\right)\right),$$

$$f\left(\frac{\frac{(k-1)A+B}{k}+\frac{(k-2)A+2B}{k}}{2}\right) = f\left(\frac{(2k-3)A+B}{2k}\right) \tag{8}$$

$$\leq \int_0^1 f\left((1-u)A+u\frac{(k-1)A+B}{k}\right)du$$
$$\leq h\left(\frac{1}{2}\right)\left(f\left(\frac{(k-1)A+B}{k}\right)+f\left(\frac{(k-2)A+2B}{k}\right)\right),$$

$$f\left(\frac{\frac{(k-2)A+2B}{k}+\frac{(k-3)A+3B}{k}}{2}\right) = f\left(\frac{(2k-5)A+5B}{2k}\right) \tag{9}$$

$$\leq \int_0^1 f\left((1-u)\frac{(k-2)A+2B}{k}+u\frac{(k-3)A+3B}{k}\right)du$$
$$\leq h\left(\frac{1}{2}\right)\left(f\left(\frac{(k-2)A+2B}{k}\right)+f\left(\frac{(k-3)A+3B}{k}\right)\right)$$

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By induction we have

$$f\left(\frac{\frac{A+(k-1)B}{k}+B}{2}\right) = f\left(\frac{A+(2k-1)B}{2k}\right) \tag{10}$$

$$\leq \int_0^1 f\left((1-u)\frac{A+(k-1)B}{k}+uB\right)du$$
$$\leq h\left(\frac{1}{2}\right)\left(f\left(\frac{A+(k-1)B}{k}\right)+f(B)\right)$$

By summing (7), (8), (9), (10) and the other inequalities between (9) and (10), we have

$$f\left(\frac{A+\frac{(k-1)A+B}{k}}{2}\right)+f\left(\frac{\frac{(k-1)A+B}{k}+\frac{(k-2)A+2B}{k}}{2}\right)$$

$$+f\left(\frac{\frac{(k-2)A+2B}{k}+\frac{(k-3)A+3B}{k}}{2}\right)+....+f\left(\frac{\frac{A+(k-1)B}{k}+B}{2}\right)$$

$$\leq k\int_0^1 f((1-t)A+tB)\,dt$$

$$\begin{aligned}
&\leq h\left(\frac{1}{2}\right)\left[\left(f(A)+f\left(\frac{(k-1)A+B}{k}\right)\right)\right.\\
&\quad+\left(f\left(\frac{(k-1)A+B}{k}\right)+f\left(\frac{(k-2)A+2B}{k}\right)\right)\\
&\quad+\left(f\left(\frac{(k-1)A+B}{k}\right)+f\left(\frac{(k-2)A+2B}{k}\right)\right)\\
&\quad\left.\ldots+\left(f\left(\frac{A+(k-1)B}{k}\right)+f(B)\right)\right]
\end{aligned} \tag{11}$$

When regulating the inequality (11), we get the desired inequality in Theorem. It is obvious from the left-hand side of the inequality (4) for $k=1$,we get $f(\frac{A+B}{2})$, and it is obvious the right-hand side of the inequality (4) is provided for $k=2$.

The second proof.

Let $x \in H$, $\|x\| = 1$ and let A and B be two self-adjoint operators with spectra in I. Define the real-valued function $\varphi_{x,A,B} : [0,1] \to \mathbb{R}$ by $\varphi_{x,A,B}(t) = \langle f((1-t)A+tB)x,x\rangle$. Since f is an operator $h-$convex, then for any $t_1,t_2 \in [0,1]$ y $\alpha,\beta \geq 0$ con $\alpha+\beta = 1$, we have

$$\varphi_{x,A,B}(\alpha t_1+\beta t_2)$$

$$\begin{aligned}
&= \langle f((1-(\alpha t_1+\beta t_2))A+(\alpha t_1+\beta t_2)B)x,x\rangle\\
&= \langle f(\alpha[(1-t_1)A+t_1B]+\beta[(1-t_2)A+t_2B])x,x\rangle\\
&\leq h(\alpha)\langle f(\alpha[(1-t_1)A+t_1B])x,x\rangle\\
&\quad+h(\beta)\langle f(\alpha[(1-t_2)A+t_2B])x,x\rangle\\
&= h(\alpha)\varphi_{x,A,B}(t_1)+h(\beta)\varphi_{x,A,B}(t_2)
\end{aligned}$$

showing that $\varphi_{x,A,B}$ is a $h-$convex function on $[0,1]$. Now we can use the Hermite-Hadamard inequality for real-valued functions

$$\begin{aligned}
g\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b g(s)\,ds\\
&\leq h\left(\frac{1}{2}\right)(g(a)+g(b))
\end{aligned}$$

to get that

$$\varphi_{x,A,B}\left(\frac{1}{2k}\right) \leq k\int_0^{1/k}\varphi_{x,A,B}(t)dt$$
$$\leq h\left(\frac{1}{2}\right)\left(\varphi_{x,A,B}(0)+\varphi_{x,A,B}\left(\frac{1}{k}\right)\right),$$

$$\varphi_{x,A,B}\left(\frac{3}{2k}\right) \leq k\int_{1/k}^{2/k}\varphi_{x,A,B}(t)dt$$
$$\leq h\left(\frac{1}{2}\right)\left(\varphi_{x,A,B}\left(\frac{1}{k}\right)+\varphi_{x,A,B}\left(\frac{1}{2k}\right)\right),$$

$$\vdots$$

$$\varphi_{x,A,B}\left(\frac{2k-1}{2k}\right) \leq k\int_{(k-1)/k}^{1}\varphi_{x,A,B}(t)dt$$
$$\leq h\left(\frac{1}{2}\right)\left(\varphi_{x,A,B}\left(\frac{k-1}{k}\right)+\varphi_{x,A,B}(1)\right).$$

By summing the inequalities above and multiplying with $(1/k)$, we get

$$\frac{1}{k}\left[\varphi_{x,A,B}\left(\frac{1}{2k}\right)+\varphi_{x,A,B}\left(\frac{3}{2k}\right)+..+\varphi_{x,A,B}\left(\frac{2k-1}{2k}\right)\right]$$

$$\leq \int_0^1\varphi_{x,A,B}(t)dt$$

$$\leq \frac{1}{k}h\left(\frac{1}{2}\right)\left[\varphi_{x,A,B}(0)+\varphi_{x,A,B}(1)+\varphi_{x,A,B}\left(\frac{1}{k}\right)\right.$$
$$\left.+..+\varphi_{x,A,B}\left(\frac{k-1}{k}\right)\right]$$

Thus, we can write

$$\frac{1}{k}\left\langle\left[f\left(\left(1-\frac{2}{k}\right)A+\frac{2}{k}B\right)+f\left(\left(1-\frac{3}{2k}\right)A+\frac{3}{2k}B\right)+..\right.\right.$$
$$\left.\left.f\left(\left(1-\frac{k-1}{k}\right)A+\frac{k-1}{k}B\right)\right]x,x\right\rangle$$
$$\leq \int_0^1\langle f((1-t)A+tB)x,x\rangle dt$$
$$\leq \frac{1}{k}h\left(\frac{1}{2}\right)\left\langle\left[f(A)+f(B)+f\left(\left(1-\frac{1}{k}\right)A+\frac{1}{k}B\right)+\ldots\right.\right.$$
$$\left.\left.f\left(\left(1-\frac{k-1}{k}\right)A+\frac{k-1}{k}B\right)\right]x,x\right\rangle.$$

By regulating these inequalities above, we get

$$\frac{1}{k}\left\langle\left[\sum_{i=0}^{k-1}f\left(\frac{(2k-2i-1)A+(2i+1)B}{2k}\right)\right]x,x\right\rangle$$

$$\leq \int_0^1\langle f((1-t)A+tB)x,x\rangle dt \tag{12}$$

$$\leq \frac{1}{k}h\left(\frac{1}{2}\right)\left\langle\left[f(A)+f(B)+\sum_{i=0}^{k-1}f\left(\frac{(k-i)A+iB}{k}\right)\right]x,x\right\rangle.$$

Finally, since by the continuity of the function f, we have

$$\int_0^1\langle f((1-t)A+tB)x,x\rangle dt$$

$$=\left\langle\left(\int_0^1 f((1-t)A+tB)dt\right)x,x\right\rangle$$

for any $x\in H$, and any two self-adjoint operators A and B with spectra in I, from (12) we get the desired result in (4).

*Remark.*If $h(t)=t$ we obtain the Theorem 4 of Bacak V. and Türkmen R. in [6].

Theorem 9.*Let $f,g: I\to\mathbb{R}$ be an operator $h-$convex function on some interval I. Then for any self-adjoint operators A and B with spectra in I, we have the inequality*

$$\int_0^1\langle f((1-t)A+tB)x,x\rangle\langle g((1-t)A+tB)x,x\rangle dt$$
$$\leq M(A,B)\int_0^1(h(t))^2dt+N(A,B)\int_0^1 h(t)h(1-t)dt,$$

where

$$M(A,B)=\langle f(A)x,x\rangle\langle g(A)x,x\rangle+\langle f(B)x,x\rangle\langle g(B)x,x\rangle$$

$$N(A,B)=\langle f(A)x,x\rangle\langle g(B)x,x\rangle+\langle f(B)x,x\rangle\langle g(A)x,x\rangle.$$

*Proof.*Let $x\in H, \|x\|=1$ and let A and B be two self-adjoint operators with spectra in . Define the real-valued functions $\varphi_{x,A,B}:[0,1]\to\mathbb{R}$ by $\varphi_{x,A,B}(t)=\langle f((1-t)A+tB)x,x\rangle$ and $\psi_{x,A,B}:[0,1]\to\mathbb{R}$ by $\psi_{x,A,B}(t)=\langle g((1-t)A+tB)x,x\rangle$. Since f and g are operator $h-$convex functions, then for every $t\in[0,1]$, we have

$$\langle f((1-t)A+tB)x,x\rangle\leq h(1-t)\langle f(A)x,x\rangle+h(t)\langle f(B)x,x\rangle \tag{13}$$

$$\langle g((1-t)A+tB)x,x\rangle\leq h(1-t)\langle g(A)x,x\rangle+h(t)\langle g(B)x,x\rangle \tag{14}$$

From 13 and 14, we obtain

$$\langle f((1-t)A+tB)x,x\rangle\langle g((1-t)A+tB)x,x\rangle \tag{15}$$
$$\leq (h(1-t))^2\langle f(A)x,x\rangle\langle g(A)x,x\rangle$$
$$+(h(t))^2\langle f(B)x,x\rangle\langle g(B)x,x\rangle \tag{16}$$
$$+h(t)h(1-t)(\langle f(A)x,x\rangle\langle g(B)x,x\rangle$$
$$+\langle f(B)x,x\rangle\langle g(A)x,x\rangle) \tag{17}$$

Since $\varphi_{x,A,B}(t)$ and $\psi_{x,A,B}(t)$ are operator $h-$convex on $[0,1]$, they are integrable on $[0,1]$ and consequently

$\varphi_{x,A,B}(t)\psi_{x,A,B}(t)$ is also integrable on $[0,1]$. Integrating both sides of the inequality (15) over $[0,1]$, we get

$$\begin{aligned}
&\int_0^1 \langle f((1-t)A+tB)x,x\rangle \langle g((1-t)A+tB)x,x\rangle dt\\
&\leq \langle f(A)x,x\rangle \langle g(A)x,x\rangle \int_0^1 (h(1-t))^2 dt\\
&+ \langle f(B)x,x\rangle \langle g(B)x,x\rangle \int_0^1 (h(t))^2 dt\\
&+ (\langle f(A)x,x\rangle \langle g(B)x,x\rangle + \langle f(B)x,x\rangle \langle g(A)x,x\rangle)\\
&\qquad \times \int_0^1 h(t)h(1-t)dt.
\end{aligned}$$

With the appropiate change of variables, we have

$$\int_0^1 (h(1-t))^2 dt = \int_0^1 (h(t))^2 dt,$$

and we can write

$$\begin{aligned}
&\int_0^1 \langle f((1-t)A+tB)x,x\rangle \langle g((1-t)A+tB)x,x\rangle dt\\
&\leq M(A,B)\int_0^1 (h(t))^2 dt + N(A,B)\int_0^1 h(t)h(1-t)dt,
\end{aligned}$$

where

$$\begin{aligned}
M(A,B) &= \langle f(A)x,x\rangle \langle g(A)x,x\rangle + \langle f(B)x,x\rangle \langle g(B)x,x\rangle\\
N(A,B) &= \langle f(A)x,x\rangle \langle g(B)x,x\rangle + \langle f(B)x,x\rangle \langle g(A)x,x\rangle .
\end{aligned}$$

*Remark.*If $h(t) = t$ we obtain

$$\int_0^1 (h(t))^2 dt = \int_0^1 t^2 dt = \frac{1}{3}$$

and

$$\int_0^1 h(t)h(1-t)dt = \int_0^1 t(1-t)dt = \frac{1}{6}$$

and so we reach the Theorem 5 of Bacak V. and Türkmen R. in [6].

Theorem 10.*Let $f,g : I \to \mathbb{R}$ be an operator $h-$convex function on some interval I. Then, for any self-adjoint operators A and B with spectra in I, we have the inequality*

$$\begin{aligned}
&\int_0^1 \langle f((1-t)A+tB)x,x\rangle \langle g((1-t)A+tB)x,x\rangle dt\\
&\leq \frac{1}{3k}M(A,B)\\
&\quad +\frac{2}{3k}\sum_{i=1}^{k-1}\left\langle f\left(\frac{(k-i)A+iB}{k}\right)x,x\right\rangle \times\\
&\qquad\qquad \left\langle g\left(\frac{(k-i)A+(i+1)B}{k}\right)x,x\right\rangle\\
&+\frac{1}{6k}\sum_{i=1}^{k-1}\left\langle f\left(\frac{(k-i)A+iB}{k}\right)x,x\right\rangle \times\\
&\qquad\qquad \left\langle g\left(\frac{(k-i-1)A+iB}{k}\right)x,x\right\rangle\\
&+\frac{1}{6k}\sum_{i=1}^{k-1}\left\langle f\left(\frac{(k-i-1)A+iB}{k}\right)x,x\right\rangle \times\\
&\qquad\qquad \left\langle g\left(\frac{(k-i)A+iB}{k}\right)x,x\right\rangle,
\end{aligned}$$

where k is the number of steps.

*Proof.*The proof is obvious from Theorems 8 ans 9.

*Remark.*If $h(t) = t$ we get the Theorem 6 of Bacak V. and Türkmen R. in [6].

4 Conclusions

In this work, we have introduced the concept of *operator h-convex functions* and we have presented some refinements of Jensens inequality and Hadamard-Hermites inequality for *h-convex function* and for *operator h-convex functions*. In addition, we have presented some applications that show how the main theorems generalize other results demonstrated in cited references. We hope that everything established here will stimulate further research in this area.

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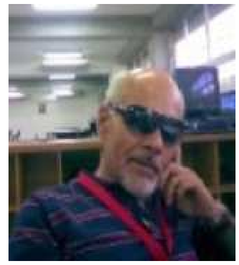

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