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Poisson Bracket Formulation for a Dissipative Two-Dimensional Anisotropic Harmonic Oscillator with Fractional Derivatives: Analysis and Applications

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Abstract: We recast the Harmonic Oscillator using fractional differential equations. to be more developed By applying the Hamiltonian formulation with fractional derivatives to the resulting Harmonic Oscillator. the canonical conjugate-momentum coordinates are defined and converted into operators that fulfill the commutation relations, which correspond to the classical theory's Poisson-bracket relations. The equations of motion are redefined in terms of the generalized brackets when these are generalized. We present a generalized dissipative two-dimensional anisotropic harmonic oscillator equation of motion with fractional derivatives. The novel method was evaluated on a single example and found to be consistent agreement with the classical fractional method.

Keywords: canonical quantization, Poisson Bracket, Dissipative Two-Dimensional Anisotropic Harmonic Oscillator, Riemann- Liouville fractional derivative.

1 Introduction

Ordinary differentiation and integration are generalized to any order in fractional calculus. The fractional derivatives are the infinitesimal generators of a family of translation invariant convolution semigroups that occur as attractors everywhere. Several applications of fractional calculus are based on replacing the time derivative in an evolution equation with a fractional order derivative. Several recent studies confirm that fractional derivatives appear to exist for crucial mathematical reasons.Over the last few decades, the fractional calculus [1–3] has begun to be used in a variety of domains, including engineering, physics, and biology, and numerous noteworthy discoveries have been published [4–19]. In recent years, there has been a lot of emphasis on fractional variational ideas [20–30] and their applications [31–36]. The fractional Lagrangian and Hamiltonian formalisms are still in their infancy for constrained systems [37]. Here, we shall look at the most fundamental models of this type: Scott-Blair and Kelvin-Voigt fractional models [38,39]. When fractional constitutive relations are employed to solve issues of vibrations of continuous structures (such as beams, bars, and so on), fractional differential equations equivalent to the equation of a forced, harmonic, damped oscillator are created [40,41]. The novel ideas presented in this manuscript have the following characteristics.

• Because the proposed method is a powerful instrument for discovering fractional new formlism, we recommend that readers consider the following points:

- The methodologies utilized in this study are applicable to a wide range of lagrangian density models.
- The method is extended to high order fractional derivatives.
- According to the findings of this study, the fractional calculus, due to the fractional derivative order and the fractional operator itself, allows for more flexible models than the traditional classical calculus. This characteristic is critical in obtaining a new formulation of the generalized Riemann-Liouville fractional derivative of a dissipative two-dimensional anisotropic harmonic oscillator.

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- The proposed method can generate a wide range of accurate solutions to generalized type differential differential equations with fractional derivative.

This paper is a generalization of for classical fields with RiemannLiouville fractional derivatives. The Poisson bracket and commutation relations are introduced for the damped harmonic oscillator. Motion equations are stated in terms of Poisson brackets, whereas Two-Dimensional Anisotropic Harmonic Oscillator equations are written in terms of commutators.
The following is how this document is structured: The definitions of fractional derivatives are briefly covered in Section.2. Section. 3 presents the lagrangian density formulation of the dissipative two-dimensional anisotropic harmonic oscillator. In Section 4, we look at fractional motion equations in terms of Euler-Lagrangian density. Section 5 is dedicated to fractional derivatives and fractional Hamiltonian density. Section 6 discusses the Lagrangian Fractional formulation of a dissipative anisotropic two-dimensional harmonic oscillator in fractional 3 derivative forms. Section 6 discusses the fractional version of the Three-Dimensional Isotropic Oscillator equation in terms of the Poisson bracket formulation. In Section 4, we look at how the Fractional Harmonic Oscillator with Fractional Derivative can be used. The work concludes with some closing notes (section8).

2 Fractional derivative definitions

In this part of study, we briefly present some properties and fundamental definitions used in this work. The Riemann-Liouville fractional derivative is given by:

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha+1} f(\tau)d\tau \quad (1)$$

The right Riemann- Liouville fractional derivative is defined as :

$$ {}_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_a^x (\tau-x)^{n-\alpha+1} f(\tau)d\tau. \quad (2)$$

where Γ denotes the Gamma function, and α is the order of the derivative such that $n-1<\alpha<n$. If α is an integer, these derivatives are defined in the usual sense, i.e.

$$ f(x) = \left(\frac{d}{dx}\right)^n f(x) \quad (3)$$

$$ {}_aD_t^\alpha f(x) = \left(\frac{d}{dx}\right)^n f(t) \qquad \alpha = 1,2,.. \quad (4)$$

$$ \overset{1}{D_t} = \frac{d}{dt} \quad (6)$$

2 - The lagrangian density formulation of the dissipative two-dimensional anisotropic harmonic oscillator.

Consider the following Lagrangian which is given by

$$ L = \sum_{i=1}^2 \left(\frac{1}{2}m\dot{q_i}^2 - 2\pi^2 m{v_i}^2{q_i}^2\right)e^{\lambda_i t} \quad (7)$$

According to Eq.(7), the equations of motion is then

$$ \sum_{i=1}^2 (\ddot{q}_i + \lambda_i q_i + 4\pi^2{v_i}^2 q_i) = 0 \quad (8)$$

Which clearly describes the Two-Dimensional Anisotropic Harmonic Oscillator dissipative. Using the transformation $x_i = \sum_{i=1}^2 q_i e^{\lambda_i t/_2}$. The lagrangian is then transformed as

$$L = \sum_{i=1}^{2} \frac{1}{2} m\dot{x}_i^2 + \frac{1}{8} m\lambda_i^2 x_i^2 - \frac{1}{4} m\lambda_i \dot{x}_i x_i - 2\pi^2 m{v_i}^2 {x_i}^2 \quad (9)$$

The teams $\dot{x}_i x_i$ does not influence the equations of motion F_i:$\left(\frac{dF_i}{dt} = 0 = \dot{x}_i x_i\right)$ implies that $F_i = \frac{x_i^2}{2}$. Thus, we have the following equivalent lagrangian:

$$L = \sum_{i=1}^{2} \frac{1}{2} m\dot{x}_i^2 + \frac{1}{8} m\lambda_i^2 x_i^2 - 2\pi^2 m{v_i}^2 {x_i}^2 \quad (10)$$

Here the subscript (i) means sum over ($x_1 = x,\ x_2 = y$).The dissipative Two-Dimensional Anisotropic Harmonic Oscillator is described by this lagrangian. The equations of motion are

$$\ddot{x} + \left(4\pi^2 {v_x}^2 - \frac{\lambda_x^2}{4}\right) x = 0 \quad (10.1)$$

$$\ddot{y} + \left(4\pi^2 {v_y}^2 - \frac{\lambda_y^2}{4}\right) y = 0 \quad (10.2)$$

The angular frequency of the dissipative Two-Dimensional Anisotropic Harmonic Oscillator is $\omega_x = 2\pi v_x$ and $\omega_y = 2\pi v_y$.

3 Equations for fractional motion in terms of Euler-Lagrangian density

The dynamics of a physical system are described in the Lagrangian as a function of the locations and velocity of all degrees of freedom that comprise the system. Pathways in the configuration space can be used to extract the dynamics. For a given direction, the position and velocity at each time, as well as the Lagrangian value, are calculated. The continuous structure of the Lagrange density is given by the dynamic field variables, generalized coordinate $\boldsymbol{q}$ and its second-order derivatives, and generalized velocities ${}_{x_\mu}D_b^\beta\, {}_{x_\sigma}D_b^\beta \boldsymbol{q}$.

$$\mathcal{L} = \mathcal{L}\left[\boldsymbol{q},\ {}_aD_{x_\mu}^\alpha \boldsymbol{q},\ {}_{x_\mu}D_b^\beta \boldsymbol{q},\ {}_aD_{x_\mu}^\alpha\ {}_aD_{x_\sigma}^\alpha\, {}_{x_\mu}D_b^\beta\, {}_{x_\sigma}D_b^\beta \boldsymbol{q}\ ,\ {}_{x_\mu}D_b^\beta\, {}_{x_\sigma}D_b^\beta \boldsymbol{q}\right] \quad (11)$$

For this Lagrangian density in fractional form, the Euler-Lagrange equation can be given as

$$\left[\frac{\partial\mathcal{L}}{\partial\boldsymbol{q}} + \frac{\partial\mathcal{L}}{\partial\ {}_aD_{x_\mu}^\alpha \boldsymbol{q}} + \frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D_b^\beta \boldsymbol{q}} + \frac{\partial\mathcal{L}}{\partial\ {}_aD_{x_\mu}^\alpha\ {}_aD_{x_\sigma}^\alpha\ {}_aD_{x_\mu}^\alpha \boldsymbol{q}} + \frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D_b^\beta\, {}_{x_\sigma}D_b^\beta\ {}_aD_{x_\mu}^\alpha \boldsymbol{q}}\right] = 0 \quad (12)$$

Using the variational principle, we can write:

$$\delta S = \int \delta\mathcal{L}\ d^4x = 0 \quad (13)$$

Using Eq. (12), the variation of $\mathcal{L}$ is:

$$\delta\mathcal{L} = \left[\begin{matrix} \frac{\partial\mathcal{L}}{\partial\boldsymbol{q}}\delta\boldsymbol{q} + \frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D_b^\beta \boldsymbol{q}}\delta\ {}_{x_\mu}D_b^\beta \boldsymbol{q} + \frac{\partial\mathcal{L}}{\partial\ {}_aD_{x_\mu}^\alpha \boldsymbol{q}}\delta\ {}_aD_{x_\mu}^\alpha \boldsymbol{q} \\ + \frac{\partial\mathcal{L}}{\partial\ {}_aD_{x_\mu}^\alpha\ {}_aD_{x_\sigma}^\alpha\ {}_aD_{x_\mu}^\alpha \boldsymbol{q}}\delta\ {}_aD_{x_\mu}^\alpha\ {}_aD_{x_\sigma}^\alpha\ {}_aD_{x_\mu}^\alpha \boldsymbol{q} + \frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D_b^\beta\, {}_{x_\sigma}D_b^\beta\ {}_aD_{x_\mu}^\alpha \boldsymbol{q}}\delta\ {}_{x_\mu}D_b^\beta\, {}_{x_\sigma}D_b^\beta\ {}_aD_{x_\mu}^\alpha \boldsymbol{q} \end{matrix}\right] d^3x \quad (14)$$

Substituting Eq. (12) into Eq. (13), and using the following commutation relation

$$\begin{bmatrix} \delta\ {}_aD^{\alpha}_{x_\mu}\boldsymbol{q} = {}_aD^{\alpha}_{x_\mu}\delta\boldsymbol{q} \\ \delta\ {}_{x_\mu}D^{\beta}_{b}\boldsymbol{q} = {}_{x_\mu}D^{\beta}_{b}\delta\boldsymbol{q} \end{bmatrix} \quad (15)$$

$$\begin{bmatrix} \delta\ {}_aD^{\alpha}_{x_\mu}\ {}_aD^{\alpha}_{x_\sigma}\boldsymbol{q} = {}_aD^{\alpha}_{x_\mu}\ {}_aD^{\alpha}_{x_\sigma}\delta\boldsymbol{q} \\ \delta\ {}_{x_\mu}D^{\beta}_{b}\ {}_{x_\sigma}D^{\beta}_{b}\boldsymbol{q} = {}_{x_\mu}D^{\beta}_{b}\ {}_{x_\sigma}D^{\beta}_{b}\delta\boldsymbol{q} \end{bmatrix} \quad (16)$$

we get,

$$\int \begin{bmatrix} \frac{\partial\mathcal{L}}{\partial\boldsymbol{q}}\delta\boldsymbol{q} + \frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{x_\mu}\boldsymbol{q}}\delta\ {}_aD^{\alpha}_{x_\mu}\boldsymbol{q} + \frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D^{\beta}_{b}\boldsymbol{q}}\delta\ {}_{x_\mu}D^{\beta}_{b}\boldsymbol{q} \\ + \frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{x_\mu}\ {}_aD^{\alpha}_{x_\sigma}\boldsymbol{q}}\delta\ {}_aD^{\alpha}_{x_\mu}\ {}_aD^{\alpha}_{x_\sigma}\boldsymbol{q} + \frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D^{\beta}_{b}\ {}_{x_\sigma}D^{\beta}_{b}\boldsymbol{q}}\delta\ {}_{x_\mu}D^{\beta}_{b}\ {}_{x_\sigma}D^{\beta}_{b}\boldsymbol{q} \end{bmatrix} d^4x = 0 \quad (17)$$

Integrating by parts the second and the third and the fourth and the fifth terms in Eq. (17) , This lead to Euler – Lagrange equations.

$$\begin{bmatrix} \frac{\partial\mathcal{L}}{\partial\boldsymbol{q}} - {}_aD^{\alpha}_{x_\mu}\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{x_\mu}\boldsymbol{q}} - {}_{x_\mu}D^{\beta}_{b}\frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D^{\beta}_{b}\boldsymbol{q}} + {}_aD^{\alpha}_{x_\mu}\ {}_aD^{\alpha}_{x_\sigma}\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{x_\mu}\ {}_aD^{\alpha}_{x_\sigma}\boldsymbol{q}} \\ + {}_{x_\mu}D^{\beta}_{b}\ {}_{x_\sigma}D^{\beta}_{b}\frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D^{\beta}_{b}\ {}_{x_\sigma}D^{\beta}_{b}\boldsymbol{q}} \end{bmatrix} = 0 \quad (18)$$

Taking $x_i = \sum_{i=1}^{2} q_i e^{\lambda_i t/2}$. as a specific instance, the Euler – Lagrange reduce to the original relations like:

$$\begin{bmatrix} \frac{\partial\mathcal{L}}{\partial x_i} - {}_aD^{\alpha}_{t}\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{t}x_i} - {}_tD^{\beta}_{b}\frac{\partial\mathcal{L}}{\partial\ {}_tD^{\beta}_{b}x_i} - {}_aD^{\alpha}_{x_i}\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{x_i}x_i} - {}_{x_i}D^{\beta}_{b}\frac{\partial\mathcal{L}}{\partial\ {}_{x_i}D^{\beta}_{b}x_i} \\ + {}_aD^{\alpha}_{t}\ {}_aD^{\alpha}_{t}\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{t}\ {}_aD^{\alpha}_{t}x_i} + {}_tD^{\beta}_{b}\ {}_tD^{\beta}_{b}\frac{\partial\mathcal{L}}{\partial\ {}_tD^{\beta}_{b}\ {}_tD^{\beta}_{b}x_i} \\ + {}_aD^{\alpha}_{x_i}\ {}_aD^{\alpha}_{x_i}\frac{\partial\mathcal{L}}{\partial\ {}_aD^{\alpha}_{x_i}\ {}_aD^{\alpha}_{x_i}x_i} + {}_{x_i}D^{\beta}_{b}\ {}_{x_i}D^{\beta}_{b}\frac{\partial\mathcal{L}}{\partial\ {}_{x_i}D^{\beta}_{b}\ {}_{x_i}D^{\beta}_{b}x_i} \end{bmatrix} = 0 \quad (19)$$

4 Fractional derivatives and fractional Hamiltonian density

The Lagrangian, which is a function of all degrees of freedom in the system, is used to represent the dynamics of a physical system. The dynamics are derived by following pathways in coordinate space. At each time step, the position and velocities of a given path are determined, as well as the Lagrangian value. The continuous system is described as $\left[\ {}_aD^{\alpha}_{t}x,\ {}_aD^{\alpha}_{t}y\right]$ with Lagrangian density specified on dynamical field variables, generalized coordinate $[x, y]$ and $\left[\ {}_aD^{\alpha}_{t}x,\ {}_aD^{\alpha}_{t}y\right]$

$$\mathcal{L} = \mathcal{L}[x, y\ ,\ {}_aD^{\alpha}_{t}x\ ,\ {}_aD^{\alpha}_{t}y\ ] \quad (20)$$

We introduce the generalized momenta as:

$$\begin{cases} \pi^1_x = \frac{\partial\mathcal{L}}{\partial(\ {}_aD^{\alpha}_{t}x)} \\ \pi^1_y = \frac{\partial\mathcal{L}}{\partial(\ {}_aD^{\alpha}_{t}y)} \end{cases}, \quad , \quad (21)$$

The Hamiltonian depending on the fractional time derivatives reads as

$$H = \pi_x^1[{}_aD_t^\alpha x] + \pi_y^1[{}_aD_t^\alpha y] - \mathcal{L}[x, y, {}_aD_t^\alpha x, {}_aD_t^\alpha y] \quad (22)$$

Calculating the total differential of this Hamiltonian, we get

$$dH = \begin{bmatrix} \pi_x^1 d[{}_aD_t^\alpha x] + {}_aD_t^\alpha x[d\pi_x^1] + \pi_y^1 d[{}_aD_t^\alpha y] + {}_aD_t^\alpha y[d\pi_y^1] - \frac{\partial L}{\partial t}dt - \frac{\partial L}{\partial x}dx \\ -\frac{\partial L}{\partial y}dy - \frac{\partial L}{\partial({}_aD_t^\alpha x)}d({}_aD_t^\alpha x) - \frac{\partial L}{\partial({}_aD_t^\alpha y)}d({}_aD_t^\alpha y) \end{bmatrix} \quad (23)$$

By comparison with the variation in L (i.e. Eq. (14), we may describe the variation of Hamiltonian induced by variations of independent variables in terms of fractional derivative as follows in ($\pi_y^1, x, \pi_x^1, y, {}_aD_{x_i}^\alpha x, {}_aD_{x_i}^\alpha y$).

Thus, the total differential of the Hamiltonian takes the form

$$dH = \begin{bmatrix} \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy + \frac{\partial H}{\partial \pi_y^1}d\pi_y^1 + \frac{\partial H}{\partial({}_aD_{x_i}^\alpha x)}d({}_aD_{x_i}^\alpha x) + \frac{\partial H}{\partial \psi^*}d\psi^* + \frac{\partial H}{\partial \pi_x^1}d\pi_x^1 \\ + \frac{\partial H}{\partial({}_aD_{x_i}^\alpha y)}d({}_aD_{x_i}^\alpha y) \end{bmatrix} \quad (24)$$

When we compare (23) and (24), we get Hamilton's equations of motion after some basic manipulations using the fractional Hamilton equations.

$$\begin{cases} \frac{\partial H}{\partial \pi_y^1} = {}_aD_t^\alpha y \\ \frac{\partial H}{\partial \pi_x^1} = {}_aD_t^\alpha x \\ \frac{\partial H}{\partial({}_aD_{x_i}^\alpha x)} = -\frac{\partial L}{\partial({}_aD_{x_i}^\alpha x)} \\ \frac{\partial H}{\partial({}_aD_{x_i}^\alpha y)} = -\frac{\partial L}{\partial({}_aD_{x_i}^\alpha y)} \end{cases} \quad (25)$$

These formulas can be rewritten using the Euler-Lagrange method, and they will take the form

$$\begin{cases} \frac{\partial H}{\partial \pi_y^1} = {}_aD_t^\alpha y \quad . \\ \frac{\partial H}{\partial \pi_x^1} = {}_aD_t^\alpha x \quad . \\ \frac{\partial H}{\partial x} = -{}_aD_t^\alpha \frac{\partial L}{\partial({}_aD_{x_i}^\alpha x)} - {}_aD_{x_i}^\alpha \frac{\partial L}{\partial({}_aD_{x_i}^\alpha x)} \\ \frac{\partial H}{\partial y} = -{}_aD_t^\alpha \frac{\partial L}{\partial({}_aD_{x_i}^\alpha y)} - {}_aD_{x^i}^\alpha \frac{\partial L}{\partial({}_aD_{x_i}^\alpha y)} \end{cases} \quad (26)$$

These new coordinates are the same as the standard canonical coordinates for integer-order derivatives.

5 The Lagrangian Fractional formulation of a dissipative anisotropic two-dimensional harmonic oscillator.

These formulas are generalized in this section so that they can be applied to continuous systems with first order fractional derivatives. The method is used to a dissipative two-dimensional anisotropic harmonic oscillator with a fractional Lagrangian density.

To rewrite the electromagnetic Lagrangian density in fractional form, we start with the definition

$$L = \sum_{i=1}^{2} \frac{1}{2} m\dot{x}_i^2 + \frac{1}{8} m\lambda_i^2 x_i^2 - 2\pi^2 m{v_i}^2 {x_i}^2 \quad (27)$$

and employ the following relationship:

$$\dot{x}_i = {}_aD_t^\alpha \boldsymbol{x_i}$$

The Lagrangian formulation is as follows:

$$L = \sum_{i=1}^{2} \frac{1}{2} m\left[{}_aD_t^\alpha x_i\right]^2 + \frac{1}{8} m\lambda_i^2 x_i^2 - 2\pi^2 m{v_i}^2 {x_i}^2 \quad (28)$$

Taking the derivative with regard to $\boldsymbol{x_i}$ from the Euler-Lagrange equation Eq.(19), we get:

$${}_aD_t^{2\alpha} \boldsymbol{x} + \left(4\pi^2 {v_x}^2 - \frac{\lambda_x^2}{4}\right) x = 0 \quad (29.1)$$

$${}_aD_t^{2\alpha} y + \left(4\pi^2 {v_y}^2 - \frac{\lambda_y^2}{4}\right) y = 0 \quad (29.2)$$

The damped oscillator has an angular frequency of $4\pi^2 {v_x}^2 - \frac{\lambda_x^2}{4}$, where ω is the angular frequency of the conservative system. Furthermore, if $\lambda = 0$, the previously mentioned Lagrangian is simplified to the Lagrangian of simple harmonic motion.

In view of equation (26), we obtain generalized Hamilton equations as:

$${}_aD_t^{2\alpha} \boldsymbol{x} + (\omega_x) x = 0 \quad (30.1)$$

$${}_aD_t^{2\alpha} y + (\omega_y) y = 0 \quad (30.2)$$

These equations are the same as the Euler equation (29). Therefore, both the methods yield the following equation. Eq. (5.) can be written as follows:

$${}_aD_t^{2\alpha} \boldsymbol{x_i} + \left(4\pi^2 {v_i}^2 - \frac{\lambda_i^2}{4}\right) x_i = 0 \quad (31)$$

Substituting ($x_i = \sum_{i=1}^{2} q_i e^{\lambda_i t/2}$.) into (5.1) we get $\sum_{i=1}^{2} (\ddot{q}_i + \lambda_i q_i + 4\pi^2 {v_i}^2 q_i) = 0$

6 The fractional form of the Three-Dimensional Isotropic Oscillator equation in terms of the Poisson bracket formulation.

Hamiltonian mechanics can be expressed in terms of Poisson brackets. In this section, an extension of the Poisson bracket has been described, which is important for generalizing fractional mechanics involving Riemann-Liouville derivatives. The Poisson brackets of Hamilton's equations of motion are then calculated as

$${}_aD_t^\alpha x = \{x, H\} = \left\{ \begin{matrix} x, \frac{1}{2\mathrm{m}}\left[{P_x}^2 + 4\pi^2 m^2 {v_x}^2 x^2 - \frac{1}{4} m\lambda_x^2 x^2\right] + \\ \left[{P_y}^2 + 4\pi^2 m^2 {v_y}^2 y^2 - \frac{1}{4} m\lambda_y^2 y^2\right] \end{matrix} \right\} = \left\{x, \frac{{P_x}^2}{2\mathrm{m}}\right\} = \frac{P_x}{\mathrm{m}} \quad (32)$$

In the same method, other Poisson bracket relations can be found:

$$ {}_aD_t^\alpha y=\{y,H\}=\left\{y,\frac{1}{2\mathrm{m}}\left(\begin{array}{c}\left(P_x{}^2+4\pi^2m^2v_x{}^2x^2-\frac{1}{4}m\lambda_x^2x^2\right)+\\ \left(P_y{}^2+4\pi^2m^2v_y{}^2y^2-\frac{1}{4}m\lambda_y^2y^2\right)\end{array}\right)\right\}=\left\{y,\frac{P_y{}^2}{2\mathrm{m}}\right\}=\frac{P_y}{\mathrm{m}} \tag{33}$$

The given equation is the same as the equation obtained using Poisson brackets (Eq. (21) in fractional form.

If $\{x,P_x\}=\{y,P_y\}=1$, similarly

$$ {}_aD_t^\alpha P_x=\{P_x,H\}=\left\{P_x,\frac{1}{2\mathrm{m}}\left(\begin{array}{c}\left(P_x{}^2+4\pi^2m^2v_x{}^2x^2-\frac{1}{4}m\lambda_x^2x^2\right)+\\ \left(P_y{}^2+4\pi^2m^2v_y{}^2y^2-\frac{1}{4}m\lambda_y^2y^2\right)\end{array}\right)\right\}=$$

$$\left\{P_x,\frac{4\pi^2m^2v_x{}^2x^2}{2\mathrm{m}}-\frac{1}{4}m\lambda_x^2x^2\right\}=-4\pi^2mv_x{}^2x+\frac{1}{4}m\lambda_x^2x \tag{34}$$

Also

$$ {}_aD_t^\alpha P_y=\{P_y,H\}=\left\{P_y,\frac{1}{2\mathrm{m}}\left(\begin{array}{c}\left(P_x{}^2+4\pi^2m^2v_x{}^2x^2-\frac{1}{4}m\lambda_x^2x^2\right)+\\ \left(P_y{}^2+4\pi^2m^2v_y{}^2y^2-\frac{1}{4}m\lambda_y^2y^2\right)\end{array}\right)\right\}=$$

$$\left\{P_y,\frac{4\pi^2m^2v_y{}^2y^2}{2\mathrm{m}}-\frac{1}{4}m\lambda_y^2y^2\right\}=-4\pi^2mv_y{}^2y+\frac{1}{4}m\lambda_y^2y \tag{35}$$

The common quantization rules that can be utilized for canonical quantization are as follows:

$[x,P_x]=[x,P_x]=i\hbar;\ [x,x]=[y,y]=[P_x,P_x]=[P_y,P_y]=0$

Position and momentum operators are indicated by $x,y,P_x$and P_y, respectively. Then the Heisenberg equations are read as:

$$D^1\langle x\rangle=\frac{i}{\hbar}\langle\left[\frac{1}{2\mathrm{m}}\left(\left(P_x{}^2+4\pi^2m^2v_x{}^2x^2-\frac{1}{4}m\lambda_x^2x^2\right)+\left(P_y{}^2+4\pi^2m^2v_y{}^2y^2-\frac{1}{4}m\lambda_y^2y^2\right)\right),x\right]\rangle$$

$$=\frac{i}{\hbar}\langle\left[\frac{P_x{}^2}{2\mathrm{m}},x\right]\rangle=\frac{\langle P_x\rangle}{m} \tag{36}$$

And

$$D^1\langle y\rangle=\frac{i}{\hbar}\langle\left[\frac{1}{2\mathrm{m}}\left(\left(P_x{}^2+4\pi^2m^2v_x{}^2x^2-\frac{1}{4}m\lambda_x^2x^2\right)+\left(P_y{}^2+4\pi^2m^2v_y{}^2y^2-\frac{1}{4}m\lambda_y^2y^2\right)\right),y\right]\rangle=\frac{i}{\hbar}\langle\left[\frac{P_y{}^2}{2\mathrm{m}},y\right]\rangle=\frac{\langle P_y\rangle}{m} \tag{37}$$

The Eq.(37)agree with Eq.(33).

$$D^1\langle P_x\rangle=\frac{i}{\hbar}\langle\left[\frac{1}{2\mathrm{m}}\left(\left(P_x{}^2+4\pi^2m^2v_x{}^2x^2-\frac{1}{4}m\lambda_x^2x^2\right)+\left(P_y{}^2+4\pi^2m^2v_y{}^2y^2-\frac{1}{4}m\lambda_y^2y^2\right)\right),P_x\right]\rangle=\frac{i}{\hbar}\langle[2\pi^2mv_x{}^2x^2,P_x]\rangle=-4\pi^2mv_x{}^2\langle x\rangle+\frac{1}{4}m\lambda_x^2\langle x\rangle \tag{38}$$

And

$$D^1 \langle P_y \rangle = \frac{i}{\hbar}\langle\Big[\frac{1}{2\mathrm{m}}\Big(\Big({P_x}^2 + 4\pi^2 m^2 {v_x}^2 x^2 - \frac{1}{4} m\lambda_x^2 x^2\Big) + \Big({P_y}^2 + 4\pi^2 m^2 {v_y}^2 y^2 - \frac{1}{4} m\lambda_y^2 y^2\Big)\Big), P_y\Big]\rangle = \frac{i}{\hbar}\langle\big[2\pi^2 m {v_y}^2 y^2, P_y\big]\rangle = -4\pi^2 m {v_y}^2 \langle y\rangle + \frac{1}{4} m\lambda_y^2 \langle y\rangle \quad (39)$$

Which also agrees with equation (35).

The classical equations of motion are obeyed by the expectation values, as expressed by equations (38,39). These equations are for predicted values $(x, y)$and hold true for every particle state. Then, using equations (33,35), we have

$${}_aD_t^{2\alpha} x = {}_aD_t^{2\alpha} \frac{P_x}{m} \qquad \Longrightarrow {}_aD_t^{2\alpha} x + \Big(4\pi {v_x}^2 - \frac{1}{4}\lambda_x^2\Big) x = 0 \quad (40)$$

$${}_aD_t^{2\alpha} y = {}_aD_t^{2\alpha} \frac{P_y}{m} \qquad \Longrightarrow {}_aD_t^{2\alpha} y + \Big(4\pi {v_y}^2 - \frac{1}{4}\lambda_y^2\Big) y = 0 \quad (41)$$

This is in full agreement with equation (5) therefore,$\{x, P_x\} = \{y, P_y\} = 1$ implies $[x, P_x] = [y, P_y] = i\hbar$.

7 Applications of the Fractional Harmonic Oscillator with Fractional Derivative

In this part, the Harmonic Oscillator will be studied using fractional calculus. A few examples are provided below.

- This method can also be used to find the relationship between the order of fractional differentiation in an equation of motion and an oscillator's ω_i-factor. The fractional calculus approach looks to be more appropriate for the stated system's physical characteristics. The use of fractional derivatives produced good agreement with the model, indicating that the fraction oscillator template may be used to represent substantially damped vibrations.
- This technique can also be used to investigate how the fractional-order derivative influences the dissipative processes in other oscillatory physical systems, such as the order fractional Harmonic Oscillator.
- The fractional derivative technique combines the use of fractional calculus skills to describe free vibration with damping. The fractional calculus has various applications in the modern theory of dissipative process modeling. A fractional oscillator is an oscillator whose motion equation includes fractional derivatives (or integrals).
- The use of fractional derivatives to the study of the effect of classical chaos on the harmonic oscillator is a fundamental tool that is widely used in all branches of Physics to comprehend more realistic systems, from classical to quantum and relativistic regimes. We know that the harmonic oscillator is integrable in Newtonian mechanics, whether driven, damped, or multidimensional. On the other hand, such an approach can be used to investigate how relativistic, one-dimensionally driven oscillators exhibit chaotic behavior.

8 Conclusions

In this paper, we analyze the fractional Hamiltonian using the Riemann-Liouville derivative and derive the Euler equations. Poisson brackets are an important aspect of Hamiltonian mechanics. The entire Hamiltonian mechanics can be stated in terms of Poisson-bracket. In light of this, a generalization of the Poisson-bracket (fractional version) is proposed. Hamilton's canonical equations (fractional case) have been represented using fractional Poisson brackets. A fractional dissipative two-dimensional anisotropic harmonic oscillator is also given in terms of Poisson brackets.

References

[1]. S.G. Samko, Fractional integrals and derivatives, theory and applications, Minsk; Nauka I Tekhnika., 1987.

[2]. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Fractional differential equations: A emergent field in applied and mathematical sciences, in: Factorization, singular operators and related problems, Springer., 151-173, 2003.

[3]. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, elsevier., 2006.

[4] J. Tenreiro Machado, A probabilistic interpretation of the fractional-order differentiation, Fractional Calculus and applied Analysis, 6 (2003) 73-79.

[5]. J. Machado, Discrete-time fractional-order controllers, Fractional Calculus and Applied Analysis., **4**,47-66, 2001 .

[6]. C.F. Lorenzo, T.T. Hartley, Fractional trigonometry and the spiral functions, Nonlinear Dynamics., **38**, 23-60, 2004.

[7]. O.P. Agrawal, Application of fractional derivatives in thermal analysis of disk brakes, Nonlinear Dynamics., **38**,191-206, 2004.

[8]. M.F. Silva, J.T. Machado, A.M. Lopes, Modelling and simulation of artificial locomotion systems, Robotica., **23**, 595-606, 2005.

[9]. O.P. Agrawal, A general formulation and solution scheme for fractional optimal control problems, Nonlinear Dynamics., **38**, 323-337, 2004.

[10]. R. Gorenflo, F. Mainardi, Fractional calculus: integral and differential equations of fractional order, arXiv preprint arXiv:0805.3823, (2008).

[11]. F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena, Chaos, Solitons & Fractals., **7**, 1461-1477, 1996.

[12]. F. Mainardi, G. Pagnini, R. Gorenflo, Mellin transform and subordination laws in fractional diffusion processes, arXiv preprint math/0702133, (2007).

[13]. A. Raspini, Simple solutions of the fractional Dirac equation of order 2/3, Physica Scripta., **64**, 20, 2001.

[14]. E.M. Rabei, T.S. Alhalholy, A. Rousan, Potentials of arbitrary forces with fractional derivatives, International journal of modern physics A., **19**, 3083-3092, 2004.

[15] M. Naber, Time fractional Schrödinger equation, Journal of Mathematical Physics., **45**, 3339-3352, 2004.

[16] S. Lim, S. Muniandy, Stochastic quantization of nonlocal fields, Physics Letters A., **324**, 396-405, 2004.

[17] R.L. Magin, Fractional calculus in bioengineering, part 1, Critical Reviews™ in Biomedical Engineering., **32**, 2004.

[18] R.L. Magin, Fractional calculus in bioengineering, part 1, Critical Reviews™ in Biomedical Engineering., 32, 2004.

[19] Magin, R.L.: Fractional calculus in bioengineering, part 3. Critic. Rev. Biomed. Eng., **32(3/4)**, 194–377, 2004.

[20] F. Riewe, Nonconservative lagrangian and hamiltonian mechanics, Physical Review E., **53**,1890, 1996.

[21] F. Riewe, Mechanics with fractional derivatives, Physical Review E., **55**, 3581, 1997.

[22] O.P. Agrawal, Formulation of Euler–Lagrange equations for fractional variational problems, Journal of Mathematical Analysis and Applications., **272**, 368-379, 2002.

[23] M. Klimek, Fractional sequential mechanics—models with symmetric fractional derivative, Czechoslovak Journal of Physics., **51** ,1348-1354, 2001.

[24] M. Klimek, Lagrangean and Hamiltonian fractional sequential mechanics, Czechoslovak Journal of Physics., **52**,1247-1253, 2002.

[26] D. Baleanu, S.I. Muslih, About Lagrangian formulation of classical fields within Riemann-Liouville fractional derivatives, in: International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, 2005, pp. 1457-1464.27. Muslih, S., Baleanu, D.: Hamiltonian formulation of systems with linear velocities within Riemann–Liouville fractional derivatives. J. Math. Anal. Appl., **304(3)**, 599–606, 2005.

[28] E.M. Rabei, K.I. Nawafleh, R.S. Hijjawi, S.I. Muslih, D. Baleanu, The Hamilton formalism with fractional derivatives, Journal of Mathematical Analysis and Applications., **327**, 891-897, 2007.

[29] S.I. Muslih, D. Baleanu, Formulation of Hamiltonian equations for fractional variational problems, Czechoslovak Journal of Physics, 55 (2005) 633-642.

[30] M Alawaideh, Y., M Alkhamiseh, B., & SM Lotayif, M. (2022). Reformulation Complex Scalar Field Interacting With the Electromagnetic Lagrangian Density by Riemann-Liouville Factional Derivative. Information Sciences Letters., **11(1)**, 8, 2022.

[31] D. Baleanu, T. Avkar, Lagrangians with linear velocities within Riemann-Liouville fractional derivatives, arXiv preprint math-ph/0405012, (2004).

[32] D. Baleanu, Constrained systems and Riemann-Liouville fractional derivative, in: Proceedings of 1st IFAC Workshop on Fractional Differentiation and its Applications., 597-602, 2004.

[33] D. Baleanu, About fractional calculus of singular Lagrangians, in: Second IEEE International Conference on Computational Cybernetics, 2004. ICCC 2004., IEEE., 379-384, 2004.

[34] D. Baleanu, S. Muslih, New trends in fractional quantization method, Intelligent Systems at the Service of Mankind., **2**, 2005.

[35] D. Baleanu, Constrained systems and Riemann-Liouville fractional derivative, in: Proceedings of 1st IFAC Workshop on Fractional Differentiation and its Applications., 597-602, 2004.

[36] A. Jalloul, J.-C. Trigeassou, K. Jelassi, P. Melchior, Fractional order modeling of rotor skin effect in induction machines, Nonlinear Dynamics., **73**, 801-813, 2013.

[36] D. Baleanu, S.I. Muslih, About fractional supersymmetric quantum mechanics, Czechoslovak Journal of Physics, **55** , 1063-1066, 2005.

[37] M. Henneaux, C. Teitelboim, Quantization of gauge systems, Princeton university press., 2020.

[38] F. Mainardi, Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models, World Scientific, 2010.

[39] R. Koeller, Applications of fractional calculus to the theory of viscoelasticity., (1984).

[40] P. Łabędzki, R. Pawlikowski, A. Radowicz, Axial vibration of bars using fractional viscoelastic material models, Vibrations in Physical Systems., **29** , 2018.

[41] O. Martin, Nonlinear dynamic analysis of viscoelastic beams using a fractional rheological model, Applied Mathematical Modelling., **43** , 351-359, 2017.