The logistic transformation of real groupoids

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Abstract: In this paper, we introduce the notion of a logistic groupoid on the real numbers \(\mathbb{R}\), and show that, given a groupoid \((\mathbb{R}, *)\) with some conditions, there exists a groupoid \((X, \oplus)\) such that \((\mathbb{R}, *)\) is the logistic groupoid of \((X, \oplus)\).

Keywords: \(d\)-algebra, logistic groupoid, logistically (Jordan, (anti-)associative.

1. Introduction

The general study of binary operations on sets has produced a substantial literature which considers various types of structures via sets of axioms, such as \(BCK\)-algebras, \(BCI\)-algebras, pseudo-\(BCK\)-algebras, \(d\)-algebras, \(MV\)-algebras, \(BCK\)-algebras, and the notion of quasi-\(d\)-algebras is a generalization of \(BCK\)-algebras, and the notion of quasi-\(d\)-algebras is a generalization of \(BCI\)-algebras. The general study of binary operations on sets has produced a substantial literature which considers various types of structures via sets of axioms, such as \(BCK\)-algebras, \(BCI\)-algebras, pseudo-\(BCK\)-algebras, \(d\)-algebras, \(MV\)-algebras, \(BCK\)-algebras, and the notion of quasi-\(d\)-algebras is a generalization of \(BCK\)-algebras, and the notion of quasi-\(d\)-algebras is a generalization of \(BCI\)-algebras.

2. Preliminaries

A (ordinary) \(d\)-algebra \(((6, 7))\) is a non-empty set \(X\) with a constant 0 and a binary operation “\(*\)” satisfying the following axioms:

\[(A)x \ast x = 0,\]
\[(B)0 \ast x = 0,\]
\[(C)x \ast y = 0 \text{ and } y \ast x = 0 \text{ imply } x = y \text{ for all } x, y \in X.\]

A \(BCK\)-algebra is a \(d\)-algebra \(X\) satisfying the following additional axioms:

\[(D)(x \ast y) \ast (x \ast z) \ast (z \ast y) = 0,\]
\[(E)(x \ast (x \ast y)) \ast y = 0 \text{ for all } x, y, z \in X.\]

An algebra \((X, *, 0)\) is said to be a pre-\(d\)-algebra if it satisfies the conditions \((A)\) and \((B)\). An algebra \((X, *, 0)\) is said to be a quasi-\(d\)-algebra if it satisfies the conditions \((A)\) and \((C)\). The notion of \(d\)-algebras is a generalization of \(BCK\)-algebras, and the notion of quasi-\(d\)-algebras is a generalization of \(BCI\)-algebras.

Example 2.1. \((3)\) Let \(X := \{0, 1, 2, \cdots\}\). Define \(x \ast y := 0\) if \(x\) is even. Let \(x \ast y := \max\{x, y\} - \min\{x, y\}\) if \(x\) and \(y\) are both odd and let \(x \ast y := x + y + 1\) if \(x\) is odd and \(y\) is even. Then \(0 \ast x = 0\), since 0 is even. Also, \(x \ast x = 0\) if \(x\) is even, and \(x \ast x = \max\{x, x\} - \min\{x, x\}\) if \(x\) is odd. Thus \((X, *, 0)\) is a pre-\(d\)-algebra.

Note that Example 2.1 is neither a \(d\)-algebra nor a quasi-\(d\)-algebra, since \(2 \ast 4 = 0 = 4 \ast 2\), but \(2 \neq 4\).
Example 2.2. ([3]) Let \( X := [0, \infty) \) and \([x]\) be the least integer greater that \( x \), let \( x \ast y := 0 \) if \( x \) is rational; \( x \ast y := \max\{x, y\} \ast \min\{x, y\} \) if both \( x \) and \( y \) are irrational; \( x \ast y := [x + y + 1] \) if \( x \) is irrational and \( y \) is rational. Also, \( x \ast 0 = x \), either because \( x \) is rational or because it is irrational and \( \max\{x, x\} = \min\{x, x\} \) = \([0]\) = 0. Of course, \( 0 \ast x = 0 \) since \( 0 \) is rational. This proves that \((X, \ast, 0)\) is a pre-d-algebra.

Note that Example 2.2 is neither a \( d \)-algebra nor a quasi-d-algebra, since \( 3 \ast 4 = 0 \ast 3 \ast 4 = 3 \neq 4 \).

J. S. Han et. al ([3]) introduced the notion of a strong \( d \)-algebra as follow: An algebra \((X; \ast, 0)\) is said to be a strong \( d \)-algebra ([3]) if it satisfies (A), (B) and (C)* where (C)* if \( x \ast y = y \ast x \), then \( x = y \).

An algebra \((X; \ast, 0)\) is said to be a strong quasi-d-algebra ([3]) if it satisfies (A) and (C)*.

Obviously, every strong \( d \)-algebra is a \( d \)-algebra, but the converse need not be true in general.

Example 2.3. ([3]) If \( X = [0, \infty) \) and if \( x \ast y := \max\{0, x - y\} \), then \((X; \ast, 0)\) is an ordinary \( d \)-algebra, since \( x \ast y = 0 \) means \( y \leq x \), and \( x \leq y, y \leq x \) means \( x = y \). We claim that \((X, \ast, 0)\) is an ordinary \( d \)-algebra. In fact, if \( x \ast y = y \ast x \) and if \( x < y \), then \( x \ast x = x - x > 0 \), so that \( x > y \) as well, which is an impossibility. Hence \((X; \ast, 0)\) is a strong \( d \)-algebra.

In the following we show an ordinary \( d \)-algebra which is not a strong \( d \)-algebra.

Example 2.4. ([3]) Let \( R \) be the set of all real numbers and define \( x \ast y := (x - y) \ast (x - e) + e \), \( x, y, e \in \mathbb{R} \), where \( \ast \) and \( \ast \) are the ordinary product and subtraction of real numbers. Then \( x \ast y = x - e \ast x - e \ast y = y \ast x = e \) yields \((x - y) \ast (x - e) = 0, (y - y) \ast (y - e) = 0 \) and \( x = y \) or \( x = e \), i.e., \((R, \ast, e)\) is a \( d \)-algebra.

However, \((R, \ast, e)\) is not a strong \( d \)-algebra. If \( x \ast y = y \ast x \Leftrightarrow (x - y) \ast (x - e) + e = (y - y) \ast (y - e) + e \Leftrightarrow (x - y) \ast (x - e) = (x - y) \ast (y - e) \Leftrightarrow (x - y) \ast (x - e + y - e) = 0 \Leftrightarrow (x - y) \ast (x + y - 2e) = 0 \Leftrightarrow (x = y \) or \( x + y = 2e \)), then there exist \( x = e + \alpha \) and \( y = c - \alpha \) such that \( x + y = 2e \), i.e., \( x \ast y = y \ast x \) and \( x \neq y \). Hence, axiom (C)* fails and thus the \( d \)-algebra \((R, \ast, e)\) is not a strong \( d \)-algebra.

3. The Logistic Transformation

One way to generate quasi-d-algebras over the real numbers is via the following mechanism. Given a groupoid \((R, \ast)\), we define an algebra \((R, \nabla, y)\) as follows:

\[
x \nabla y := \frac{e^{x+y}}{e^{x+y} + e^{y+x}} - \frac{1}{2}
\]

for any \( x, y \in \mathbb{R} \). We call \((R, \nabla)\) the logistic groupoid of \((R, \ast)\). Obviously, we have \(-\frac{1}{2} < x \nabla y < \frac{1}{2} \) for any \( x, y \in \mathbb{R} \). We denote the notation \( \nabla \ast \) by \( \nabla \) if there is no confusion.

Example 3.1. Let \((R, \ast)\) be a groupoid defined by

\[
x \ast y := \begin{cases} -1 & \text{if } y < x, \\ 0 & \text{if } x = y, \\ 1 & \text{if } y > x \end{cases}
\]

for any \( x, y \in \mathbb{R} \). If \( x < y \), then \( x \ast y = 1, y \ast x = 1 \) so that \( x \nabla y = \frac{e^{x+y}}{e^{x+y} + e^{y+x}} - \frac{1}{2} = \frac{1}{2} \). If \( x = y \), then \( x \ast y = y \ast x = 0 \) so that \( x \nabla y = 0 \). If \( x > y \), then \( x \ast y = 1 \) and \( y \ast x = -1 \) so that \( x \nabla y = \frac{1}{2} \). Hence we obtain

\[
x \nabla y := \begin{cases} \frac{e^{x+y}}{e^{x+y} + e^{y+x}} - \frac{1}{2} & \text{if } x < y, \\ 0 & \text{if } x = y, \\ \frac{e^{x+y}}{e^{x+y} + e^{y+x}} - \frac{1}{2} & \text{if } y > x \end{cases}
\]

for any \( x, y \in \mathbb{R} \). The groupoid \((R, \nabla)\) is the logistic groupoid of \((R, \ast)\). It follows that \(-\frac{1}{2} < x \nabla y < \frac{1}{2} \) and \( x \nabla y + y \nabla x = 0 \) for any \( x, y \in \mathbb{R} \).

The following theorem shows that a groupoid with a special condition can be the logistic groupoid of a groupoid.

Theorem 3.2. Given a groupoid \((R, \ast)\) with \(-\frac{1}{2} < x \ast y < \frac{1}{2}, \forall x, y \in \mathbb{R} \), if we define a binary operation \( \ast \) on \( R \) which satisfies

\[
x \ast y + y \ast x := \ln \left[ \frac{1 - 2(x \ast y)}{1 + 2(x \ast y)} \right],
\]

then \((R, \ast)\) is a logistic groupoid of \((R, \ast)\), i.e., \( \ast = \nabla \ast \).

Proof. If we let \( \alpha := x \ast y - y \ast x \), then \( e^\alpha = e^{e^{x+y} - e^{y+x}} = e^{2(x+y)} \) and \( 2(e^{x+y} - 1)(x \ast y - y \ast x) = 1 - e^\alpha \). It follows that \( x \ast y = \frac{1}{2} \ast x \nabla y \), \forall x, y \in \mathbb{R} \). This proves that \((R, \ast)\) is the logistic groupoid of \((R, \ast)\). \( \Box \)

Remark 1. The groupoid \((R, \nabla)\) in Example 3.1 is the logistic groupoid of \((R, \ast)\) satisfying the condition: \(-\frac{1}{2} < x \nabla y < \frac{1}{2} \) for any \( x, y \in \mathbb{R} \). If we define a binary operation \( \ast \) on \( \mathbb{R} \) which satisfies the condition:

\[
x \ast y - y \ast x := \ln \left[ \frac{1 - 2(x \ast y)}{1 + 2(x \ast y)} \right]
\]

as in Theorem 3.2, then it satisfies the following condition:

\[
x \ast y - y \ast x := \begin{cases} 2, & \text{if } x < y, \\ 0, & \text{if } x = y, \\ -2, & \text{if } y > x \end{cases}
\]

for any \( x, y \in \mathbb{R} \). In fact, if \( x < y \), then \( x \nabla y = \frac{x - y}{2(1 + e^{x+y})} \) and hence \( x \nabla y - y \nabla x = 2 \). If \( x = y \), then \( x \nabla y = 0 \) and hence \( x \nabla y - y \nabla x = 0 \). Similarly, if \( x > y \), then \( x \nabla y = \frac{x - y}{2(1 + e^{x+y})} \) and \( x \nabla y - y \nabla x = -2 \).
2. Define a binary operation “∗α” on R by
\[
x ∗_α y := \begin{cases} 
-1 + α & \text{if } x < y, \\
α & \text{if } x = y, \\
1 + α & \text{if } x > y
\end{cases}
\]
for any \(x, y \in R\) where \(α\) is a non-zero element of R. By routine calculations we obtain that the groupoid \(\langle R, ∇_α \rangle\) is equal to \(\langle R, ∇_α \rangle\) for any non-zero \(α \in R\), i.e., \(\langle R, ∇_α \rangle\) is the groupoid of any groupoid \(\langle R, ∗\rangle\) by Theorem 3.2 and Example 3.1.

**Proposition 3.3.** Let \(\langle R, ∇ \rangle\) be a groupoid of a groupoid \(\langle R, ∗\rangle\). Then \(\langle R, ∗\rangle\) is commutative, i.e., \(x ∗ y = y ∗ x\) if and only if \(x ∇ y = 0\) for any \(x, y \in R\).

**Proof.** If we assume that \(x ∗ y = y ∗ x\) where \(x, y \in R\), then
\[
\begin{align*}
(x ∗ y) = (y ∗ x) = e^{x∗y} = 1 + \frac{1}{2} + \frac{1}{2} = 2.
\end{align*}
\]
Conversely, if we assume that \(x ∇ y = 0\), then \(e^{x∗y} = e^{y∗x} = 2\), and hence \(x ∗ y = y ∗ x\). Since the function \(e^x\) is a one-one function, we obtain \(x ∗ y = y ∗ x\).

**Corollary 3.4.** Let \(\langle R, ∗\rangle\) be a groupoid. Then \(x ∇ x = 0\) for any \(x \in R\).

**Proof.** Since \(x ∗ x = x ∗ x\) for any \(x \in R\), it follows from Proposition 3.3 that \(x ∇ x = 0\).

**Proposition 3.5.** Let \(\langle R, ∗\rangle\) be a groupoid. Then \(x ∇ y + y ∇ x = 0\) for any \(x \in R\).

**Proof.** Straightforward.

**Proposition 3.6.** Let \(\langle R, ∇ \rangle\) be a groupoid of a groupoid \(\langle R, ∗\rangle\). Then \(\langle R, ∗\rangle\) satisfies the condition \((C)^*\) if and only if \(\langle R, ∇ \rangle\) satisfies the condition \((C)^*\).

**Proof.** \((\Rightarrow)\) If \(x ∗ y = y ∗ x\) where \(x, y \in R\), then \(x ∇ y = 0 = y ∇ x\) by Proposition 3.3. By assumption, we obtain \(x = y\), proving that \(\langle R, ∗\rangle\) satisfies the condition \((C)^*\).

\((\Leftarrow)\) If \(x ∇ y = y ∇ x\) where \(x, y \in R\), then \(0 = x ∇ y + y ∇ x = 2x ∇ y\) by Proposition 3.5, and hence \(x ∇ y = 0 = y ∇ x\). By Proposition 3.3, we obtain \(x ∗ y = y ∗ x\). By assumption, we obtain \(x = y\), proving that \(\langle R, ∇ \rangle\) satisfies the condition \((C)^*\).

**Proposition 3.7.** Let \(\langle R, ∇ \rangle\) be the groupoid of a groupoid \(\langle R, ∗\rangle\) and let \(x, y \in R\). If \(x ∗ y = 0\), then \(y ∗ x = \ln(\frac{1}{2} \cdot 2x ∇ y)\). \(\Box\)

**Proof.** Let \(x ∗ y = 0\). Then \(x ∇ y = \frac{1}{2} + \frac{1}{2} = 1\), whence \(y ∗ x = \ln(\frac{1}{2} \cdot 2x ∇ y)\).

**Proposition 3.8.** If \(\langle R, ∗\rangle\) is a strong quasi-d-algebra, then \(\langle R, ∇ \rangle\) is a strong quasi-d-algebra.

**Proof.** It follows immediately from Corollary 3.4 and Proposition 3.6.

Let \(\langle R, ∇ \rangle\) be a groupoid of a groupoid \(\langle R, ∗\rangle\). The groupoid \(\langle R, ∗\rangle\) is said to be logistically associative if \(\langle R, ∇ \rangle\) is associative.

**Proposition 3.9.** If the groupoid \(\langle R, ∗\rangle\) is commutative, then it is logistically associative.

**Proof.** If the groupoid \(\langle R, ∗\rangle\) is commutative, then \(x ∇ y = 0\) for any \(x, y \in R\) by Proposition 3.3. This means that \((x ∇ y) ∇ z = 0 = x ∇ (y ∇ z)\) for all \(x, y, z \in R\).

Note that the converse of Proposition 3.9 need not be true in general.

**Example 3.10.** Given \(x \in R\), if we define a map \(q : R → R\) by
\[
q(x) := \begin{cases} 
-1 & \text{if } x < |x| ≤ \frac{1}{2}, \\
0 & \text{otherwise}
\end{cases}
\]
then \(q(3.75) = 3.75 \text{ and } q(2.25) = 2.25 \text{ and } q(0.25) = 0.25.\)

Let \(x, y, z \in R\) with \(|x| > \frac{1}{2}\). Then \(x ∗ (y ∇ z) = x ∗ (y ∇ z)\) for \(x, y, z \in R\) and \(y ∇ z = (y ∇ z) ∗ (y ∇ z)\) for \(x, y, z \in R\) and \(q(x ∇ y ∇ z) = x ∇ y ∇ z\).

Define a binary operation “∗” on R by
\[
x ∗ y := \begin{cases} 
(x(y − q(y)))(y − q(y)) & \text{if } |x| > \frac{1}{2}, \\
(x − q(x))(y − q(y)) & \text{otherwise}
\end{cases}
\]
Let \(x, y, z \in R\) with \(|x| > \frac{1}{2}\). Then \(x ∗ (y ∇ z) = x ∗ (y ∇ z)\) for \(x, y, z \in R\) and \(y ∇ z = (y ∇ z) ∗ (y ∇ z)\) for \(x, y, z \in R\) and \(q(x ∇ y ∇ z) = x ∇ y ∇ z\).

**Theorem 3.11.** If the groupoid \(\langle R, ∗\rangle\) is logistically associative, then
\[
(x ∇ (y ∇ z)) ∇ z = x ∇ (y ∇ z)
\]
for all \(x, y, z \in R\).

**Proof.** For any \(x, y, z \in R\), by applying Proposition 3.3, we obtain \((x ∇ (y ∇ z)) ∇ z = (y ∇ z) ∇ x\) for all \(x, y, z \in R\). Moreover, we obtain \((x ∇ y) ∇ z = (y ∇ z) ∇ x\) for all \(x, y, z \in R\) and \((y ∇ z) ∇ x = (y ∇ z) ∇ x\) for all \(x, y, z \in R\), proving the theorem.

Note that, if \(\langle R, ∗\rangle\) is logistically associative, by applying Proposition 3.3 and Theorem 3.11, we obtain \(x ∗ (y ∇ z) = (y ∇ z) ∗ x\) for all \(x, y, z \in R\).

Let \(\langle R, ∇ \rangle\) be a groupoid of a groupoid \(\langle R, ∗\rangle\). The groupoid \(\langle R, ∗\rangle\) is said to be logistically Jordan if
\[
(x ∇ y) ∇ z + (z ∇ x) ∇ y + (y ∇ z) ∇ x = 0
\]

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for any \(x, y, z \in \mathbb{R}\).

**Corollary 3.12.** If the groupoid \((\mathbb{R}, \ast)\) is logistically associative, then it is logistically Jordan.

**Proof.** It follows immediately from Theorem 3.11. \(\square\)

Let \((\mathbb{R}, \triangledown)\) be a logistic groupoid of a groupoid \((\mathbb{R}, \ast)\).
The groupoid \((\mathbb{R}, \ast)\) is said to be logistically anti-associative if
\[(x \triangledown y) \triangledown (y \triangledown z) = [x \triangledown (y \triangledown z)]\]
for any \(x, y, z \in \mathbb{R}\).

**Corollary 3.13.** If \((\mathbb{R}, \ast)\) is logistically associative, then it is logistically anti-associative as well.

**Proof.** It follows immediately from Theorem 3.11. \(\square\)

Let \((\mathbb{R}, \triangledown)\) be a logistic groupoid of a groupoid \((\mathbb{R}, \ast)\).
The groupoid \((\mathbb{R}, \ast)\) is said to be logistically medial if
\[(x \triangledown y) \triangledown (y \triangledown z) = 0\]
for any \(x, y, z \in \mathbb{R}\).

**Corollary 3.14.** If \((\mathbb{R}, \ast)\) is logistically associative, then it is logistically medial.

**Proof.** By applying Theorem 3.11, we have \(x \triangledown (y \triangledown z) = 0\) for all \(x, y, z \in \mathbb{R}\). If we replace \(x\) by \(x \triangledown y\) then \((x \triangledown y) \triangledown (y \triangledown z) = 0\). \(\square\)

**Theorem 3.15.** If \((\mathbb{R}, \ast)\) is logistically anti-associative, then
\[(x \triangledown y) \triangledown (y \triangledown z) = (z \triangledown x) \triangledown y = (y \triangledown z) \triangledown x\]
for any \(x, y, z \in \mathbb{R}\).

**Proof.** Let \((\mathbb{R}, \ast)\) be a logistically anti-associative groupoid. Given \(x, y, z \in \mathbb{R}\), by Proposition 3.5, we have \((x \triangledown y) \triangledown z = (z \triangledown x) \triangledown y + (y \triangledown z) \triangledown x = 3[(x \triangledown y) \triangledown z] = 0\), and \((x \triangledown y) \triangledown z = -[x \triangledown (y \triangledown z)] = (y \triangledown z) \triangledown x\). \(\square\)

**Corollary 3.16.** If \((\mathbb{R}, \ast)\) is both logistically Jordan and logistically anti-associative, then it is logistically associative.

**Proof.** If it is logistically Jordan, then \((x \triangledown y) \triangledown z + (z \triangledown x) \triangledown y + (y \triangledown z) \triangledown x = 3[(x \triangledown y) \triangledown z] = 0\), and \((x \triangledown y) \triangledown z = -[x \triangledown (y \triangledown z)]\) and hence \(z \triangledown (x \triangledown y) = 0\) for any \(x, y, z \in \mathbb{R}\). It follows that \((\mathbb{R}, \ast)\) is logistically associative. \(\square\)

**Proposition 3.17.** If \((\mathbb{R}, \ast)\) is logistically anti-associative and \(x \triangledown y = y \triangledown x\) for any \(x, y \in \mathbb{R}\), then it is logistically medial.

**Proof.** Given \(x, y, z \in \mathbb{R}\), \((x \triangledown y) \triangledown z = (y \triangledown x) \triangledown z = -[y \triangledown (x \triangledown z)]\) and \((x \triangledown y) \triangledown z = z \triangledown (y \triangledown x) = -[z \triangledown (x \triangledown y)] = -[(x \triangledown y) \triangledown x] = y \triangledown (x \triangledown z)\). Hence \(y \triangledown (x \triangledown z) = -[y \triangledown (x \triangledown z)]\) and \(y \triangledown (x \triangledown z) = 0\) for all \(x, y, z \in \mathbb{R}\), proving that \((\mathbb{R}, \ast)\) is logistically associative. By Corollary 3.14, we obtain that \((\mathbb{R}, \ast)\) is logistically medial. \(\square\)

**Proposition 3.18.** Let \((\mathbb{R}, \ast)\) and \((\mathbb{R}, \bullet)\) be two groupoids with the same logistic groupoid \((\mathbb{R}, \ast)\). If we define a binary operation \(\boxdot\) on \(\mathbb{R}\) by
\[x \boxdot y := x \ast y - x \bullet y, \quad \forall x, y \in \mathbb{R},\]
then \((\mathbb{R}, \boxdot)\) is commutative and hence it is logistically associative.

**Proof.** Since \((\mathbb{R}, \ast)\) and \((\mathbb{R}, \bullet)\) have the same logistic groupoid \((\mathbb{R}, \ast)\), we obtain
\[x \ast y - y \ast x = x \bullet y - y \bullet x = \ln\left[\frac{1 + 2(x \ast y)}{1 - 2(x \ast y)}\right]\]
Hence \(x \boxdot y - y \boxdot x = (x \ast y - x \bullet y) - (y \ast x - y \bullet x) = (x \ast y - y \bullet x) - (x \bullet y - y \ast x) = 0\), proving that \(x \boxdot y = y \boxdot x\), i.e., \((\mathbb{R}, \boxdot)\) is commutative. It follows from Proposition 3.9 that \((\mathbb{R}, \boxdot)\) is logistically associative. \(\square\)

**Remark.** Note that \((\mathbb{R}, \boxdot)\) does not have \((\mathbb{R}, \ast)\) as its logistic groupoid in Proposition 3.18. In Example 3.1 and Remark 2, we see that \((\mathbb{R}, \ast)\) and \((\mathbb{R}, \bullet)\) have the same logistic groupoid \((\mathbb{R}, \triangledown)\). Let \(x \triangle y := x \ast y - x \ast y, \forall x, y \in \mathbb{R}\). If \(x < y\), then \(x \triangle y = -1 - 0 = -1\). If \(x = y\), then \(x \triangle y = 0 - 1 = -1\). If \(x > y\), then \(x \triangle y = 1 - 2 = -1\), i.e., \(x \triangle y = -1\) for any \(x, y \in \mathbb{R}\). Hence \(x \triangledown y = e^{x \triangle y} - \frac{1}{2} = 0\) for any \(x, y \in \mathbb{R}\). This proves that \((\mathbb{R}, \boxdot)\) does not have \((\mathbb{R}, \triangledown)\) as its logistic groupoid.

Of course, Proposition 3.18 does not provide a solution \((\mathbb{R}, \ast)\) for a given groupoid \((\mathbb{R}, \ast)\) which is its prescribed logistic groupoid. The answer to this problem is contained in Theorem 3.19.

**Theorem 3.19.** Given a groupoid \((\mathbb{R}, \ast)\) with \(-\frac{1}{2} < x \ast y < \frac{1}{2}\) and \(x \ast y + y \ast x = 0\), if we define a groupoid \((\mathbb{R}, \odot)\) by
\[x \odot y := \begin{cases} \alpha & \text{if } x \neq y, \\ x - y \ln \left[\frac{1 - 2(x \ast y)}{1 + 2(x \ast y)}\right] & \text{if } x = y \end{cases}\]
where \(\alpha \in \mathbb{R}\), then \((\mathbb{R}, \ast)\) is the logistic groupoid of \((\mathbb{R}, \odot)\).

**Proof.** Given \(x, y \in \mathbb{R}\), if \(x \neq y\), since \(x \ast y + y \ast x = 0\), then we have
\[y \odot x = \begin{cases} y & \text{if } x = y, \\ -y & \text{if } x \neq y \end{cases}\]
\[x \odot y = \begin{cases} x - y \ln \left[\frac{1 + 2(x \ast y)}{1 - 2(x \ast y)}\right] & \text{if } x \neq y, \\ y - x \ln \left[\frac{1 - 2(x \ast y)}{1 + 2(x \ast y)}\right] & \text{if } x = y \end{cases}\]
Hence
x ⊗ y − y ⊙ x = \frac{x}{x - y} \ln \left[ \frac{1 - 2(x * y)}{1 + 2(x * y)} \right] - \frac{y}{x - y} \ln \left[ \frac{1 - 2(x * y)}{1 + 2(x * y)} \right] = \ln \left[ \frac{1 - 2(x * y)}{1 + 2(x * y)} \right].

If x = y, since x * y + y * x = 0, we have x * y = 0 and hence ln \left[ \frac{1 - 2(x * y)}{1 + 2(x * y)} \right] = ln 1 = 0 = a - a = x ⊕ x - x ⊙ x. By applying Theorem 3.2 we prove that (R, ∗) is the logistic groupoid of (R, ⊕).

**Corollary 3.20.** Given a groupoid (R, ∗) with \(-\frac{1}{2} < x * y < \frac{1}{2}\) and \(x * y + y * x = 0\), if (R, ∗) is the logistic groupoid of (R, ⊕) described in Theorem 3.19, then (R, ∗) is the logistic groupoid of a groupoid (R, ⋆), where \(x ⋆ y := x ⊕ y + x ⊙ y\) and \(x ⋙ y = x ⊕ y\) for any \(x, y \in R\).

**Proof.** Since \(x ⋙ y = y ⋙ x\) for any \(x, y \in R\), we have
\[x ⋇ y - y ⋇ x = (x ⊕ y + x ⊙ y) - (y ⊕ y + y ⊙ y) = (x ⊕ y - y ⊕ y) + (x ⋙ y - y ⋙ y) = x ⊕ y - y ⊕ y \circledast x.

By applying Theorem 3.19, we obtain that (R, ∗) is the logistic groupoid of a groupoid (R, ⋆). □

Note that this is an analogue to the problem of solving a first order linear differential equation.

**Theorem 3.21.** Let (R, ∗) and (R, ⋆) be groupoids. If we define a binary operation “\(\bowtie\)” on R by \(x \bowtie y := x * y - x * y, \forall x, y \in R\) and if this operation is commutative, then the logistic groupoid of (R, ∗) is equal to the logistic groupoid of (R, ⋆).

**Proof.** Let (R, ⊖) and (R, ⊖) be logistic groupoids of groupoids (R, ∗) and (R, ⋆), respectively. Then we have
\[x ⊖ y = \frac{e^{x+y} - 1}{e^{x+y} + 1} - \frac{1}{2} \quad \text{and} \quad x ⊖ y = \frac{e^{x*}\cdot e^{y*} - 1}{e^{x*}\cdot e^{y*} + 1} - \frac{1}{2}
\]
for any \(x, y \in R\). Since \(x \bowtie y = y \bowtie x\), we obtain \(x * y - y * x = x * y - y * x\). Hence \(x ⊖ y = \frac{e^{x+y} - 1}{e^{x+y} + 1} - \frac{1}{2} = \frac{1}{1 + e^{x+y}} - \frac{1}{2} = x \bowtie y\). Hence (R, ⊖) and (R, ⊖) have the same logistic groupoid. □

**References**


