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Some New Semi-Normed Triple Sequence Spaces Defined by a Sequence Of Moduli

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Abstract: In this paper we study some properties of the sequence space $\chi^3_f(p,q,u)$ and obtain some inclusion relations.

Keywords: gai sequence, analytic sequence, triple sequence, Orlicz function.

1 Introduction

Throughout w,χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^3 for the set of all complex triple sequences (x_{mnk}), where $m,n,k\in\mathbb{N}$, the set of positive integers.Then, w^3 is a linear space under the coordinate wise addition and scalar multiplication.

Let (x_{mnk}) be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is said to be convergent if and only if the triple sequence (S_{mnk}) is convergent, where

$$S_{mnk}=\sum_{i,j,q=1}^{m,n,k} x_{ijq}\ \ (m,n,k=1,2,3,...).$$

A sequence $x=(x_{mnk})$ is said to be triple analytic if

$$\sup_{m,n,k}|x_{mnk}|^{\frac{1}{m+n+k}}<\infty.$$

The vector space of all triple analytic sequences are usually denoted by Λ^3. A sequence $x=(x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}}\rightarrow 0 \text{ as } m,n,k\rightarrow\infty.$$

The vector space of all triple entire sequences are usually denoted by Γ^3. The space Λ^3 and Γ^3 is a metric space with the metric

$$d(x,y)=\sup_{m,n,k}\left\{|x_{mnk}-y_{mnk}|^{\frac{1}{m+n+k}}: m,n,k:1,2,3,...\right\}, \quad (1)$$

for all $x=\{x_{mnk}\}$ and $y=\{y_{mnk}\}$ in Γ^3. Let $\phi=\{\text{finite sequences}\}$.

Consider a triple sequence $x=(x_{mnk})$. The $(m,n,k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]}=\sum_{i,j,q=0}^{m,n,k} x_{ijq}\delta_{ijq}$ for all $m,n,k\in\mathbb{N}$,

$$\delta_{mnk}=\begin{pmatrix} 0 & 0 & ... & 0 & 0 & ... \\ 0 & 0 & ... & 0 & 0 & ... \\ . \\ . \\ . \\ 0 & 0 & ... & 1 & 0 & ... \\ 0 & 0 & ... & 0 & 0 & ... \end{pmatrix}$$

with 1 in the $(m,n,k)^{th}$ position and zero otherwise.

A sequence $x=(x_{mnk})$ is called triple gai sequence if $((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}}\rightarrow 0$ as $m,n,k\rightarrow\infty$. The triple gai sequences will be denoted by χ^3.

Consider a triple sequence $x=(x_{mnk})$. The $(m,n,k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]}=\sum_{i,j,q=0}^{m,n,k} x_{ijq}\Im_{ijq}$ for all $m,n,k\in\mathbb{N}$; where $\Im_{ijq}$ denotes the triple sequence whose only non zero term is a $\frac{1}{(i+j+k)!}$ in the $(i,j,k)^{th}$ place for each $i,j,k\in\mathbb{N}$.

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An FK-space(or a metric space) X is said to have AK property if $(\Im_{mnk})$ is a Schauder basis for X, or equivalently $x^{[m,n,k]} \to x$.

An FDK-space is a triple sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings are continuous.

If X is a sequence space, we give the following definitions:

(i) X' is continuous dual of X;

(ii) $X^{\alpha} = \left\{ a = (a_{mnk}) : \sum_{m,n,k=1}^{\infty} |a_{mnk} x_{mnk}| < \infty, \text{ for each } x \in X \right\}$;

(iii) $X^{\beta} = \left\{ a = (a_{mnk}) : \sum_{m,n,k=1}^{\infty} a_{mnk} x_{mnk} \text{ is convergent, for each } x \in X \right\}$;

(iv) $X^{\gamma} = \left\{ a = (a_{mn}) : \sup_{m,n \geq 1} \left| \sum_{m,n,k=1}^{M,N,K} a_{mnk} x_{mnk} \right| < \infty, \text{ for each } x \in X \right\}$;

(v) Let X be an FK-space $\supset \phi$; then $X^{f} = \left\{ f(\Im_{mnk}) : f \in X' \right\}$;

(vi) $X^{\delta} = \left\{ a = (a_{mnk}) : \sup_{m,n,k} |a_{mnk} x_{mnk}|^{1/m+n+k} < \infty, \text{ for each } x \in X \right\}$;

$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called $\alpha-$ (or Köthe-Toeplitz) dual of $X, \beta-$(or generalized-Köthe-Toeplitz) dual of $X, \gamma-$ dual of X, $\delta-$dual of X respectively. X^{α} is defined by Gupta and Kamptan [10]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\alpha} \subset X^{\gamma}$ does not hold.

2 Definitions and Preliminaries

A sequence $x = (x_{mnk})$ is said to be triple analytic if $\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty$. The vector space of all triple analytic sequences is usually denoted by Λ^3. A sequence $x = (x_{mnk})$ is called triple entire sequence if $|x_{mnk}|^{\frac{1}{m+n+k}} \to 0$ as $m,n,k \to \infty$. The vector space of triple entire sequences is usually denoted by Γ^3. A sequence $x = (x_{mnk})$ is called triple gai sequence if $((m+n+k)! |x_{mnk}|)^{\frac{1}{m+n+k}} \to 0$ as $m,n,k \to \infty$. The vector space of triple gai sequences is usually denoted by χ^3. The space χ^3 is a metric space with the metric

$$d(x,y) = \sup_{m,n,k} \left\{ ((m+n+k)! |x_{mnk} - y_{mnk}|)^{\frac{1}{m+n+k}} : m,n,k : 1,2,3,... \right\} \quad (2)$$

for all $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\}$ in χ^3.

Let p, q be semi norms on a vector space X. Then p is said to be stronger that q if whenever (x_{mnk}) is a sequence such that $p(x_{mnk}) \to 0$, then also $q(x_{mnk}) \to 0$. If each is stronger than the others, then p and q are said to be equivalent.

A sequence space E is said to be solid or normal if $(\alpha_{mnk} x_{mnk}) \in E$ whenever $(x_{mnk}) \in E$ and for all sequences of scalars (α_{mnk}) with $|\alpha_{mnk}| \leq 1$, for all $m,n,k \in \mathbb{N}$.

A sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.

A sequence E is said to be convergence free if $(y_{mnk}) \in E$ whenever $(x_{mnk}) \in E$ and $x_{mnk} = 0$ implies that $y_{mnk} = 0$.

Let $p = (p_{mnk})$ be a sequence of positive real numbers with $0 < p_{mnk} < \sup p_{mnk} = G$ and Let $D = \max(1, 2^{G-1})$. Then for $a_{mnk}, b_{mnk} \in \mathbb{C}$, the set of complex numbers for all $m,n,k \in \mathbb{N}$ we have

$$|a_{mnk} + b_{mnk}|^{p_{mnk}} \leq D\left\{ |a_{mnk}|^{p_{mnk}} + |b_{mnk}|^{p_{mnk}} \right\}, \quad (3)$$

where $D = max\left(1, 2^{H-1}\right), H = \sup_{m,n,k} p_{mnk}$

By $S(X)$ we denote the linear space of all sequences $x = (x_{mnk})$ with $(x_{mnk}) \in X$ and the usual coordinate wise operations: $\alpha x = (\alpha x_{mnk})$ and $x + y = (x_{mnk} + y_{mnk})$, for each $\alpha \in \mathbb{C}$. If $\lambda = (\lambda_{mnk})$ is a scalar sequence and $x \in S(X)$ then we shall write $\lambda x = (\lambda_{mnk} x_{mnk})$

Let U be the set of all sequences $u = (u_{mnk})$ such that $u_{mnk} \neq 0$ and complex for all $m,n,k = 1,2,3,\cdots$.

Following Ruckle [11] and Maddox [12] we recall that a function $f : [0,\infty) \to [0,\infty)$ such that modulus f is (i) $f(x) = 0$ if and only if $x = 0$, (ii) $f(x+y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$, (iii) f is increasing, (iv) f is continuous from the right of 0. It follows from (ii) and (iv) f must be continuous everywhere on $[0,\infty)$. For a sequence of moduli $f = (f_{mnk})$ we give the following conditions: (v) $\sup_{m,n,k} f_{mnk}(t) < \infty$ for all $t \geq 0$, (vi) $\lim_{t \to 0} f_{mnk}(t) = 0$ uniformly in $m,n,k \geq 1$. We remark that in case $f_{mnk} = f(m,n,k \geq 1)$, where f is a modulus function, the conditions (v) and (vi) are automatically fulfilled.

Let (X,q) be a semi normed space over the field $\mathbb{C}$ of complex numbers with the semi norm q. The symbol $\chi_f^3(X)$ denotes the spaces of all triple gai sequences defined over X. We define the following sequence space:

$$\chi_f^3(p,q,u) = \left\{ x \in S(X) : u_{mnk} \left[f_{mnk} \left(q \left(((m+n+k)! |x_{mnk}|)^{1/m+n+k} \right) \right) \right]^{p_{mnk}} \to 0 \text{ as } m,n,k \to \infty \right\}$$

We get the following sequence spaces from $\chi_f^2(p,q,u)$ on giving particular values to p and u. Taking $p_{mnk} = 1$ for all $m,n,k \in \mathbb{N}$ we have

$$\chi_f^3(q,u) = \left\{ x \in S(X) : u_{mnk} \left[f_{mnk} \left(q \left(((m+n+k)! |x_{mnk}|)^{1/m+n+k} \right) \right) \right] \to 0 \text{ as } m,n,k \to \infty \right\}$$

If we take $u_{mnk}=1$, then we have

$$\chi_f^3(p,q)=\Big\{x\in S(X): \left[f_{mnk}\left(q\left(((m+n+k)!\,|x_{mnk}|)^{1/m+n+k}\right)\right)\right]^{p_{mnk}}\to 0 \text{ as } m,n,k\to\infty\Big\}$$

If we take $p_{mnk}=1$ and $u_{mnk}=1$ for all $m,n,k\in\mathbb{N}$, then we have

$$\chi_f^3(q)=\Big\{x\in S(X): \left[f_{mnk}\left(q\left(((m+n+k)!\,|x_{mnk}|)^{1/m+n+k}\right)\right)\right]\to 0 \text{ as } m,n,k\to\infty\Big\}$$

In addition to the above sequence spaces, we have $\chi_f^3(p,q,u)=\chi_f^3(p)$, on taking $u_{mnk}=1$ for all $m,n,k\in\mathbb{N}$, $q(x)=|x|,(f_{mnk})=f$ for all $m,n,k\in\mathbb{N}$ and $X=\mathbb{C}$. In this chapter we introduce the sequence spaces $\chi_f^3(p,q,u)$, using an modulus function f and defined over a semi normed space (X,q), semi normed by q. We study some properties of these sequence spaces and obtain some inclusion relations.

Lemma 2.1. Let p and q be semi norms on a linear space X. Then p is stronger than q if and only if there exists a constant M such that $q(x)\le Mp(x)$ for all $x\in X$.

Remark 2.2. From the two above definitions it is clear that a sequence space E is solid implies that E is monotone.

3 Main Results

Theorem 3.1. If $f=(f_{mnk})$ be a sequence of moduli, then $\chi_f^3(p,q,u)$ are linear spaces over the set of complex numbers.

Proof: It is routine verification. Therefore the proof is omitted.

Theorem 3.2. $\chi_f^3(p,q,u)$ are paranormed spaces with the paranorm g defined by

$$g(x)=\sup_{m,n,k}u_{mnk}\left[f_{mnk}\left(q\left(((m+n+k)!\,|x_{mnk}|)^{1/m+n+k}\right)\right)\right].$$

Proof: Clearly $g(x)=g(-x)$ and $g(\theta)=0$, where θ is the zero sequence. It can be easily verified that $g(x+y)\le g(x)+g(y)$. Next $x\to\theta,\lambda$ fixed implies $g(\lambda x)\to 0$. Also $x\to\theta$ and $\lambda\to 0$ implies $g(\lambda x)\to 0$. The case $\lambda\to 0$ and x fixed implies that $g(\lambda x)\to 0$ follows from the following expressions.

$$g(\lambda x)=\sup_{m,n,k}u_{mnk}\left[f_{mnk}\left(q\left(((m+n+k)!\,|\lambda_{mnk}x_{mnk}|)^{1/m+n+k}\right)\right)\right].$$

$$g(\lambda x)=\Big\{|\lambda|^{1/m+n+k}: \sup_{m,n,k}u_{mnk}\left[f_{mnk}\left(q\left(((m+n+k)!\,|x_{mnk}|)^{1/m+n+k}\right)\right)\right]\Big\}.$$

Hence $\chi_f^3(p,q,u)$ is a paranormed space. This completes the proof.

Theorem 3.3. Let $f=(f_{mnk})$ and $T=(T_{mnk})$ be a two sequence of moduli. Then

$$\chi_f^3(p,q,u)\bigcap\chi_T^3(p,q,u)\subseteq\chi_{f+T}^3(p,q,u)$$

Proof: The proof is easy, so omitted.

Remark 3.4. Let $f=(f_{mnk})$ be a sequence of moduli q_1 and q_2 be two semi norms on X, we have

(i) $\chi_f^3(p,q_1,u)\cap\chi_f^3(p,q_2,u)\subseteq\chi_f^3(p,q_1+q_2,u)$

(ii) If q_1 is stronger than q_2 then $\chi_f^3(p,q_1,u)\subseteq\chi_f^3(p,q_2,u)$

(iii) If q_1 is equivalent to q_2 then $\chi_f^3(p,q_1,u)=\Gamma_f^3(p,q_2,u)$

Theorem 3.5. (i) Let $0\le p_{mnk}\le r_{mnk}$ and $\left\{\frac{r_{mnk}}{p_{mnk}}\right\}$ be bounded. Then $\chi_f^3(r,q,u)\subset\chi_f^3(p,q,u)$

(ii) $u_1\le u_2$ implies $\chi_f^3(p,q,u_1)\subset\chi_f^3(p,q,u_2)$

Proof: Let

$$x\in\chi_f^3(r,q,u) \tag{4}$$

$$u_{mnk}\left[f_{mnk}\left(q\left(((m+n+k)!\,|x_{mnk}|)^{1/m+n+k}\right)\right)\right]^{r_{mnk}}\to 0 \quad \text{as } m,n,k\to\infty \tag{5}$$

Let $t_{mnk}=u_{mnk}\left[f_{mnk}\left(q\left(((m+n+k)!\,|x_{mnk}|)^{1/m+n+k}\right)\right)\right]^{r_{mnk}}$ and $\lambda_{mnk}=\frac{p_{mnk}}{r_{mnk}}$. Since $p_{mnk}\le r_{mnk}$, we have $0\le\lambda_{mnk}\le 1$. Take $0<\lambda<\lambda_{mnk}$.

Define $u_{mnk}=t_{mnk}(t_{mnk}\ge 1); u_{mnk}=0(t_{mnk}<1)$; and $v_{mnk}=0(t_{mnk}\ge 1); v_{mnk}=t_{mnk}(t_{mnk}<1)$; $t_{mnk}=u_{mnk}+v_{mnk}$; $t_{mnk}^{\lambda_{mnk}}+v_{mnk}^{\lambda_{mnk}}$. Now it follows that

$$u_{mnk}^{\lambda_{mnk}}\le t_{mnk} \text{ and } v_{mnk}^{\lambda_{mnk}}\le v_{mnk}^{\lambda} \tag{6}$$

i.e $t_{mnk}^{\lambda_{mnk}}=u_{mnk}^{\lambda_{mnk}}+v_{mnk}^{\lambda_{mnk}};\ t_{mnk}^{\lambda_{mnk}}\le t_{mnk}+v_{mnk}^{\lambda}$ by (6)

$$u_{mnk}\left[f_{mnk}\left(q\left(((m+n+k)!\,|x_{mnk}|)^{1/m+n+k}\right)\right)^{r_{mnk}}\right]^{\lambda_{mnk}}\le u_{mnk}\left[f_{mnk}\left(q\left(((m+n+k)!\,|x_{mnk}|)^{1/m+n+k}\right)\right)\right]^{r_{mnk}}$$

$$u_{mnk}\left[f_{mnk}\left(q\left(((m+n+k)!\,|x_{mnk}|)^{1/m+n+k}\right)\right)^{r_{mnk}}\right]^{p_{mnk}/r_{mnk}}\le u_{mnk}\left[f_{mnk}\left(q\left(((m+n+k)!\,|x_{mnk}|)^{1/m+n+k}\right)\right)\right]^{r_{mnk}}$$

$$u_{mnk}\left[f_{mn}\left(q\left(\frac{((m+n)!\,|x_{mn}|)^{1/m+n+k}}{\rho}\right)\right)\right]^{p_{mnk}}$$
$$\leq u_{mnk}\left[f_{mnk}\left(q\left(((m+n+k)!\,|x_{mnk}|)^{1/m+n+k}\right)\right)\right]^{r_{mnk}}$$

But

$$u_{mnk}\left[f_{mnk}\left(q\left(((m+n+k)!\,|x_{mnk}|)^{1/m+n+k}\right)\right)\right]^{r_{mnk}}\to 0 \quad \text{as} \quad m,n,k\to\infty.$$

By (5), we have

$$u_{mnk}\left[f_{mnk}\left(q\left(((m+n+k)!\,|x_{mnk}|)^{1/m+n+k}\right)\right)\right]^{p_{mnk}}\to 0 \quad \text{as} \quad m,n,k\to\infty.$$

Hence

$$x\in\chi_f^3(p,q,u) \tag{7}$$

From (4) and (7) we get $\chi_f^3(r,q,u)\subset\chi_f^3(p,q,u)$. This completes the proof.

Proof (ii): The proof is easy, so omitted.

Theorem 3.6. The space $\chi_f^3(p,q,u)$ is solid, hence is monotone.

Proof: Let $(x_{mnk})\in\chi^3_{f_{mnk}}(p,q,u)$ and (α_{mnk}) be a sequence of scalars such that $|\alpha_{mnk}|^{1/m+n+k}\leq 1$ for all $m,n,k\in\mathbb{N}$. Then

$$u_{mnk}\left[f_{mnk}\left(q\left(((m+n+k)!\,|\alpha_{mnk}x_{mnk}|)^{1/m+n+k}\right)\right)\right]^{p_{mnk}}$$
$$\leq u_{mnk}\left[f_{mnk}\left(q\left(((m+n+k)!\,|x_{mnk}|)^{1/m+n+k}\right)\right)\right]^{p_{mnk}} \quad \text{for all } m,n,k\in\mathbb{N}$$

$$\left[f_{mnk}\left(q\left(((m+n+k)!\,|\alpha_{mnk}x_{mnk}|)^{1/m+n+k}\right)\right)\right]^{p_{mnk}}$$
$$\leq\left[f_{mnk}\left(q\left(((m+n+k)!\,|x_{mnk}|)^{1/m+n+k}\right)\right)\right]^{p_{mnk}} \quad \text{for all } m,n,k\in\mathbb{N}.$$

This completes the proof.

Result 3.7. The space $\chi_f^3(p,q,u)$ are not convergence free in general.

Proof: The proof follows from the following example.

Example. Consider the sequences $(x_{mnk}),(y_{mnk})\in\chi_f^3(p,q,u)$. Defined as $(x_{mnk})=\frac{1}{(m+n+k)!}\left(\frac{1}{m+n+k}\right)^{m+n+k}$ and $(y_{mnk})=\frac{1}{(m+n+k)!}\left(\frac{m-n-k}{m+n+k}\right)^{m+n+k}$. Hence $u_{mnk}\left[f_{mnk}\left(q\left(\frac{1}{(m+n+k)}\right)\right)\right]^{p_{mnk}}\to 0$ as $m,n,k\to\infty$. Which implies $(x_{mnk})=0$. Also $u_{mnk}\left[f_{mnk}\left(q\left(\frac{m-n-k}{(m+n+k)}\right)\right)\right]^{p_{mnk}}\to 0$ as $m,n,k\to\infty$. But $(y_{mnk})\not\to 0$. Hence the space $\chi_f^3(p,q,u)$ are not convergence free in general. This completes the proof.

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