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The S-Transform on Sobolev Spaces

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Abstract: In this paper, we study the Parseval's formula and inversion formula for the S-transform analogous to the wavelet transform. We also investigate the boundedness and continuity results for the S-transform on the Sobolev spaces.

Keywords: Distribution, Sobolev spaces, S-transform, Fourier transformation, Parseval's formula.

1 Introduction

The S-transform was first used by Stockwell *et al.*[1] in 1996 and since then has seen several interesting applications. It has been successfully used in several disciplines including geophysics [2,3], medicine [4,5], hydrogeology [6] and mechanical engineering [7]. If $\omega(t,\xi)$ is a window function, then the continuous S-transform of f with respect to ω is defined as [8]

$$(S_\omega f)(\tau,\xi) = \int_{\mathbb{R}^n} f(t)\ \omega(\tau - t,\xi)\ e^{-i2\pi\langle\xi,t\rangle}\ dt, \qquad (1)$$

for $\tau,\xi \in \mathbb{R}^n$, provided the integral exists.

In signal analysis, at least in dimension $n = 1$, $\mathbb{R}^{2n}$ is called the time-frequency plane, and in physics $\mathbb{R}^{2n}$ is called the phase space. Equation(1) can be rewritten as a convolution

$$(S_\omega f)(\tau,\xi) = \left(f(\cdot)e^{-i2\pi\langle\xi,\cdot\rangle} * \omega(\cdot,\xi)\right)(\tau). \qquad (2)$$

Applying the convolution property for the Fourier transform in (2), we obtain

$$(S_\omega f)(\tau,\xi) = \mathscr{F}^{-1}\left\{\hat{f}(\cdot+\xi)\ \hat{\omega}(\cdot,\xi)\right\}(\tau), \qquad (3)$$

where $\hat{f}(\eta) = (\mathscr{F}f)(\eta) = \int_{\mathbb{R}^n} f(t)\ e^{-i2\pi\langle\eta,t\rangle}dt$ is the Fourier transform of f.

2 Parseval's relation for the S-transform

In this section, we give the Parseval's formula for the S-transform which is more general and practicable than that given in [9].

Now, assume that $\xi \in \mathbb{R}^n \setminus \{\mathbf{0}\} = \mathbb{R}^n_0$. The space $\mathscr{L}^2\left(\mathbb{R}^n \times \mathbb{R}^n_0, \frac{d\tau d\xi}{(1+|\xi|)^{n+1}}\right)$ is defined as the set of function $F(\tau,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^n_0$ such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n_0} |F(\tau,\xi)|^2 \frac{d\tau d\xi}{(1+|\xi|)^{n+1}} < \infty. \qquad (4)$$

This is a Hilbert space with inner product

$$\langle F,G\rangle_{\mathscr{L}^2} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n_0} F(\tau,\xi)\ \overline{G(\tau,\xi)}\ d\mu(\tau,\xi), \qquad (5)$$

where $d\mu(\tau,\xi) = \frac{d\tau d\xi}{(1+|\xi|)^{n+1}}$.

Theorem 1(Parseval's formula). *Suppose $\omega_1(\cdot,\xi),\omega_2(\cdot,\xi)$ are window functions such that*

$$0 < C_{\omega_1,\omega_2} := \int_{\mathbb{R}^n_0} \frac{\widehat{\omega}_1(\cdot,\xi)\ \overline{\widehat{\omega}_2(\cdot,\xi)}}{(1+|\xi|)^{n+1}}\ d\xi < \infty. \qquad (6)$$

Let $f,g \in L^2(\mathbb{R}^n)$ and $(S_{\omega_1} f)$, $(S_{\omega_2} g)$ be the S-transforms of f and g, respectively. Then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n_0} (S_{\omega_1} f)(\tau,\xi)\overline{(S_{\omega_2} g)(\tau,\xi)}\ \frac{d\tau d\xi}{(1+|\xi|)^{n+1}} = C_{\omega_1,\omega_2}\int_{\mathbb{R}^n} f(t)\overline{g(t)}\ dt. \qquad (7)$$

This immediately implies the Plancherel formula

$$\|S_\omega f\|_{\mathscr{L}^2(\mathbb{R}^n\times\mathbb{R}^n_0)} = (C_{\omega_1,\omega_2})^{1/2}\ \|f\|_{L^2(\mathbb{R}^n)}. \qquad (8)$$

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*Proof.*Using (3), we have

$$\int_{\mathbb{R}^n}(S_{\omega_1}f)(\tau,\xi)\overline{(S_{\omega_2}g)(\tau,\xi)}d\tau$$
$$=\int_{\mathbb{R}^n} d\tau\,(S_{\omega_1}f)(\tau,\xi)\,\overline{\int_{\mathbb{R}^n} d\eta\,\hat{g}(\eta+\xi)\,\hat{\omega}_2(\eta,\xi)\,e^{i2\pi\langle\tau,\eta\rangle}}$$
$$=\int_{\mathbb{R}^n} d\eta\,\overline{\hat{g}(\eta+\xi)}\,\overline{\hat{\omega}_2(\eta,\xi)}\int_{\mathbb{R}^n} d\tau\,(S_{\omega_1}f)(\tau,\xi)\,e^{-i2\pi\langle\tau,\eta\rangle}$$
$$=\int_{\mathbb{R}^n} d\eta\,\overline{\hat{g}(\eta+\xi)}\,\overline{\hat{\omega}_2(\eta,\xi)}\,\hat{f}(\eta+\xi)\,\hat{\omega}_1(\eta,\xi)$$
$$=\int_{\mathbb{R}^n} dx\,\hat{\omega}_1(x-\xi,\xi)\,\overline{\hat{\omega}_2(x-\xi,\xi)}\,\overline{\hat{g}(x)}\,\hat{f}(x).$$

Hence,

$$\int_{\mathbb{R}^n_0}\int_{\mathbb{R}^n}(S_{\omega_1}f)(\tau,\xi)\overline{(S_{\omega_2}g)(\tau,\xi)}\frac{d\tau d\xi}{(1+|\xi|)^{n+1}}$$
$$=\int_{\mathbb{R}^n_0}\int_{\mathbb{R}^n}\hat{\omega}_1(x-\xi,\xi)\,\overline{\hat{\omega}_2(x-\xi,\xi)}\,\overline{\hat{g}(x)}\,\hat{f}(x)\frac{dxd\xi}{(1+|\xi|)^{n+1}}$$
$$=\int_{\mathbb{R}^n} dx\,\hat{f}(x)\,\overline{\hat{g}(x)}$$
$$\times\left(\int_{\mathbb{R}^n_0}\hat{\omega}_1(x-\xi,\xi)\,\overline{\hat{\omega}_2(x-\xi,\xi)}\,\frac{d\xi}{(1+|\xi|)^{n+1}}\right)$$
$$=C_{\omega_1,\omega_2}\int_{\mathbb{R}^n} dx\,\hat{f}(x)\,\overline{\hat{g}(x)}$$
$$=C_{\omega_1,\omega_2}\int_{\mathbb{R}^n} f(t)\,\overline{g(t)}\,dt.$$

The proof of (8) is obvious.

Theorem 2(Inversion formula). *If $f\in L^2(\mathbb{R}^n)$ and window function ω satisfies the condition*

$$0<C_\omega:=\int_{\mathbb{R}^n_0}\frac{|\widehat{\omega}(\cdot,\xi)|^2}{(1+|\xi|)^{n+1}}\,d\xi<\infty. \tag{9}$$

Then

$$f(t)=\frac{1}{C_\omega}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n_0}(S_\omega f)(\tau,\xi)\,\overline{\omega(\tau-t,\xi)}\,e^{i2\pi\langle t,\xi\rangle}\times\frac{d\tau d\xi}{(1+|\xi|)^{n+1}}.$$

*Proof.*By the above theorem we can write

$$C_\omega\int_{\mathbb{R}^n} f(t)\,\overline{g(t)}\,dt$$
$$=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n_0}\frac{d\tau d\xi}{(1+|\xi|)^{n+1}}\,(S_\omega f)(\tau,\xi)\overline{(S_\omega g)(\tau,\xi)}$$
$$=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n_0}\frac{d\tau d\xi}{(1+|\xi|)^{n+1}}\,(S_\omega f)(\tau,\xi)$$
$$\times\overline{\int_{\mathbb{R}^n} g(t)\,\omega(\tau-t,\xi)\,e^{-i2\pi\langle\xi,t\rangle}\,dt}$$
$$=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n_0}\frac{d\tau d\xi}{(1+|\xi|)^{n+1}}\,(S_\omega f)(\tau,\xi)$$
$$\times\int_{\mathbb{R}^n} dt\,\overline{g(t)}\,\overline{\omega(\tau-t,\xi)}\,e^{i2\pi\langle\xi,t\rangle}$$
$$=\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n_0}\frac{d\tau d\xi}{(1+|\xi|)^{n+1}}\,(S_\omega f)(\tau,\xi)\,\overline{\omega(\tau-t,\xi)}\right.$$
$$\left.e^{i2\pi\langle\xi,t\rangle}\right)\times\overline{g(t)}dt.$$

Therefore, we get

$$f(t)=\frac{1}{C_\omega}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n_0}(S_\omega f)(\tau,\xi)\,\overline{\omega(\tau-t,\xi)}\,e^{i2\pi\langle t,\xi\rangle}\times\frac{d\tau d\xi}{(1+|\xi|)^{n+1}}.$$

The S-transform and fractional S-transform have been studied on distribution spaces by Singh *et al.* [10, 11, 12, 13, 14, 15, 16, 17].

3 S-transform on Sobolev spaces

Sobolev space play an important role in the theory of partial differential equations, specially those related to quantum mechanics or physics. The theory of Sobolev spaces were facilitated by the theory of distributions and Fourier analysis. We now recall the definitions of Sobolev spaces.

Definition 1(The Sobolev space $W^{m,p}(\Omega)$). *Let $\Omega\subset\mathbb{R}^n$ be open, $1\le p\le\infty$, and m be a nonnegative integer. The Sobolev space $W^{m,p}(\Omega)$ is defined by*

$$W^{m,p}(\Omega)=\left\{f\in\mathscr{D}'(\Omega):D^\alpha f\in L^p(\Omega)\ \text{for all}\ |\alpha|\le m\right\}, \tag{10}$$

and equipped with the norm

$$\|f\|_{m,p}=\left(\sum_{|\alpha|\le m}\|D^\alpha f\|_{L^p}^p\right)^{1/p}\quad \text{for}\ 1\le p<\infty, \tag{11}$$

and

$$\|f\|_{m,\infty}=\sup_{|\alpha|\le m}\|D^\alpha f\|_{L^\infty} \tag{12}$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n), \alpha_i \geq 0$, α_i *integers*, $|\alpha| = (\alpha_1 + \alpha_2 + \cdots + \alpha_n)$, *and partial derivatives* $D^\alpha = \prod_{j=i}^{n} \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}$ *in the distributional sense.*

Definition 2(The Sobolev space $H^s(\mathbb{R}^n)$). *For* $s \in \mathbb{R}$, *the Sobolev space* $H^s(\mathbb{R}^n)$ *is defined as the set of all ultradistributions* $f \in \mathscr{S}'$ *such that*

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}^n} (1+|\eta|^2)^s \, |\hat{f}(\eta)|^2 d\eta\right)^{1/2} < \infty. \tag{13}$$

Theorem 3.*Let* $\Omega \subset \mathbb{R}^n$ *be open. Suppose* $\omega(\cdot,\xi) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, *then for fixed finite* $\xi \in \mathbb{R}_0^n$, *the operator* $S_\omega : W^{m,p}(\Omega) \to W^{m,p}(\Omega)$ *is continuous and*

$$\begin{aligned} \|(S_\omega f)(\cdot,\xi)\|_{W^{m,p}} &= \left(\sum_{|\alpha|\leq m} \|D_\tau^\alpha (S_\omega f)(\tau,\xi)\|_{L^p}^p\right)^{1/p} \\ &\leq \|\omega(\cdot,\xi)\|_{L^1} \left\|M_{-\xi} f(\cdot)\right\|_{W^{m,p}}, \end{aligned} \tag{14}$$

where $M_{-\xi} f(\cdot) = e^{-i2\pi\langle\xi,\cdot\rangle} f(\cdot)$.

*Proof.*If $f \in W^{m,p}(\Omega)$ then $D^\alpha f \in L^p(\Omega)$ for all $|\alpha| \leq m$, and hence for fixed finite ξ, $D^\alpha M_{-\xi} f \in L^p(\Omega)$ for all $|\alpha| \leq m$. Therefore, by using Young's inequality we have

$$\begin{aligned} \|D_\tau^\alpha (S_\omega f)(\tau,\xi)\|_{L^p}^p &= \left\|D_\tau^\alpha (M_{-\xi} f(\cdot) * \omega(\cdot,\xi))(\tau)\right\|_{L^p}^p \\ &\leq \left\|D_\tau^\alpha (M_{-\xi} f)(\tau)\right\|_{L^p}^p \|\omega(\cdot,\xi)\|_{L^1}^p . \end{aligned}$$

In particular,

$$\|(S_\omega f)(\cdot,\xi)\|_{L^p}^p \leq \left\|M_{-\xi} f\right\|_{L^p}^p \|\omega(\cdot,\xi)\|_{L^1}^p .$$

Furthermore,

$$\begin{aligned} &\|(S_\omega f)(\cdot,\xi)\|_{W^{m,p}} \\ &= \left(\sum_{|\alpha|\leq m} \|D_\tau^\alpha (S_\omega f)(\tau,\xi)\|_{L^p}^p\right)^{1/p} \\ &\leq \left(\sum_{|\alpha|\leq m} \left\|D_\tau^\alpha (M_{-\xi} f)(\tau)\right\|_{L^p}^p \|\omega(\cdot,\xi)\|_{L^1}^p\right)^{1/p} \\ &= \|\omega(\cdot,\xi)\|_{L^1} \left(\sum_{|\alpha|\leq m} \left\|D_\tau^\alpha (M_{-\xi} f)(\tau)\right\|_{L^p}^p\right)^{1/p} \\ &= \|\omega(\cdot,\xi)\|_{L^1} \left\|M_{-\xi} f(\cdot)\right\|_{W^{m,p}} . \end{aligned}$$

The generalized Sobolev spaces and weighted Sobolev spaces are very powerful tools to solve the degenerate elliptic equations. Pathak *et al.* [18] have studied multiwavelets in the generalized Sobolev space. Also construction of wavelets in the generalized Sobolev space can be found in [19]. Section 4 and 5 are devoted to studied the S-transform in the weighted Sobolev spaces and generalized Sobolev spaces.

4 The S-transform on weighted Sobolev space

Definition 3(The weighted L^p space). *Let* k *be a weight function (i.e., a locally integrable function on* $\mathbb{R}^n$ *such that* $k(x) > 0$ *for a.e.* $x \in \mathbb{R}^n$*) and* $\Omega \subset \mathbb{R}^n$ *be open. For* $1 \leq p < \infty$, *the weighted* $L_k^p(\Omega)$ *space is defined as the set of all measurable function* f *on* Ω *such that*

$$\|f\|_{L_k^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, k(x) dx\right)^{1/p} < \infty. \tag{15}$$

Definition 4(The weighted Sobolev space $W_k^{m,p}(\Omega)$). *Let* $\Omega \subset \mathbb{R}^n$ *be open,* $1 \leq p < \infty$, *and* m *be a nonnegative integer. The weighted Sobolev space* $W_k^{m,p}(\Omega)$ *is defined as the set of all* $f \in \mathscr{D}'(\Omega)$ *with distributional derivatives* $D^\alpha f \in L_k^p(\Omega)$ *for* $|\alpha| \leq m$. *The norm of* f *in* $W_k^{m,p}(\Omega)$ *is given by*

$$\|f\|_{W_k^{m,p}} = \left(\sum_{|\alpha|\leq m} \|D^\alpha f\|_{L_k^p}^p\right)^{1/p}, \tag{16}$$

The space $W_k^{m,p}(\Omega)$ reduces to Sobolev space $W^{m,p}(\Omega)$ for $k = 1$.

Theorem 4.*Let* $\Omega \subset \mathbb{R}^n$ *be open. Suppose* $\omega(\cdot,\xi) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ *and* k *be a weight function. Then for fixed finite* $\xi \in \mathbb{R}_0^n$, *the operator* $S_\omega : W_k^{m,p}(\Omega) \to W_k^{m,p}(\Omega)$ *is continuous and*

$$\begin{aligned} \|(S_\omega f)(\cdot,\xi)\|_{W_k^{m,p}} &= \left(\sum_{|\alpha|\leq m} \|D^\alpha (S_\omega f)(\cdot,\xi)\|_{L_k^p}^p\right)^{1/p} \\ &\leq \|\omega(\cdot,\xi)\|_{L^1} \left\|M_{-\xi} f(\cdot)\right\|_{W_k^{m,p}}, \end{aligned} \tag{17}$$

where $M_{-\xi} f(\cdot) = e^{-i2\pi\langle\xi,\cdot\rangle} f(\cdot)$.

*Proof.*If $f \in W_k^{m,p}(\Omega)$ then $D^\alpha f \in L_k^p(\Omega)$ for all $|\alpha| \leq m$, and hence for fixed finite ξ, $D^\alpha M_{-\xi} f \in L_k^p(\Omega)$ for all $|\alpha| \leq m$. Therefore,

$$\begin{aligned} \|D_\tau^\alpha (S_\omega f)(\tau,\xi)\|_{L_k^p}^p &= \left\|D_\tau^\alpha (M_{-\xi} f(\cdot) * \omega(\cdot,\xi))(\tau)\right\|_{L_k^p}^p \\ &\leq \left\|D_\tau^\alpha (M_{-\xi} f)(\tau)\right\|_{L_k^p}^p \|\omega(\cdot,\xi)\|_{L^1}^p . \end{aligned}$$

Furthermore,

$$\begin{aligned} &\|(S_\omega f)(\cdot,\xi)\|_{W_k^{m,p}} \\ &= \left(\sum_{|\alpha|\leq m} \|D_\tau^\alpha (S_\omega f)(\tau,\xi)\|_{L_k^p}^p\right)^{1/p} \\ &\leq \left(\sum_{|\alpha|\leq m} \left\|D_\tau^\alpha (M_{-\xi} f)(\tau)\right\|_{L_k^p}^p \|\omega(\cdot,\xi)\|_{L^1}^p\right)^{1/p} \\ &= \|\omega(\cdot,\xi)\|_{L^1} \left(\sum_{|\alpha|\leq m} \left\|D_\tau^\alpha (M_{-\xi} f)(\tau)\right\|_{L_k^p}^p\right)^{1/p} \\ &= \|\omega(\cdot,\xi)\|_{L^1} \left\|M_{-\xi} f(\cdot)\right\|_{W_k^{m,p}} . \end{aligned}$$

Corollary 1. *If $\omega_1(\cdot,\xi),\ \omega_2(\cdot,\xi) \in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and $f,g \in W_k^{m,p}(\Omega)$, then for fixed finite $\xi \in \mathbb{R}_0^n$, the following estimate holds*

$$\begin{aligned}&\|(S_{\omega_1} f)(\cdot,\xi) - (S_{\omega_2} g)(\cdot,\xi)\|_{W_k^{m,p}}\\ &\leq \|(\omega_1-\omega_2)(\cdot,\xi)\|_{L^1}\left\|M_{-\xi} f(\cdot)\right\|_{W_k^{m,p}}\\ &+\|\omega_2(\cdot,\xi)\|_{L^1}\left\|M_{-\xi}(f-g)(\cdot)\right\|_{W_k^{m,p}}.\end{aligned}$$

Proof.

$$\begin{aligned}&\|(S_{\omega_1} f)(\cdot,\xi) - (S_{\omega_2} g)(\cdot,\xi)\|_{W_k^{m,p}}\\ &\leq \|(S_{\omega_1} f)(\cdot,\xi) - (S_{\omega_2} f)(\cdot,\xi)\|_{W_k^{m,p}}\\ &+\|(S_{\omega_2} f)(\cdot,\xi) - (S_{\omega_2} g)(\cdot,\xi)\|_{W_k^{m,p}}\\ &=\|(S_{\omega_1-\omega_2} f)(\cdot,\xi)\|_{W_k^{m,p}} + \|(S_{\omega_2}(f-g))(\cdot,\xi)\|_{W_k^{m,p}}\\ &\leq \|(\omega_1-\omega_2)(\cdot,\xi)\|_{L^1}\left\|M_{-\xi} f(\cdot)\right\|_{W_k^{m,p}}\\ &+\|\omega_2(\cdot,\xi)\|_{L^1}\left\|M_{-\xi}(f-g)(\cdot)\right\|_{W_k^{m,p}}.\end{aligned}$$

5 S-transform on generalized Sobolev spaces

We now recall the definition of generalized Sobolev spaces [20,21]. Let $\mathscr{M}$ be the set of continuous real valued function κ on $\mathbb{R}^n$ satisfying the following conditions:

(i)
$$0=\kappa(0)\leq \kappa(\xi+\eta)\leq \kappa(\xi)+\kappa(\eta), \tag{18}$$

(ii)
$$\int_{\mathbb{R}^n}\frac{\kappa(\xi)}{(1+|\xi|)^{n+1}}d\xi\ <\infty, \tag{19}$$

(iii)
$$\kappa(\xi)\geq a+b\,\log(1+|\xi|), \tag{20}$$

for some real number a and positive real number b.

We denote by $\mathscr{M}_c$ the set of all $\kappa\in\mathscr{M}$ such that $\kappa(\xi)=\sigma(|\xi|)$ with σ concave on $[0,\infty)$.

Definition 5. *For $\kappa\in\mathscr{M}_c$, the Björck-space $\mathscr{S}_\kappa(\mathbb{R}^n)$ is the set of all functions $\phi\in L^1(\mathbb{R}^n)$ such that $\phi,\hat\phi\in C^\infty$ and for each multi-index α and each non-negative number λ*

$$p_{\alpha,\lambda}(\phi)=\sup_x e^{\lambda\kappa(x)}\,|D^\alpha\phi(x)|<\infty, \tag{21}$$

and

$$\pi_{\alpha,\lambda}(\phi)=\sup_\xi e^{\lambda\kappa(\xi)}\left|D^\alpha\hat\phi(\xi)\right|<\infty. \tag{22}$$

The dual of $\mathscr{S}_\kappa$ is denoted by $\mathscr{S}'_\kappa$, the elements of which are called ultradistributions. For its various properties, see [20].

Definition 6. *Let $\Omega\subset\mathbb{R}^n$ be open and weight function κ satisfy (18). The space $\mathscr{D}_\kappa(\Omega)$ is defined as the set of all $\phi\in L^1(\mathbb{R}^n)$ such that ϕ has compact support in Ω and*

$$\int_{\mathbb{R}^n}|\hat\phi(\xi)|\,e^{\lambda\kappa(\xi)}d\xi<\infty,\qquad \forall\,\lambda>0. \tag{23}$$

The dual of $\mathscr{D}_\kappa$ is denoted by $\mathscr{D}'_\kappa$ and called Beurling generalized distribution space.

Now, let w be a continuous positive function on $\mathbb{R}^n$ with the following properties. There exist $\lambda>0$ and $C,D,E>0$, such that, for all $\xi,\eta\in\mathbb{R}^n, t\in\mathbb{R}, |t|<1$, and $\kappa\in\mathscr{M}_c$

$$\begin{aligned}w(\xi)&\leq Ce^{\lambda\kappa(\xi)},\\ w(\xi+\eta)&<D(w(\xi)+w(\eta)),\\ w(t\xi)&<Ew(\xi).\end{aligned} \tag{24}$$

Definition 7(The generalized Sobolev space $H_w^\kappa(\mathbb{R}^n)$). *The generalized Sobolev space $H_w^\kappa(\mathbb{R}^n)$ is defined as the set of all ultradistributions $f\in\mathscr{S}'_\kappa$ such that*

$$\|f\|^2_{H_w^\kappa}=\int_{\mathbb{R}^n}|\hat f(\eta)|^2\,w(\eta)d\eta<\infty. \tag{25}$$

The space $H_w^\kappa(\mathbb{R}^n)$ reduces to Sobolev space $H^s(\mathbb{R}^n)$ for $w(\eta)=(1+|\eta|^2)^s,\ s\in\mathbb{R}$.

Theorem 5. *If $w(x)\geq 1$, for all $x\in\mathbb{R}^n$ and $\omega(\cdot,\xi)\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. Then for fixed finite $\xi\in\mathbb{R}_0^n$, the operator $S_\omega : H_w^\kappa\to H_w^\kappa$ is continuous and*

$$\|(S_\omega f)(\cdot,\xi)\|^2_{H_w^\kappa}\leq D\,(1+w(-\xi))\,\|\omega(\cdot,\xi)\|^2_{L^1}\,\|f\|^2_{H_w^\kappa}. \tag{26}$$

Proof. By using (3) and (24) , we have

$$\begin{aligned}&\|(S_\omega f)(\cdot,\xi)\|^2_{H_w^\kappa}\\ &=\int_{\mathbb{R}^n}w(\eta)|\mathscr{F}((S_\omega f)(\cdot,\xi))(\eta)|^2d\eta\\ &=\int_{\mathbb{R}^n}w(\eta)|\hat f(\eta+\xi)\,\hat\omega(\eta,\xi)|^2d\eta\\ &\leq\int_{\mathbb{R}^n}w(\eta)|\hat f(\eta+\xi)|^2\,\|\omega(\cdot,\xi)\|^2_{L^1}d\eta\\ &=\|\omega(\cdot,\xi)\|^2_{L^1}\int_{\mathbb{R}^n}w(x-\xi)|\hat f(x)|^2dx\\ &\leq\|\omega(\cdot,\xi)\|^2_{L^1}\int_{\mathbb{R}^n}D(w(x)+w(-\xi))|\hat f(x)|^2dx\\ &=D\|\omega(\cdot,\xi)\|^2_{L^1}\left(\int_{\mathbb{R}^n}w(x)|\hat f(x)|^2dx+w(-\xi)\int_{\mathbb{R}^n}|\hat f(x)|^2dx\right)\\ &\leq D\|\omega(\cdot,\xi)\|^2_{L^1}\left(\|f\|^2_{H_w^\kappa}+w(-\xi)\|f\|^2_{H_w^\kappa}\right)\\ &=D\,(1+w(-\xi))\|\omega(\cdot,\xi)\|^2_{L^1}\|f\|^2_{H_w^\kappa}.\end{aligned}$$

This completes the proof of the theorem.

Corollary 2. *Let $\omega(\cdot,\xi)\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. For fixed finite $\xi\in\mathbb{R}_0^n$, the operator $S_\omega : H^s\to H^s$ is continuous and*

$$\|(S_\omega f)(\cdot,\xi)\|_{H^s}\leq 2^{s/2}(1+|\xi|^2)^{s/2}\|\omega(\cdot,\xi)\|_{L^1}\|f\|_{H^s}. \tag{27}$$

Proof.

$$\begin{aligned}
&\|(S_\omega f)(\cdot,\xi)\|_{H^s}^2\\
&=\int_{\mathbb{R}^n}(1+|\eta|^2)^s\,|\mathscr{F}((S_\omega f)(\cdot,\xi))(\eta)|^2 d\eta\\
&=\int_{\mathbb{R}^n}(1+|\eta|^2)^s\,|\hat{f}(\eta+\xi)\,\hat{\omega}(\eta,\xi)|^2 d\eta\\
&\le\int_{\mathbb{R}^n}(1+|\eta|^2)^s\,|\hat{f}(\eta+\xi)|^2\,\|\omega(\cdot,\xi)\|_{L^1}^2\,d\eta\\
&=\|\hat{\omega}(\cdot,\xi)\|_{L^1}^2\int_{\mathbb{R}^n}(1+|x-\xi|^2)^s\,|\hat{f}(x)|^2dx\\
&=\|\omega(\cdot,\xi)\|_{L^1}^2\int_{\mathbb{R}^n}(1+|x|^2+|\xi|^2-2\langle x,\xi\rangle)^s\,|\hat{f}(x)|^2dx\\
&\le\|\omega(\cdot,\xi)\|_{L^1}^2\int_{\mathbb{R}^n}(1+2|x|^2+2|\xi|^2)^s\,|\hat{f}(x)|^2dx\\
&\le\|\omega(\cdot,\xi)\|_{L^1}^2\int_{\mathbb{R}^n}2^s(1+|x|^2)^s\,(1+|\xi|^2)^s\,|\hat{f}(x)|^2dx\\
&=2^s(1+|\xi|^2)^s\,\|\omega(\cdot,\xi)\|_{L^1}^2\,\|f\|_{H^s}^2.
\end{aligned}$$

This completes the proof.

Corollary 3. *Suppose $w(x)\ge 1$, for all $x\in\mathbb{R}^n$ and $\omega_1(\cdot,\xi),\ \omega_2(\cdot,\xi)\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. Let $f,g\in H_w^\kappa(\mathbb{R}^n)$, then for fixed $\xi\in\mathbb{R}_0^n$, the following estimate holds*

$$\begin{aligned}
&\|(S_{\omega_1}f)(\cdot,\xi)-(S_{\omega_2}g)(\cdot,\xi)\|_{H_w^\kappa}\\
&\le\sqrt{D\,(1+w(-\xi))}\,\Big(\|(\omega_1-\omega_2)(\cdot,\xi)\|_{L^1}\,\|f\|_{H_w^\kappa}\\
&+\|\omega_2(\cdot,\xi)\|_{L^1}\,\|f-g\|_{H_w^\kappa}\Big).
\end{aligned}$$

Proof.

$$\begin{aligned}
&\|(S_{\omega_1}f)(\cdot,\xi)-(S_{\omega_2}g)(\cdot,\xi)\|_{H_w^\kappa}\\
&\le\|(S_{\omega_1}f)(\cdot,\xi)-(S_{\omega_2}f)(\cdot,\xi)\|_{H_w^\kappa}\\
&+\|(S_{\omega_2}f)(\cdot,\xi)-(S_{\omega_2}g)(\cdot,\xi)\|_{H_w^\kappa}\\
&=\|(S_{\omega_1-\omega_2}f)(\cdot,\xi)\|_{H_w^\kappa}+\|(S_{\omega_2}(f-g))(\cdot,\xi)\|_{H_w^\kappa}.
\end{aligned}\tag{28}$$

Now, using (3) and (24), we have

$$\begin{aligned}
&\|(S_{\omega_1-\omega_2}f)(\cdot,\xi)\|_{H_w^\kappa}^2\\
&=\int_{\mathbb{R}^n}|\hat{f}(\eta+\xi)|^2\,|(\hat{\omega}_1-\hat{\omega}_2)(\eta,\xi)|^2w(\eta)d\eta\\
&\le\|(\omega_1-\omega_2)(\cdot,\xi)\|_{L^1}^2\int_{\mathbb{R}^n}|\hat{f}(\eta+\xi)|^2\,w(\eta)d\eta\\
&=\|(\omega_1-\omega_2)(\cdot,\xi)\|_{L^1}^2\int_{\mathbb{R}^n}|\hat{f}(x)|^2\,w(x-\xi)dx\\
&\le\|(\omega_1-\omega_2)(\cdot,\xi)\|_{L^1}^2\int_{\mathbb{R}^n}|\hat{f}(x)|^2\,D(w(x)+w(-\xi))\,dx\\
&\le D\,(1+w(-\xi))\,\|(\omega_1-\omega_2)(\cdot,\xi)\|_{L^1}^2\,\|f\|_{H_w^\kappa}^2.
\end{aligned}\tag{29}$$

Similarly,

$$\begin{aligned}
&\|(S_{\omega_2}(f-g))(\cdot,\xi)\|_{H_w^\kappa}^2\\
&\le D\,(1+w(-\xi))\,\|\omega_2(\cdot,\xi)\|_{L^1}^2\,\|(f-g)\|_{H_w^\kappa}^2.
\end{aligned}\tag{30}$$

Using (29) and (30) in (28), we get the desired result.

Corollary 4. *If $\omega_1(\cdot,\xi),\ \omega_2(\cdot,\xi)\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and $f,g\in H^s(\mathbb{R}^n)$, then for any fixed $\xi\in\mathbb{R}_0^n$, the following estimate holds*

$$\begin{aligned}
&\|(S_{\omega_1}f)(\cdot,\xi)-(S_{\omega_2}g)(\cdot,\xi)\|_{H^s}\\
&\le\ 2^{s/2}\,(1+|\xi|^2)^{s/2}\Big(\|(\omega_1-\omega_2)(\cdot,\xi)\|_{L^1}\,\|f\|_{H^s}\\
&+\|\omega_2(\cdot,\xi)\|_{L^1}\,\|f-g\|_{H^s}\Big).
\end{aligned}$$

Definition 8. *We define the space $\mathscr{L}_w^p$ of all measurable functions f on $\mathbb{R}^n\times\mathbb{R}_0^n$ such that*

$$\|f\|_{\mathscr{L}_w^p}=\left(\int_{\mathbb{R}_0^n}\|f(\cdot,\xi)\|_{H_w^\kappa}^p\,\frac{d\xi}{(1+|\xi|^2)^{n+1}}\right)^{1/p}<\infty,$$

for $1\le p<\infty,\xi\in\mathbb{R}_0^n$.

Theorem 6. *Suppose $w(x)\ge 1$, for all $x\in\mathbb{R}^n$ and $\omega(\cdot,\xi)\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ such that*

$$C_{\omega,p}:=\int_{\mathbb{R}_0^n}w(-\xi)\,\frac{|\hat{\omega}(\cdot,\xi)|^p}{(1+|\xi|^2)^{n+1}}d\xi<\infty.$$

If $S_\omega f(\tau,\xi)$ is the S-transform of the function $f\in H_w^\kappa(\mathbb{R}^n)$, then we have the following estimate holds

$$\|S_\omega f\|_{\mathscr{L}_w^p}\le(2D\,C_{\omega,p})^{1/p}\,\|f\|_{H_w^\kappa}.$$

Proof.

$$\begin{aligned}
&\|(S_\omega f)(\cdot,\xi)\|_{H_w^\kappa}^p\\
&=\int_{\mathbb{R}^n}|\mathscr{F}((S_\omega f)(\cdot,\xi))(\eta)|^p\,w(\eta)\,d\eta\\
&=\int_{\mathbb{R}^n}|\hat{f}(\eta+\xi)\hat{\omega}(\eta,\xi)|^p\,w(\eta)\,d\eta\\
&=\int_{\mathbb{R}^n}|\hat{f}(x)\hat{\omega}(x-\xi,\xi)|^p\,w(x-\xi)\,dx\\
&\le\int_{\mathbb{R}^n}|\hat{f}(x)|^p\,|\hat{\omega}(x-\xi,\xi)|^p\,D(w(x)+w(-\xi))\,dx.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&\|S_\omega f\|^p_{\mathscr{L}^p_w} \\
&= \int_{\mathbb{R}^n_0} \|(S_\omega f)(\cdot,\xi)\|^p_{H^\kappa_w} \frac{d\xi}{(1+|\xi|^2)^{n+1}} \\
&\le \int_{\mathbb{R}^n_0}\int_{\mathbb{R}^n} |\hat{f}(x)|^p \, |\hat{\omega}(x-\xi,\xi)|^p \, D(w(x)+w(-\xi)) \\
&\times \frac{dx d\xi}{(1+|\xi|^2)^{n+1}} \\
&= D\int_{\mathbb{R}^n} |\hat{f}(x)|^p \, w(x) \left(\int_{\mathbb{R}^n_0} \frac{|\hat{\omega}(x-\xi,\xi)|^p}{(1+|\xi|^2)^{n+1}} d\xi\right) dx \\
&+ D\int_{\mathbb{R}^n} |\hat{f}(x)|^p \left(\int_{\mathbb{R}^n_0} w(-\xi) \, \frac{|\hat{\omega}(x-\xi,\xi)|^p}{(1+|\xi|^2)^{n+1}} d\xi\right) dx \\
&\le 2D\, C_{\omega,p} \, \|f\|^p_{H^\kappa_w}.
\end{aligned}$$

Corollary 5.*Suppose $\omega(\cdot,\xi) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that*

$$C_{\omega,2} := \int_{\mathbb{R}^n_0} \frac{|\hat{\omega}(\cdot,\xi)|^2}{(1+|\xi|^2)^{n-s+1}} d\xi < \infty\,; \qquad 0 < s \le n.$$

If $S_\omega f(\tau,\xi)$ is the S-transform of the function $f \in H^s(\mathbb{R}^n)$, then for $w(\eta) = (1+|\eta|^2)^s$ and for any $\xi \in \mathbb{R}^n_0$, the following estimate holds

$$\|S_\omega f\|_{\mathscr{L}^2_w} \le (2^s\, C_{\omega,2})^{1/2} \, \|f\|_{H^s}.$$

6 Conclusion

In this paper, we have derived the Parsevals formula and inversion formula for the S-transform in Section 2. In Section 3, we have given some new results for Sobolev spaces by making use of the S-transform that extend the scope of the study of S-transform to a function and distribution spaces.

Dedication

This article is dedicated to late Professor R. S. Pathak.

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