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Family of T-Inverse Exponential $\{Y\}$ Distributions: Properties and Applications

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Abstract: Statisticians have been exploring the generalization and extension of families of distributions in order to make them flexible for data analysis. In this study, a new family of inverse exponential distribution called T-IE$\{$Y$\}$ family of distributions has been developed. Statistical properties and estimation of parameters are explored. Three special cases of the family of distributions namely, Log-logistic-IE$\{$Weibull$\}$, Weibull-IE$\{$Lomax$\}$ and Gumbel-IE$\{$Logistic$\}$ distributions were developed. By varying the values of the parameters the density functions assume various shapes such as approximately symmetric, positively skewed, negatively skewed, J-shape and reversed-J shape. The hazard rate function on the other hand may take both monotonic and non-monotonic shapes namely increasing, decreasing and inverted bathtub shape. Empirical applications of the special distributions to real life data sets are included.

Keywords: Inverse exponential distribution, Weibull distribution, Lomax distribution, logistic distribution, hazard, quantile.

1 Introduction

The selection of appropriate distribution for modelling is very vital in statistical data analysis. This is because, data from different fields of study may exhibit different traits and when an inadequate model is selected; it may lead to loss of information or reduce the power and efficiency associated with that data set. Choosing a distribution that better describes the characteristics of a given data set is often a challenging task notwithstanding the existence of many statistical distributions. This is because, no single distribution can model all kinds of data sets perfectly. It is common therefore for researchers to generalize or extend the existing distributions in order to make them more flexible to handle wide variety of data sets.
One of the methods used recently by researchers is the quantile based approach pioneered by [2]. This approach is an extension of the transformed-transform family introduced by [3].
Suppose the cummulative distribution function (CDFs) of random variable T, R and Y are $G_T(x), G_R(x)$ and $G_Y(x)$ respectivly. Also, let their probability density functions (PDFs) be $g_T(x)$, $g_R(x)$ and $g_Y(x)$ respectively. If the quantile function of random variable Z is $Q_Z(u)$=$inf\{z : G_Z(z) \geq u\}$,$0 < u < 1$, then, the corresponding quantile functions for the random variables T, R and Y are $Q_T(u), Q_R(u)$ and $Q_Y(u)$ respectively. The CDF of the random variable X in the T-X$\{$Y$\}$ frame work of [2] is defined as;

$$G_X(x) = \int_a^{Q_Y(G_R(x))} g_T(t)dt = G_T(Q_Y(G_R(x))). \quad (1)$$

The corresponding PDF and hazard rate functions are;

$$g_X(x) = g_R(x) \times \frac{g_T\left(Q_Y(G_R(x))\right)}{g_Y\left(Q_Y(G_R(x))\right)}. \quad (2)$$

and

$$h_X(x) = h_R(x) \times \frac{h_T\left(Q_Y(G_R(x))\right)}{h_Y\left(Q_Y(G_R(x))\right)}. \quad (3)$$

respectively. The T-R$\{$Y$\}$ technique has been adopted and used by a number of researchers to develop families of distributions. Some of these include T-normal$\{$Y$\}$ family [4]; T-gamma$\{$Y$\}$ family [5]; T-exponential$\{$Y$\}$family [22]; Weibull-R family [7], Lomax-R $\{$Y$\}$family [12], T-Burr family [13], Weibull Burr XII family [14], T-Pareto $\{$Y$\}$ family [8], T-Burr III $\{$Y$\}$ family [9] and T-Nadarajah- Haghighi $\{$Y$\}$ family [15] of distributions.

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This study therefore proposes another new class of distribution called the T-inverse exponential {Y} (T-IE{Y}) family of distributions. The remaining part of the article are presented as follows: the T-IE{Y} family is presented in section 2, the statistical properties of T-IE{Y} family is presented in section 3, the special distributions are given in section 4, section 5 presents parameter estimation of the proposed distribution, section 6 presents the simulation results, section 7 presents the empirical applications of the proposed distributions and the conclusions are given in section 8.

2 T-IE{Y} Family of Distributions

Suppose R is an IE random variable with PDF $g_R(x) = \frac{\lambda}{x^2}\exp(-\lambda/x)$ and CDF $G_R(x) = \exp(-\lambda/x)$, where $x > 0$ and $\lambda > 0$ is a scale parameter. Using the technique by [2], the CDF of T-IE{Y} family of distribution is define as

$$G_X(x) = \int_a^{Q_Y(\exp(-\frac{\lambda}{x}))} g_T(t)dt = G_T(Q_Y[\exp(-\lambda/x)]). \tag{4}$$

Differentiating equation (4) with respect to x gives the corresponding PDF as

$$g_X(x) = \frac{\lambda}{x^2}\exp(-\lambda/x)Q_Y'(\exp(-\lambda/x))g_T(Q_Y(\exp(-\lambda/x))). \tag{5}$$

The hazard rate function is given by

$$h_X(x) = \frac{\frac{\lambda}{x^2}\exp(-\lambda/x)Q_Y'(\exp(-\lambda/x))g_T(Q_Y(\exp(-\lambda/x)))}{1-G_T(Q_Y(\exp(-\lambda/x)))}. \tag{6}$$

Remark 1. If X follows the T-IE{Y} family of distributions given by equation (4), we have the following:

i. $Q_X(u) = -\frac{\lambda}{\log[G_Y(Q_T(u))]}, u \in (0,1)$.

ii. $X = -\frac{\lambda}{\log[G_Y(T)]}$.

iii. If the distribution of Y follows IE and X follows the T-IE{Y} family of distributions then $X \overset{d}{=} T$.

2.1 Some Sub-families of T-IE{Y} Family of Distributions

The T-IE{Y} family given in equation (5) can generate several different extended IE families. In this section, three sub-families of T-IE{Y} family namely T-IE{Weibull}, T-IE{Logistic} and T-IE{Lomax} families of distributions are discussed.

2.1.1 The T-IE{Weibull} Family

If Y follows Weibull distribution with quantile function $Q_Y(u) = (-\log(1-u))^{\frac{1}{\gamma}}$. Then the CDF of T-IE{Weibull} family is given by;

$$G_X(x) = G_T[(-\log(1-\exp(-\lambda/x)))^{\frac{1}{\gamma}}], \tag{7}$$

where $T \in (0,\infty)$. The corresponding PDF is given by

$$g_X(x) = \frac{\lambda\exp(-\lambda/x)}{\gamma x^2(1-\exp(-\lambda/x))}\{-\log(1-\exp(-\lambda/x))\}^{\frac{1}{\gamma}-1}\times g_T[\{-\log(1-\exp(-\lambda/x))\}^{\frac{1}{\gamma}}], x>0. \tag{8}$$

The T-IE{Weibull} family reduces to T-IE{exponential} family when $\gamma = 1$ and T-IE{Rayleigh} family when $\gamma = 2$.

2.1.2 The T-IE{logistic} Family

Assuming Y follows logistics distribution with quantile function $Q_Y(u) = -\frac{1}{\alpha}\log(u^{-1}-1)$. Then, the CDF of T-IE{logistic} Family is given by;

$$G_X(x) = G_T[-\frac{1}{\alpha}\log(\exp(\lambda/x)-1)], \tag{9}$$

where the random variable $T \in (-\infty,\infty)$. The corresponding PDF is given by;

$$g_X(x) = \frac{\lambda\exp(\lambda/x)}{\alpha x^2(\exp(\lambda/x)-1)}g_T[-\frac{1}{\alpha}\log(\exp(\lambda/x)-1)]. \tag{10}$$

2.1.3 T-IE {Lomax} Family

Assuming Y follows Lomax distribution with the Quantile function $Q_Y(u) = ((1-u)^{-1/\gamma}-1)$. If $T \in (0,\infty)$, then the CDF of the T-IE {Lomax} family if given by;

$$G_X(x) = G_T[(1-\exp(-\lambda/x))^{-\frac{1}{\gamma}}-1]. \tag{11}$$

The corresponding PDF is is given by;

$$g_X(x) = \frac{\lambda\exp(-\lambda/x)}{\gamma x^2(1-\exp(-\lambda/x))^{\frac{1}{\gamma}-1}}g_T[(1-\exp(-\lambda/x))^{-\frac{1}{\gamma}}-1]. \tag{12}$$

3 Statistical Properties of the T-IE {Y} Family

Some statistical properties of the T-IE{Y} family have been investigated and presented in this section.

3.1 Mode

This sub-section presents the mode of T-IE{Y} family.
Proposition 1. The mode of the T-IE{Y} family is obtained by finding the solution to the equation

$$\zeta[Q_Y'(\exp(-\lambda/x)] + \zeta[g_T(Q_Y(\exp(-\lambda/x)))] + \frac{\lambda}{x^2} - \frac{2}{x} = 0, \quad (13)$$

where $\zeta(q) = \frac{q'}{q}$.

Proof. Differentiating the logarithm of equation (5) with respect to x and equating it to zero completes the proof.

3.2 Transformation

Lemma 1. If T is a random variable with CDF $G_T(x)$, then the random variable:

i. $X=-\frac{\lambda}{\log(1-\exp^{T^{\gamma}})}$ follows the T-IE{Weibull} family of distribution.

ii. $X= -\frac{\lambda}{\log(1+\exp^{-\alpha T}}\big)^{-1}$ follows the T-IE{Logistic} family of distribution.

iii. $X= -\frac{\lambda}{\log(1-(1+T)^{-\gamma})}$ follows the T-IE{Lomax} family of distribution.

Proof. The proof follows from Remark 1 (ii).

3.3 Quantile Function

Quantile functions are used to generate random samples from a given distribution. They can be used to describe the characteristics such skewness and kurtosis of a distribution.

Lemma 2. The quantile functions for T-IE{Weibull}, T-IE{Logistics} and T-IE{Lomax}are respectively given by;

i. $Q_X(u) = -\frac{\lambda}{\log[1-\exp(-(Q_T(u))^{\gamma})]}, u \in (0,1)$.

ii. $Q_X(u) = \frac{\lambda}{\log[\exp(-\alpha Q_T(u))+1]}$, u $\in (0,1)$.

iii. $Q_x(u) = -\frac{\lambda}{\log[1-(Q_T(u)+1)^{-\gamma}]}, u \in (0,1)$.

Proof. The proof follows from Remark 1 (i).

3.4 Moments

Moments are used for finding skewness, kurtosis, measures of variation, measures of central tendency among others. The moments of T-IE{Y} family are presented in this sub-section.

Preposition 2. The n^{th} non-central moment of the T-IE{Y} family of distribution is given as;

$$\mu_n' = (-1)^n \lambda^n E[\log G_Y(T)]^{-n}, n = 1,2,\ldots \quad (14)$$

Proof. The proof follows from Remark 1 (ii).

Corollary 1. The n^{th} moment of the T-IE{Weibull}, T-IE{Logistic}and T-IE{Lomax} family are respectively given by;

i. $\mu_n' = E(X^n) = (-1)^n \lambda^n E[\log(1-\exp(-T^{\gamma}))]^{-n}$,n = 1,2,..

ii. $\mu_n' = E(X^n) = (-1)^n \lambda^n E[\log(1+\exp(-\alpha T))]^{-n}, n = 1,2,\ldots$

iii. $\mu_n' = E(X^n) = (-1)^n \lambda^n E[\log(1-(1+T))^{\gamma}]^{-n}, n = 1,2,\ldots$

Proof. The proof follows from Lemma 1.

3.5 Shannon Entropy

This section presents the Shannon entropy of the T-IE{Y} family. Entropy is a measure of variation or uncertainty. The entropy of random variable X with PDF $g_X(x)$ is given by $\vartheta_X = -E(\log g_X(x))$ [21].

Proposition 3. The Shannon entropy of T-IE{Y} family of distributions is given by;

$$\vartheta_X = \vartheta_T + E(\log g_Y(T)) - \log(\lambda) + 2E(\log(X)) + \lambda E(\tfrac{1}{X}), \quad (15)$$

where ϑ_T is the Shannon entropy for the random variable T.
Proof. As $X \overset{d}{=} Q_R(G_Y(T))$, then, $T \overset{d}{=} Q_Y(G_R(X))$. Thus, $g_X(x) = \frac{g_T(t)}{g_Y(t)} g_R(x)$.

This implies that

$\vartheta_X = \vartheta_T + E(\log g_Y(T)) - E(\log g_R(x))$.

Substituting the PDF of R and simplifying completes the proof.

Corollary 2. The Shannon entropies of the T-IE{Weibull}, T-IE{Logistic} and T-IE{Lomax} family of distributions are respectively given by;

i. $\vartheta_x = \vartheta_T + \log(\alpha/\lambda) + (\alpha - 1)E(\log(T)) - E(T^{\alpha}) + 2E(\log(X)) + \lambda E(1/X)$.

ii. $\vartheta_x = \vartheta_T + \log(\alpha/\lambda) - \alpha E(T) - 2E(\log(1 + \exp(-\alpha T))) + 2E(\log(X)) + \lambda E(1/x)$.

iii. $\vartheta_x = \vartheta_T + \log(\gamma/\lambda) - (\gamma+1)E(\log(1+T)) + 2E(\log(X)) + \lambda E(1/x)$.

Proof. The proof of corollary 2 follows by substituting the PDFs of Weibull, Logistics and Lomax distributions in Proposition 3.

4 Special Distributions

In this section, three special distributions in the T-IE{Y} family of distributions have been developed. These three distributions are Log-Logistic-IE{Weibull}(LLIEW), Gumbel-IE{Logistic}(GIEL) and Weibull-IE{Lomax}(WIEL).

4.1 The Log-Logistic-IE{Weibull} Distribution

Let $T \sim \mathrm{Log-Logistic}(\alpha, \theta)$ distribution with the CDF $G_T(x) = 1 - [1 + (x/\alpha)^\theta]^{-1}$ and the PDF $g_T(x) = \frac{\theta}{\alpha^\theta} x^{\theta-1}[1 + (x/\alpha)^\theta]^{-2}$. Then, the CDF and PDF of LLIEW distribution are respectively defined as;

$$G_X(u) = 1 - [1 + (-\alpha^{-1}[\log(1 - \exp(-\lambda/x))]^{1/\gamma})^\theta]^{-1}. \tag{16}$$

and

$$g_X(x) = \frac{\lambda\theta\exp(-\lambda/x)(-\log(1-\exp(-\lambda/x)))^{1/\gamma-1}}{\alpha^\theta\gamma x^2(1-\exp(-\lambda/x))} \times \frac{[(-\log(1-\exp(-\lambda/x)))^{1/\gamma}]^{\theta-1}}{[1+(-\alpha^{-1}(\log(1-\exp(-\lambda/x)))^{1/\gamma})]^2}, \tag{17}$$

where $x > 0, \alpha > 0, \gamma > 0, \theta > 0, \lambda > 0$.

The plot of the PDF of LLIEW distribution for some given parameter values is shown in Figure 1. The PDF can be positively skewed, negatively skewed, unimodal, J-shape and reversed-J shape.

The hazard rate function of LLIEW distribution is given by;

$$h_X(x) = \frac{\lambda\theta\exp(-\lambda/x)(-\log(1-\exp(-\lambda/x)))^{1/\gamma-1}}{\alpha^\theta\gamma x^2(1-\exp(-\lambda/x))} \times \frac{[(-\log(1-\exp(-\lambda/x)))^{1/\gamma}]^{\theta-1}}{[1+(-\alpha^{-1}(\log(1-\exp(-\lambda/x)))^{1/\gamma})]}. \tag{18}$$

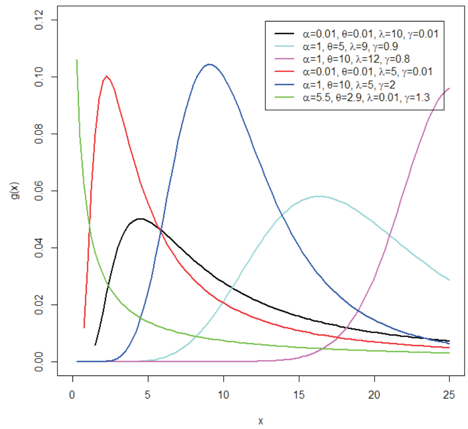

Fig. 1: PDF of the Log-logistic-IE {Weibull}

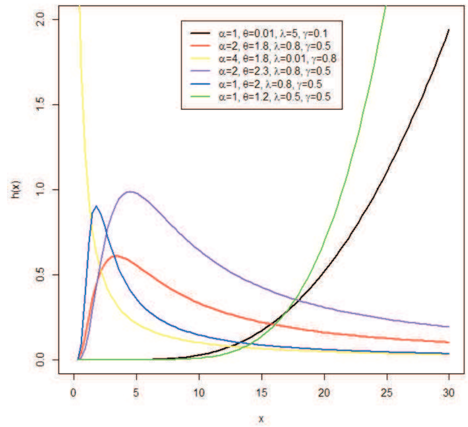

Fig. 2: Hazard rate function of the Log-logistic -IE {Weibull} distribution

Figure 2 shows the plots of the hazard rate function of LLIEW for some selected parameter values. The plots of hazard rate function give various shapes such as decreasing, increasing and inverted bathtub.

The quantile function for the LLIEW distribution is given by;

$$Q_X(u) = -\frac{\lambda}{\log[1+\exp[\alpha((1-u)^{-1}-1)^{1/\theta}]^\gamma} \tag{19}$$

4.2 The Gumbel-IE{Logistic}(GIEL) Distribution

This section presents the GIEL distribution. If $T \sim \text{Gumbel}(0,1)$ with CDF $G_T(x) = \exp(-\exp(x))$ and PDF $g_T(x) = \exp(-x-\exp(-x))$. Then the CDF and the PDF of the GIEL distribution are respectively given by;

$$G_X(x) = \exp(-(\exp(\lambda/x)-1)^{1/\alpha}) \tag{20}$$

and

$$g_X(x) = \frac{\lambda \exp(\lambda/x)}{\alpha x^2}(\exp(\lambda/x)-1)^{\frac{1}{\alpha}-1} \times \exp[-(\exp(\lambda/x)-1)^{\frac{1}{\alpha}}], x>0, \alpha>0, \lambda>0. \tag{21}$$

The plots of the PDF of the GIEL distribution for some selected parameter values are shown in Figure 3. The PDF can be positively skewed and reversed J-shape.

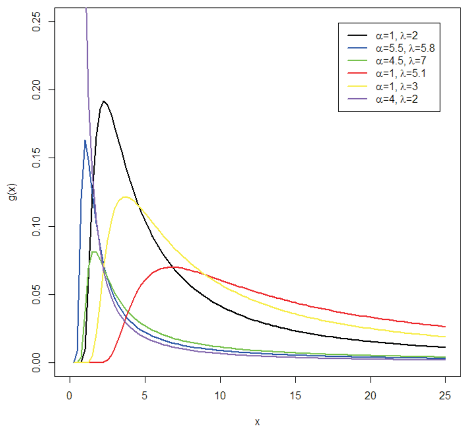

Fig. 3: PDF of the GIEL distribution

The hazard rate function of GIEL distribution is given by;

$$h_X(x) = \frac{\lambda \exp(\lambda/x)[\exp(\lambda/x)-1]^{\frac{1}{\alpha}-1}}{\alpha x^2} \times \frac{\exp[-(\exp(\lambda/x)-1)^{\frac{1}{\alpha}}]}{1-(\exp[-(\exp(\lambda/x)-1)^{\frac{1}{\alpha}}])}, x>0. \tag{22}$$

The plot of the hazard rate function for some selected parameter values are shown in Figure 4. When the parameter values were varied, it was noticed that the shape of the hazard rate function of the GIEL distribution shows decreasing, increasing and inverted bathtub shape.

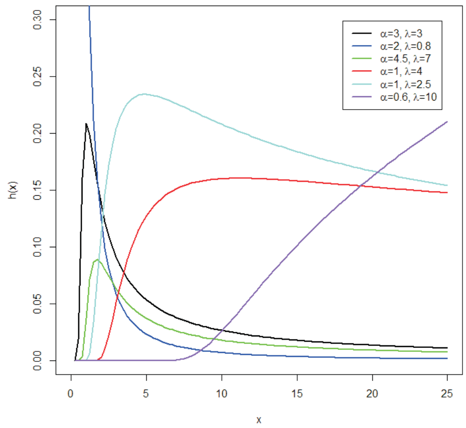

Fig. 4: Hazard function of the GIEL distribution

In order to simulate random samples from GIEL distribution, it is necessary to develop its quantile function. The quantile function of GIEL distribution is given by;

$$Q_X(u) = -\frac{\lambda}{\log[1-\alpha \log u]}, u \in (0,1). \tag{23}$$

4.3 Weibull-IE{Lomax} Distribution

If $T \sim \text{Weibull}(\alpha,\beta)$ distribution with CDF $G_T(x) = 1-\exp(-\alpha x^\beta)$ and PDF $g_T(x) = \alpha\beta x^{\beta-1}\exp(-\alpha x^\beta)$. Then the CDF and the PDF are respectively given by;

$$G_X(x) = 1-\exp(-\alpha((1-\exp(-\lambda/x))^{-1/\gamma}-1)^\beta), \quad x>0, \alpha>0, \lambda>0, \gamma>0. \tag{24}$$

and

$$g_X(x) = \frac{\alpha\beta\lambda\exp(-\lambda/x)}{\gamma x^2[1-\exp(-\lambda/x)]^{\frac{1}{\gamma}-1}}[\{1-\exp(-\lambda/x)\}^{-\frac{1}{\gamma}}-1]^{\beta-1} \times \exp[-\alpha\{1-\exp(-\lambda/x)\}^{-\frac{1}{\gamma}}-1]^\beta, x>0. \tag{25}$$

The plot of the PDF of the WIEL distribution for some selected parameter values is shown in Figure 5. The PDF of WIEL distribution can be positively skewed, approximately symmetric and unimodal with varying degree of kurtosis.

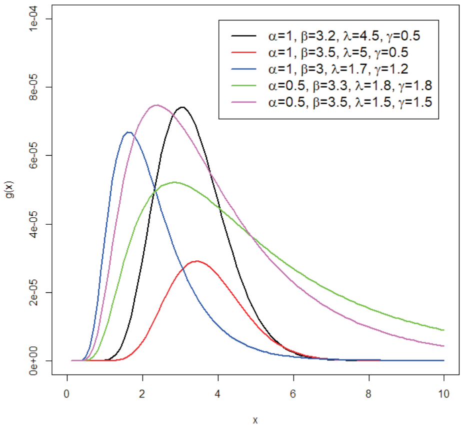

Fig. 5: PDF of the WIEL distribution

The hazard rate function of WIEL distribution is given by;

$$h_X(x) = \frac{\alpha\beta\lambda\exp(-\lambda/x)}{\gamma x^2(1-\exp(-\lambda/x)^{\frac{1}{\gamma}-1})}\times$$
$$(\{1-\exp(-\lambda/x)\}^{-\frac{1}{\gamma}}-1)^{\beta-1}, x>0. \quad (26)$$

Figure 6 shows the graphical presentation of the hazard rate function of WIEL distribution. It can be observed that, with varying parameter values, the hazard rate function has inverted bathtub shape and decreasing shape.

It is important to develop the quantile function in order to simulate random samples from WIEL distribution. The quantile function of WIEL distribution is given as;

$$Q_X(u) = -\frac{\lambda}{\log[1+(\alpha^{-1}(1+\log(1-u)^{1/\beta}))^{\gamma}]}, u\in(0,1). \quad (27)$$

5 Methods of Parameter Estimation

This section presents three parameter estimation procedures for estimating the parameters of GIEL distribution. These estimators are: Maximum Likelihood estimators, Ordinary Least Square estimators and Weighted Least Square estimators.

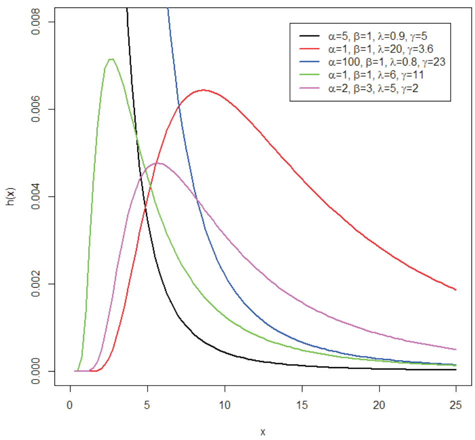

Fig. 6: Hazard rate function of the WIEL distribution

5.1 Maximum Likelihood Estimation

This subsection present the maximum likelihood estimation procedure for both complete and incomplete data sets.

5.1.1 Maximum likelihood Estimation for complete Samples

In this section we estimated the parameters $\theta = (\lambda,\alpha)^T$ of the GIEL distribution, using the method of maximum likelihood estimation. Let $x_1, x_2, \ldots x_n$ be a random sample of size n from the GIEL distribution, then the total log-likelihood function is given by;

$$\ell = n\log(\frac{\lambda}{\alpha}) + \sum_{i=1}^{n}\frac{\lambda}{x_i} - 2\sum_{i=1}^{n}\log x_i + (\frac{1}{\alpha}-1)\times$$
$$\sum_{i=1}^{n}\log(\exp(\frac{\lambda}{x_i})-1) - \sum_{i=1}^{n}(\exp(\frac{\lambda}{x_i})-1)^{\frac{1}{\alpha}} \quad (28)$$

Differentiating equation (28) with respect to the parameters λ and α respectively gives;

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\frac{1}{x_i} + (\frac{1}{\alpha}-1)\sum_{i=1}^{n}\frac{\exp(\frac{\lambda}{x_i})}{(\exp(\frac{\lambda}{x_i})-1)x_i} -$$
$$\sum_{i=1}^{n}\frac{\exp(\frac{\lambda}{x_i})(\exp(\frac{\lambda}{x_i})-1)^{\frac{1}{\alpha}-1}}{\alpha x_i} \quad (29)$$

and

$$\frac{\partial \ell}{\partial \alpha} = -\frac{n}{\alpha} - \sum_{i=1}^{n} \frac{\log(\exp(\frac{\lambda}{x_i}) - 1)}{\alpha^2} + \sum_{i=1}^{n} \frac{(\exp(\frac{\lambda}{x_i})^{\frac{1}{\alpha}} - 1)\log(\exp(\frac{\lambda}{x_i}) - 1)}{\alpha^2}. \quad (30)$$

Equating $\frac{\partial \ell}{\partial \lambda}$ and $\frac{\partial \ell}{\partial \alpha}$ to zero and solving for the parameters λ and α in the system of equations yields the maximum likelihood estimates of the parameters.

5.1.2 Maximum likelihood Estimation for Incomplete Samples

Suppose we observed the first r failed items $X_1, X_2, \ldots X_n$. Let $z_i = \exp(\lambda/x_i) - 1$, then, the total log-likelihood function for GIEL distribution with right censored data is given by;

$$\ell = \sum_{i=1}^{n} r_i \times \log\left[\frac{\lambda \exp(\frac{\lambda}{x_i})}{\alpha \times x_i^2} \times (z_i)^{\frac{1}{\alpha}-1} \times \exp(-z_i)^{\frac{1}{\alpha}}\right] + \sum_{i=1}^{n} (1-r_i) \times \log(1 - \exp(-z_i)^{\frac{1}{\alpha}}). \quad (31)$$

Differentiating ℓ with respect to λ and α gives

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{n} \frac{\exp(-z_i)^{\frac{1}{\alpha}} \log(-z_i)^{\frac{1}{\alpha}}(1-r_i)}{(1-\exp(-z_i)^{\frac{1}{\alpha}})\alpha^2} - \sum_{i=1}^{n} (\lambda^{-1}\alpha r_i(z_i)^{1-\frac{1}{\alpha}}(\frac{\lambda(z_i)^{\frac{1}{\alpha}-1}(\exp(\frac{\lambda}{x_i}) + (-z_i)^{\frac{1}{\alpha}})}{\alpha^2 x_i^2}) - \quad (32)$$

$$\sum_{i=1}^{n} (\lambda^{-1}\alpha r_i(z_i)^{1-\frac{1}{\alpha}}(\frac{\lambda \log(z_i)(z_i)^{\frac{1}{\alpha}-1}(\exp(\frac{\lambda}{x_i}) + (-z_i)^{\frac{1}{\alpha}})}{\alpha^3 x_i^2}) + \frac{\lambda \log(-z_i)(-z_i)^{\frac{1}{\alpha}}(z_i)^{\frac{1}{\alpha}-1}(\exp(\frac{\lambda}{x_i}) + (-z_i)^{\frac{1}{\alpha}})}{\alpha^3 x_i^2}$$

and

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{n} \lambda^{-1}\alpha r_i {x_i}^2 (z_i)^{1-\frac{1}{\alpha}}(\exp((\frac{-\lambda}{x_i}) - (-z_i)^{\frac{1}{\alpha}})) \times \quad (33)$$

$$(\frac{\lambda(z_i)^{\frac{1}{\alpha}-1}\exp((\frac{\lambda}{x_i}) + (-z_i)^{\frac{1}{\alpha}})}{\alpha {x_i}^2} + \frac{(z_i)^{\frac{1}{\alpha}-1}\exp((\frac{\lambda}{x_i}) + (z_i)^{\frac{1}{\alpha}})}{\alpha {x_i}^2}).$$

Equating equations (31), (32) and (33) to zero and solving simultaneously results in the estimates.

5.2 Ordinary and Weighted Least Square Estimators

The OLS estimates α_{OLS} and λ_{OLS} , can be obtained by minimizing the function:

$$L(x_{(i)}|\Theta) = \sum_{i=1}^{n} [\exp[-(\exp(\lambda/x_{(i)}) - 1)^{\frac{1}{\alpha}}] - \tfrac{i}{n+1}]^2, \quad (34)$$

with respect to α and λ. Where $x_(i)$ is the ordered statistics of the random sample of size n. Equivalently, the estimates can be obtained by solving the following non-linear equations

$$\sum_{i=1}^{n} [\exp[-(\exp(\lambda/x_{(i)}) - 1)^{\frac{1}{\alpha}}] - \tfrac{i}{n+1}]\Psi_1(x_{(i)}|\lambda,\alpha) = 0 \quad (35)$$

and

$$\sum_{i=1}^{n} [\exp[-(\exp(\lambda/x_{(i)}) - 1)^{\frac{1}{\alpha}}] - \tfrac{i}{n+1}]\Psi_2(x_{(i)}|\lambda,\alpha) = 0, \quad (36)$$

where

$$\Psi_1(x_{(i)}|\lambda,\alpha) = \frac{\partial}{\partial \lambda} G_X(x_{(i)}) \quad (37)$$

$$= -\frac{\exp(\frac{\lambda}{x_i})(\exp(\frac{\lambda}{x_i}) - 1)^{\frac{1}{\alpha}^{-1}}}{\alpha x_(i)} \exp(-(\exp(\frac{\lambda}{x_(i)}) - 1)^{\frac{1}{\alpha}})$$

and

$$\Psi_2(x_{(i)}|\lambda,\alpha) = \frac{\partial}{\partial \alpha} G_X(x_{(i)}) = -(\exp(\frac{\lambda}{x_(i)}) - 1)^{\frac{1}{\alpha}} \times \log(\exp(\frac{\lambda}{x_(i)}) - 1)\exp(-(\exp(\frac{\lambda}{x_(i)}))^{\frac{1}{\alpha}}). \quad (38)$$

The WLS estimates can be obtained by minimizing the following function

$$W(x_{(i)}|\Theta) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}[\exp[-(\exp(\lambda/x_{(i)}) - 1)^{\frac{1}{\alpha}}] - \tfrac{i}{n+1}]^2 \quad (39)$$

with respect to α and λ. Equivalently, the WLS estimates can obtained by solving the following non-linear equations

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}[\exp[-(\exp(\lambda/x_{(i)}) - 1)^{\frac{1}{\alpha}}] - \tfrac{i}{n+1}]\Psi_1(x_{(i)}|\lambda,\alpha) = 0 \quad (40)$$

and

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}[\exp[-(\exp(\lambda/x_{(i)}) - 1)^{\frac{1}{\alpha}}] - \tfrac{i}{n+1}]\Psi_2(x_{(i)}|\lambda,\alpha) = 0. \quad (41)$$

where $\Psi_1(x_{(i)}|\lambda,\alpha) = 0$ and $\Psi_2(x_{(i)}|\lambda,\alpha)= 0$ are defined in (37) and (38) respectively.

6 Simulation

In this section, simulation results for the estimators of the parameters of the GIEL distribution are presented. The simulation experiment was replicated for $N = 1500$ each with sample size of $n = 20, 50, 100, 300$ and 600; parameter values $(\alpha, \lambda) = (0.4, 4.4)(1.5, 1.2)(2.3, 0.5)$ and $(1.2, 1.5)$. Table 1 and Table 2 present the average absolute bias (ABias) and mean square error (MSE) of the MLE, OLS and WLS estimators for some selected parameter values. The MSEs and ABias of MLE decreases as the sample size increases. However, the MSEs and ABias of OLS and WLS decreases intermittently as the sample size increases. It can be concluded that, MLE is the best methods in estimating GIEL distribution.

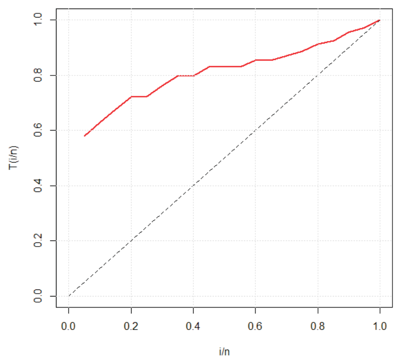

Fig. 7: TTT plot for analgesic data

7 Empirical Applications

This section presents the applications of the GIEL distribution using three real data sets to illustrate its flexibility. The maximum likelihood estimation (MLE) procedure was used to estimate model parameters. Model selection was done using log-likelihood value, Akaike information criterion (AIC), corrected Akaike information criterion (AICc), Bayesian information criterion (BIC), Cramér-Von Mises minimum distance (W*), and Kolmogorove-Smirnov (K-S) statistics.

7.1 Uncensored data sets

7.1.1 Analgesic data

The first data set is a complete data that represents the relief times of patients receiving an analgesic. The data set can be found in [17] and consists of 20 observations as follows: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2. The distribution of the data is positively skewed (skewness=1.59) with an increasing hazard rate function as shown in Figure 7.
The performance of GIEL distribution was compared with Kumaraswamy inverse exponential (KIE) distribution [16], inverse exponential (IE) distribution [10], Burr type XII (BXII) distribution [6] and generalized inverse exponential (GIE) distribution [1]. The estimates of the parameters, standard errors and 95 percent confidence intervals (CI) are presented in Table 3. The log-likelihood values and the goodness of fit statistics are presented in Table 4. The estimates in Table 4 indicate that the GIEL distribution provides a better fit to the data set compare with the other models. Figure 8 displays the histogram, fitted PDFs, empirical CDFs and fitted CDFs for analgesic data set. From the plots, GIEL distribution mimics the shapes of the data set better than the other fitted distributions.

7.1.2 Roofing sheet data

The second data set was retrieved from [19]. The data set represents coating weight (gm/m^2) by chemical method on top center side (TCS) of roofing sheets from ALAF industry, Tanzania. The observations are; 36.8, 47.2, 35.6, 36.7, 55.8, 58.7, 42.3, 37.8, 55.4, 45.2, 31.8, 48.3, 45.3, 48.5, 52.8, 45.4, 49.8, 48.2, 54.5, 50.1, 48.4, 44.2, 41.2, 47.2, 39.1, 40.7, 4.03, 41.2, 30.4, 42.8, 38.9, 34, 33.2, 56.8, 52.6, 40.5, 40.6, 45.8, 58.9, 28.7, 37.3, 36.8, 40.2, 58.2, 59.2, 42.8, 46.3, 61.2, 58.4, 38.5, 34.2, 41.3, 42.6, 43.1, 42.3, 54.2, 44.9, 42.8, 47.1, 28.9, 42.8, 29.4, 32.7, 40.1, 33.2, 31.6, 36.2, 33.6, 32.9,34.5, 33.7,39.9. The data is positively skewed (skewness=0.42) and has an increasing hazard rate as display by Figure 9.
The performance of the GIEL distribution is compare with the two parameter Weibull (WD) [18], BXII, inverse Rayleigh (IRD) [20] and Rayleigh (RD) distributions. The estimates of the parameters, their corresponding standard errors and 95 percent confidence level (CI) are presented in Table 5. Table 6 presents the model selection criteria for the TCS data set. It can be realized from Table 6 that, the GIEL distribution provide a better fit to the TCS data better than the other candidate models. Figure 10 displays the histogram, fitted PDFs, empirical CDFs and fitted CDFs for the TCS data set. From the plots, GIEL distribution mimics the shapes of the data set better than the other fitted distributions.

7.2 Censored data set

This section presents the application of the GIEL distribution to a censored data set. The performance of GIEL distribution was compared with inverse Rayleigh (IRD) distribution, Rayleigh (RD) distribution and Burr type XII (BXII) distribution using the log-likelihood values, AIC, AICc and BIC. These data represent the recurrent times of infection at the point of inserting a catheter for patients undergoing kidney dialysis. The data was retrieved from [11] and has 38 observations as follows: 16, 13*, 28, 318, 12, 245, 9, 30, 196, 154, 333, 8*, 38, 70*, 25*, 4*, 177, 114, 459*, 108*, 562, 24*, 66, 46*, 40, 201, 156, 30, 25, 26, 58, 43, 30, 5*, 8, 16*, 78, 8*. Data values with asterisks are censored data. The distribution of the data set has a decreasing failure rate as shown by the TTT plot in Figure 11.

Table 1: Simulation results for $(\alpha = 0.4, \lambda = 4.4) and (\alpha = 1.5, \lambda = 1.2)$

Parameter	n	MLE		OLS		WLS	
		ABias	MSE	ABias	MSE	ABias	MSE
$\alpha = 0.4$	20	0.122	0.023	0.19	0.05	0.194	0.05
	50	0.078	0.009	0.183	0.041	0.204	0.054
	100	0.054	0.004	0.188	0.039	0.203	0.046
	300	0.032	0.002	0.194	0.039	0.206	0.044
	600	0.022	8.00E-04	0.194	0.038	0.206	0.043
$\lambda = 4.4$	20	0.464	0.362	13.31	183.82	13.533	189.7
	50	0.3	0.148	13.26	178.447	13.49	184.77
	100	0.199	0.063	13.2	175.56	13.456	182.31
	300	0.122	0.023	13.12	172.497	13.369	179.15
	600	0.083	0.011	13.14	172.743	13.388	179.44
$\alpha = 1.5$	20	0.248	0.095	0.35	0.171	0.364	0.178
	50	0.156	0.039	0.325	0.133	0.363	0.156
	100	0.108	0.018	0.332	0.125	0.379	0.155
	300	0.064	0.006	0.344	0.124	0.397	0.161
	600	0.045	0.003	0.344	0.121	0.401	0.162
$\lambda = 1.2$	20	0.226	0.092	3.982	18.124	4.171	19.689
	50	0.141	0.034	3.889	15.951	4.109	17.756
	100	0.094	0.014	3.835	15.064	4.065	16.907
	300	0.056	0.005	3.775	14.376	4.016	16.258
	600	0.038	0.002	3.782	14.358	4.027	16.282

The maximum likelihood estimates of the parameters, standard errors and the 95 percent confidence interval (CI) are given in Table 7. All the estimates of the GIEL distribution are significant at 5 percent level of significance.

Table 8 presents the model selection criteria and goodness-of-fit statistics for the dialysis data set. The results indicate that, the GIEL distribution provides a better fit to the data set compare to the other distributions since GIEL distribution has the highest log-likelihood value and smallest goodness-of-fit statistics.

8 Conclusion

This study presents a new family of distributions called the T-IE{Y} family using the T-R{Y} framework pioneered by [2]. Statistical properties such as mode, quantile function, moments and Shannon entropy of the new family were derived. Three special distributions, namely the LLIEW, WIEL and GIEL distributions were proposed and the shapes of their PDFs and hazard rate functions for some given parameter values studied. The plots of the hazard rate functions revealed that these families of distributions can exhibit different types of monotonic and non-monotonic failure rates. These make them suitable for modeling datasets that exhibit both monotonic and non-monotonic failure rates. Three estimations techniques; MLE, OLS and WLS were employed in estimating the

Table 2: Simulation results for $(\alpha = 2.3, \lambda = 0.5) and (\alpha = 1.2, \lambda = 1.5)$

Parameter	n	MLE		OLS		WLS	
		ABias	MSE	ABias	MSE	ABias	MSE
$\alpha = 2.3$	20	0.408	0.256	0.539	0.411	0.572	0.441
	50	0.268	0.109	0.491	0.309	0.579	0.396
	100	0.175	0.049	0.498	0.287	0.613	0.405
	300	0.103	0.017	0.517	0.281	0.653	0.435
	600	0.074	0.009	0.518	0.275	0.663	0.445
$\lambda = 0.5$	20	0.13	0.034	1.96	5.389	2.14	6.116
	50	0.081	0.011	1.841	3.822	2.057	4.737
	100	0.052	0.005	1.784	3.359	2.013	4.263
	300	0.03	0.001	1.733	3.062	1.975	3.972
	600	0.02	0.0007	1.735	3.038	1.985	3.973
$\alpha = 1.2$	20	0.192	0.057	0.281	0.11	0.29	0.113
	50	0.122	0.023	0.264	0.087	0.289	0.098
	100	0.084	0.011	0.271	0.083	0.3	0.097
	300	0.049	0.004	0.28	0.082	0.311	0.099
	600	0.035	0.002	0.281	0.08	0.319	0.108
$\lambda = 1.5$	20	0.232	0.094	4.753	24.569	4.901	26.091
	50	0.147	0.037	4.689	22.75	4.867	24.481
	100	0.098	0.015	4.647	21.926	4.832	23.693
	300	0.059	0.005	4.594	21.217	4.786	23.028
	600	0.04	0.003	4.601	21.224	4.799	23.093

parameters of the GIEL distribution and Monte Carlo simulations performed to examine the performance of these methods. The results of the simulations revealed that the MSEs and ABias of MLE decreases as the sample size increases. However, the MSEs and ABias of OLS and WLS decreases intermittently as the sample size increases. It can be concluded that, MLE is the best method in estimating the parameters of the GIEL distribution. Empirical applications with real data sets revealed the GIEL distribution performs better than the other candidate models.

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Table 3: Maximum likelihood estimates for relief time of analgesic data set

Model	Estimates	Standard error	CI
GIEL	$\alpha = 0.3418$	6.0361×10^{-2}	[0.2235, 0.4601]
	$\lambda = 1.0728$	6.1835×10^{-2}	[0.9516, 1.1940]
KIE	a=96.8838	4.2346×10^{-3}	[96.8755, 96.8921]
	b=20.7664	1.0919×10^{1}	[0, 42.1676]
	$\theta = 0.0637$	1.1193×10^{-2}	[0.0615, 0.0659]
GIE	a=20.8142	1.0964×10^{1}	[0, 42.3036]
	$\theta = 6.1757$	1.0865×10^{1}	[4.0462, 8.3052]
BXII	$\alpha = 25.1031$	2.7244×10^{1}	[0, 78.5013]
	$\beta = 0.0676$	7.4788×10^{-2}	[0.0529, 0.0823]
IE	$\lambda = 1.7248$	3.8567×10^{-1}	[0.9689, 2.4807]

Table 4: Model selection criteria for analgesic data set

Model	Log-likelihood	AIC	AICc	BIC	W*	K-S	P-value
GIEL	-15.5400	35.0856	37.7915	37.0770	0.0327	0.1164	0.9491
GIE	-17.1000	38.2091	38.9150	40.2006	0.0806	0.1543	0.7278
KIE	-17.1050	40.2091	41.7091	43.1963	0.0805	0.1543	0.7280
IE	-32.6690	67.337	67.5596	68.331	0.0490	0.3872	0.0050

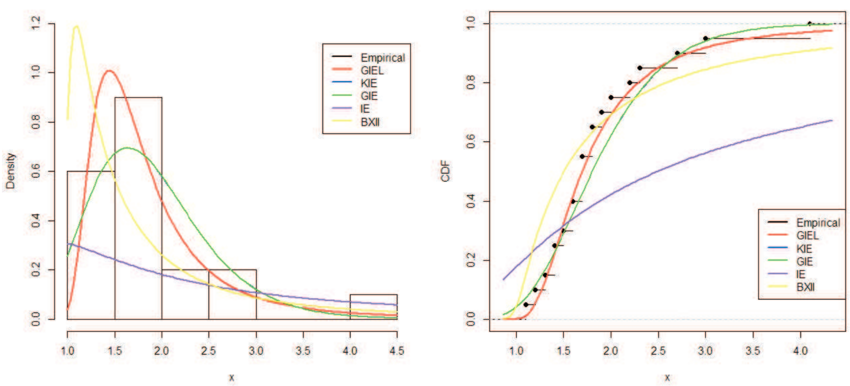

Fig. 8: PDFs and CDFs for analgesic data set

Table 5: Maximum likelihood estimates for TCS data

Model	Estimates	Standard error	CI
GIEL	$\alpha = 0.2421$	2.1784×10^{-2}	[0.1994, 0.2851]
	$\lambda = 26.5815$	5.7754×10^{-1}	[25.4495, 27.7135]
WD	$\beta = 0.9825$	1.1579×10^{-2}	[0.7556, 1.2094]
	$\theta = 43.0989$	5.1252×10^{0}	[33.0535, 53.1443]
BXII	$\alpha = 2.7783$	4.9440×10^{0}	[0, 12.4685]
	$\beta = 0.0961$	1.7131×10^{-1}	[0, 0.4319]
IRD	$\delta = 40.8748$	2.4086×10^{0}	[36.1539, 45.5957]
RD	$\delta = 31.0159$	1.8276×10^{0}	[27.4338, 34.6340]

Table 6: Model selection

Model	**Log-likelihood**	**AIC**	**AICc**	**BIC**	**W***	**K-S**	**P-value**
GIEL	-255.700	515.463	515.637	520.016	0.138	0.103	0.4296
IRD	-296.800	595.604	595.661	597.880	0.036	0.365	9.035×10^{-9}
RD	-296.890	595.789	595.846	598.066	0.070	0.363	1.122×10^{-8}
WD	-342.950	689.897	690.071	694.450	0.050	0.480	7.661×10^{-15}
BXII	-436.760	877.517	877.691	882.070	0.034	0.592	2.2×10^{-16}

Table 7: Maximum likelihood estimates for the dialysis data

Model	Estimates	Standard error	CI
GIEL	$\alpha = 1.7009$	2.8971×10^{-1}	[1.331, 2.2687]
	$\lambda = 22.5010$	4.1303×10^{0}	[14.4052, 30.5964]
BXII	$\alpha = 3.2910$	7.5129×10^{0}	[0, 18.0163]
	$\beta = 0.0545$	1.2474×10^{-1}	[0, 0.2990]
IRD	$\delta = 139.8074$	1.3731×10^{1}	[112.8946, 166.7201]
RD	$\delta = 25.1519$	2.1962×10^{0}	[20.8473, 29.4565]

Table 8: Model selection criteria for the dialysis data

Model	**Log-likelihood**	**AIC**	**AICc**	**BIC**
GIEL	-152.230	316.455	316.798	319.730
RD	-175.610	353.228	353.340	354.866
BXII	-177.900	359.827	360.170	363.102
IRD	-180.360	362.710	362.821	364.348

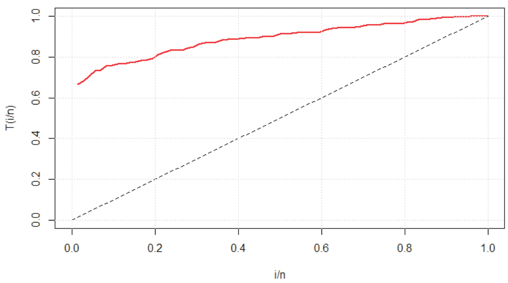

Fig. 9: TTT plot for TCS data

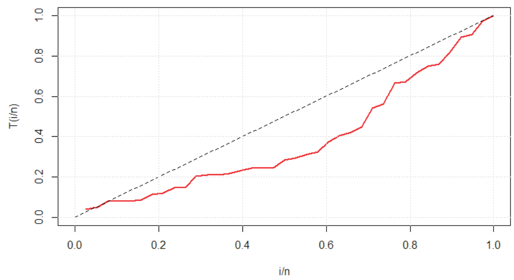

Fig. 11: TTT plot for dialysis data

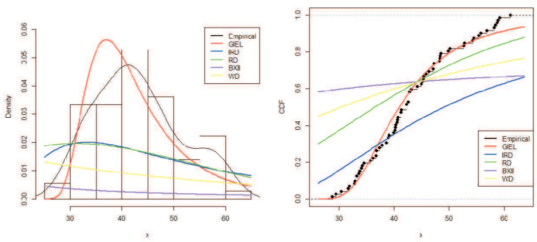

Fig. 10: PDFs and CDFs for the TCS data

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