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# Approximate Solution of an Unknown Coefficients in Parabolic Equation

*Mine Aylin Bayrak*[1,*] *and Emine Can*[2]

[1] Kocaeli University, Department of Mathematics, Izmit, Kocaeli, Turkey
[2] Kocaeli University, Department of Physics, Izmit, Kocaeli, Turkey

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**Abstract:** In this paper, an inverse parabolic equation is solved by using the homotopy analysis method (HAM) and the homotopy perturbation method (HPM). The approximation solution of this equation is calculated in the form of series which its components are computed easily. HPM is shown not always to generate a continuous family of solutions in terms of the homotopy parameter. By the convergence-control parameter this can however be prevented to occur in the HAM. Illustrative examples are presented to exhibit a comparison between the HAM and the HPM.

**Keywords:** Inverse parabolic equation; Homotopy analysis method; Homotopy perturbation method; Convergence

## 1 Introduction

Consider the following inverse problem of simultaneously finding unknown coefficients $p(t)$ and $u(x,t)$ from the following parabolic equation

$$u_t(x,t) = u_{xx}(x,t) + p(t)u(x,t) + qu_x(x,t) + f(x,t), \quad x \in (0,1), t \in (0,T] \tag{1}$$

with the initial-boundary conditions

$$u(x,0) = g(x), x \in (0,1) \tag{2}$$

$$u(0,t) = h_1(t), t \in (0,T] \tag{3}$$

$$u(1,t) = h_2(t), t \in (0,T] \tag{4}$$

and the additional specification

$$u(x^*,t) = E(t), x^* \in (0,1), t \in (0,T] \tag{5}$$

where $f(x,t), g(x), h_1(t), h_2(t)$ and $E(t) \neq 0$ are known functions, $q$ is a known constant and $x^*$ is a fixed prescribed interior point in $(0,1)$. Physically, the inverse coefficient problem is the reconstruction of a property of a medium in some bounded region by using state mesurements taken on the boundary. Inverse coefficient problems for parabolic equations have been studied by many people, for example, by [1,2,3,4,5,6,7]. The homotopy method is a combination of the classical perturbation technique and homotopy concept as used in topology. Liao in [8,9], proposed for the first time this technique, named the homotopy analysis method (HAM). This method has been successfully applied in many engineering [10,11,12,13]. Afterwards, He in [14] gave the homotopy perturbation method (HPM). Different from other methods, the HAM provides a simple way to control and adjust the convergence region of solution series by means of an auxiliary parameter [15,16]. Unfortunately, Sajid and Hayat [17] pointed out that the so-called homotopy perturbation has nothing new except its new name, because the HPM is only a special case of the homotopy analysis method (HAM) so that all results given by the HPM can also be obtained by the HAM as a special case.

## 2 Homotopy methods

Homotopy methods as applied to the nonlinear equations were first proposed by the Chinese mathematician Liao [18]. The essential idea of this method is to introduce a homotopy parameter, say $p$, which varies continuously from 0 to 1 and a nonzero auxiliary parameter so-called

* Corresponding author e-mail: aylin@kocaeli.edu.tr

the convergence control parameter $\hbar$. At $p=0$, the system of equations usually is reduced to a simplified form which normally admits a rather simple solution. As $p$ gradually increases continuously toward 1, the system goes through a sequence of deformations, and the solution at each stage is close to that at the previous stage of the deformation. Eventually, at $p=1$, the system takes the original form of the equation and the final stage of the deformation gives the desired solution. To illuminate the application of the homotopy method on the nonlinear differential equations, consider the general problem

$$N(u)=0, B(u,\frac{du}{dn})=0 \quad (6)$$

where $u$ is the function to be solved under the boundary constraints given in B. The homotopy on $u(r,p): R\times[0,1]\to R$ can be constructed so that

$$H(u,p)=(1-p)[L(u)-L(u_0)]+p\hbar N(u) \quad (7)$$

where $L$ is a suitable auxiliary linear operator to approximate the solution and $u_0$ is an initial approximation of Eq.(6) satisfying exactly the boundary conditions. It is obvious from Eq. (7) that

$$H(u,0)=L(u)-L(u_0), H(u,1)=N(u). \quad (8)$$

Thus, as $p$ moves from 0 to 1, $u(r,p)$ deforms from $u_0(r)$ to $u(r)$. Besides, the solution of Eq.(7) can be expressed as a power series in $p$ at $p=0$ in the form

$$u(r,p)=u_0(r)+\sum_{k=0}^{\infty}u_k(r)p^k. \quad (9)$$

At $p=1$, on the condition that the convergence of series (9) is guaranteed, an analytic series solution is determined in the form

$$u(r,p)=\sum_{k=0}^{\infty}u_k(r), \quad (10)$$

where

$$u_k(r)=\frac{\partial^k u(r,p)}{\partial p^k}|_{p=0}, \quad (11)$$

Hence, the approximate solution to (6) can always be written at the $M$th-order of approximation by truncating the series (10)

$$u(r)=\sum_{k=0}^{M}u_k(r), \quad (12)$$

It should be remarked that when $\hbar$ is set to unity in (7), solutions will be independent of $\hbar$ and the resulting homotopy is termed the homotopy perturbation method (HPM), whereas as a general case of $\hbar$, it is named the homotopy analysis method (HAM).

**Theorem** Suppose that $A\subset R$ be a Banach space donated with a suitable norm, say $\|.\|_\infty$, over which the sequence $u_k(r)$ of (9) is defined for a prescribed value of $\hbar$. Assume also that the initial approximation $u_0(r)$ remains inside the ball of the solution $u(r)$. Taking $\gamma\in R$ as a constant, the following statements hold.

(i)If there exists some $\gamma\in[0,1]$, such that for all $k$, we have $\|u_{k+1}(r)\|\leqslant\gamma\ |u_k(r)\|$, then the series solution $u(r,p)=\sum_{k=0}^{\infty}u_k(r)p^k$, converges absolutely to (10), at $p=1$, over the domain of definition of $r$.

(ii)If there exists some $\gamma>1$, such that for all $k$, we have $\|u_{k+1}(r)\|\leqslant\gamma\ |u_k(r)\|$, then the series solution $u(r,p)=\sum_{k=0}^{\infty}u_k(r)p^k$, diverges at $p=1$, over the domain of definition of $r$.[13]

## 3 Analysis of the methods

In this section, the homotopy method is considered for solving problem $(1)-(5)$. Applying a pair of transformations [1] as follows

$$w(x,t)=u(x,t)exp(\frac{q}{2}x)r(t), x\in(0,1), t\in(0,T] \quad (13)$$

$$w(x,0)=g(x)exp(\frac{q}{2}x), x\in(0,1) \quad (14)$$

$$w(0,t)=r(t)h_1(t), t\in(0,T] \quad (15)$$

$$w(1,t)=r(t)h_2(t)exp(\frac{q}{2}), t\in(0,T] \quad (16)$$

subject to

$$r(t)=\frac{w(x^*,t)}{E(t)}exp(-\frac{q}{2}x^*), x\in(0,1), t\in(0,T].$$

It is easy to show that the original inverse problem $(1)-(5)$ is equivalent to the auxiliary problem $(13)-(16)$. Firstly, let choose the linear operator

$$L[u(x,t;p)]=\frac{\partial u(x,t;p)}{\partial t} \quad (17)$$

with the property $L[c]=0$, where $c$ is a constant. Also let define

$$N[u(x,t;p)]=\frac{\partial w}{\partial t}-\frac{\partial^2 w}{\partial x^2}-r(t)exp(\frac{q}{2}x)f(x,t) \quad (18)$$

Using the above definition, the zeroth deformation is constructed

$$(1-p)L[u(x,t;p)-w_0(x,t)]=p\hbar N[u(x,t;p)] \quad (19)$$

where from (14). Obviously, when $p=0$ and $p=1$,

$$u(x,t;0)=w_0(x,t), u(x,t;1)=w(x,t) \quad (20)$$

Thus, the $m$th-order deformation equation is obtained

$$L[w_m(x,t)-\kappa_m w_{m-1}(x,t)]=\hbar R_m(w_{m-1}) \quad (21)$$

where

$$R_m(w_{m-1})=\frac{\partial w}{\partial t}-\frac{\partial^2 w}{\partial x^2}-r(t)exp(\frac{q}{2}x)f(x,t)$$

and

$$\kappa_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & \text{Otherwise} \end{cases}$$

Now the solution of the $m$th-order deformation equation is for $m \geqslant 1$

$$w_m(x,t) = \kappa_m w_{m-1}(x,t) + \hbar \int_0^t R_m(w_{m-1}(x,t))d\tau + C \tag{22}$$

where the constant of integration $C$ is determined by the initial condition $w_m(x,0)$.

## 4 Numerical Examples

In order to assess the advantages and the accuracy of the homotopy method for solving an inverse parabolic equation, numerical examples are presented to determine the error of the approximations.

**Example 4.1** Consider problem $(1)-(5)$ with the following conditions [19]

$u(x,0) = x,$
$u(0,t) = 0,$
$u(1,t) = exp(t),$
$f(x,t) = -(2+xt^2)exp(t),$
$E(t) = \frac{1}{2}exp(t),$
$q = 2$
with $x^* = \frac{1}{2}$. The exact solution of this problem is
$u(x,t) = xexp(t)$
and
$p(t) = 1+t^2.$
Using the HAM, it follows that
$w_0(x,t) = xexp(x),$
$w_1(x,t) = xexp(x)\frac{1}{3}\hbar t(t^2-3),$
$w_2(x,t) = xexp(x)\frac{1}{18}\hbar t(t^2-3)(\hbar t^3 - 3\hbar t + 6\hbar + 6),$
$w_3(x,t) =$

$$xexp(x)\frac{1}{162}\hbar t(t^2-3)\begin{pmatrix} \hbar^2t^6 - 6\hbar^2t^4 + 18\hbar^2t^3 + 9\hbar^2t^2 \\ -54\hbar^2 t + 54\hbar^2 + 18\hbar t^3 \\ -54\hbar t + 108\hbar + 54 \end{pmatrix}$$

$\vdots$

Hence, with the third-order approximation by HAM

$$w(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + w_3(x,t) + \ldots$$

$$= xexp(x)\begin{pmatrix} \frac{\hbar^3t^9}{162} - \frac{\hbar^3t^7}{18} + \frac{\hbar^3t^6}{9} + \frac{\hbar^3t^5}{6} - \frac{2\hbar^3t^4}{3} + \frac{\hbar^3t^3}{6} \\ +\hbar^3t^2 - \hbar^3 t + \frac{\hbar^2t^6}{6} - \hbar^2t^4 + \hbar^2t^3 \\ +\frac{3\hbar^2t^2}{2} - 3\hbar^2 t + \hbar t^3 - 3\hbar t + 1 \end{pmatrix}$$

Homotopy perturbation method (HPM) solution is a special case of Homotopy analysis method (HAM) solution when $\hbar = -1$. This fact has been pointed out by many researchers such as Abbasbandy [20], Liao et al.[16], Bataineh et al. [21], Van Gorder et al. [22], Hayat and Sajid [23], Turkyilmazoglu [24], Das and Gupta [25]. A comparison between the absolute error for HAM ($\hbar = -1.09$), HPM and the exact solution at $t = 1$ and various values of $x$ is presented in Table 1.

Suppose that $N : R \times [0,1] \to R$ and taking $v_0(x,t) = w_0(x,t) = w(x,0)$ and using Eq. (22) we have

$\|v_0 - w\| = \|1 - exp(t - \frac{t^3}{3})\|$

$\|V_1 - w\| = \left\|1 + \frac{1}{3}\hbar t(t^2-3) - exp(t - \frac{t^3}{3})\right\|$ Since, for all

$\leqslant \left\|1 - exp(t - \frac{t^3}{3})\right\| \left\|1 + \frac{\hbar t(t^2-3)}{3 - 3exp(t - \frac{t^3}{3})}\right\|$

$t \in [0,1]$ we have

$\left\|1 + \frac{\hbar t(t^2-3)}{3-3exp(t-\frac{t^3}{3}))}\right\| \leqslant \gamma < 1$

where $\gamma = 0.0011$ for HAM ($\hbar = -1.42$) and $\gamma = 0.2966$ for HPM ($\hbar = -1$).

Therefore,

$\|V_1 - w\| \leqslant \gamma \left\|1 - exp(t - \frac{t^3}{3})\right\| = \gamma \|v_0 - w\|$

$$\|V_2 - w\| = \left\| \begin{matrix} 1 + \frac{1}{3}\hbar t(t^2-3) - exp(t - \frac{t^3}{3}) \\ + \frac{1}{18}\hbar t(t^2-3)(\hbar t^3 - 3\hbar t + 6\hbar + 6) \end{matrix} \right\|$$

$\forall t \in [0,1], \left\|1 + \frac{\hbar t(t^2-3)(\hbar t^3 - 3\hbar t + 6\hbar + 6)}{18 + 6\hbar t(t^2-3)}\right\| \leqslant \gamma_1 < \gamma$

where $\gamma_1 = 0.0051$ for HAM ($\hbar = -1.16$) and $\gamma_1 = 0.2094$ for HPM ($\hbar = -1$).

Thus,

$\|V_2 - w\| \leqslant \gamma^2 \|v_0 - w\|$

$\|V_3 - w\| =$

$$\left\| \begin{matrix} 1 + \frac{1}{3}\hbar t(t^2-3) \\ + \frac{1}{18}\hbar t(t^2-3)(\hbar t^3 - 3\hbar t + 6\hbar + 6) - exp(t - \frac{t^3}{3}) \\ + \frac{1}{162}\hbar t(t^2-3)\begin{pmatrix} \hbar^2t^6 - 6\hbar^2t^4 + 18\hbar^2t^3 + 9\hbar^2t^2 \\ -54\hbar^2 t + 54\hbar^2 + 18\hbar t^3 - 54\hbar t \\ +108\hbar t + 54 \end{pmatrix} \end{matrix} \right\|$$

$\forall t \in [0,1],$

$$\left\|1 + \frac{\hbar t(t^2-3)\begin{pmatrix} \hbar^2t^6 - 6\hbar^2t^4 + 18\hbar^2t^3 + 9\hbar^2t^2 \\ -54\hbar^2 t + 54\hbar^2 + 18\hbar t^3 \\ -54\hbar t + 108\hbar t + 54 \end{pmatrix}}{162 + 54(\hbar t(t^2-3)) + 9(\hbar t(t^2-3)(\hbar t^3 - 3\hbar t + 6\hbar + 6))}\right\| \leqslant \gamma_2 < \gamma$$

where $\gamma_2 = 0.0058$ for HAM ($\hbar = -1.09$) and $\gamma_2 = 0.1608$ for HPM ($\hbar = -1$).

Thus,

$\|V_3 - w\| \leqslant \gamma^3 \|v_0 - w\|$

Therefore, $\|V_n - w\| \leqslant \gamma^n \|v_0 - w\|$

**Example 4.2** Consider problem $(1)-(5)$ with the following conditions [19]

$u(x,0) = sin(\frac{\pi}{2}x),$
$u(0,t) = 0,$
$u(1,t) = exp(t),$
$f(x,t) = ((\frac{\pi^2}{4} - t)sin(\frac{\pi}{2}x) - \pi cos(\frac{\pi}{2}x))exp(t),$
$E(t) = \frac{\sqrt{2}}{2}exp(t),$
$q = 2,$
with $x^* = \frac{1}{2}$. The exact solution of this problem is
$u(x,t) = sin(\frac{\pi}{2}x)exp(t)$
and
$p(t) = 1+t$
Using the HAM, it follows that
$w_0(x,t) = sin(\frac{\pi}{2}x)exp(x),$
$w_1(x,t) = sin(\frac{\pi}{2}x)exp(x)\frac{1}{2}\hbar t(t-2),$
$w_2(x,t) = sin(\frac{\pi}{2}x)exp(x)\frac{1}{8}\hbar t(t-2)(\hbar t^2 - 2\hbar t - 4\hbar + 4),$

$w_3(x,t) =$

$$sin(\tfrac{\pi}{2}x)exp(x)\tfrac{1}{48}\hbar t(t-2)\begin{pmatrix} \hbar^2t^4 - 4\hbar^2t^3 + 16\hbar^2t^2 \\ -24\hbar^2t + 24\hbar^2 + 12\hbar t^2 \\ -24\hbar t + 48\hbar + 24 \end{pmatrix}$$

$\vdots$

Hence, with the third-order approximation by HAM
$w(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + w_3(x,t) + \ldots$

$$= sin(\tfrac{\pi}{2}x)exp(x)\begin{pmatrix} \frac{\hbar^3t^6}{48} - \frac{\hbar^3t^5}{8} + \frac{\hbar^3t^4}{2} - \frac{7\hbar^3t^3}{6} + \frac{3\hbar^3t^2}{2} \\ +\frac{3\hbar^2t^4}{8} - \hbar^3t - \frac{3\hbar^2t^3}{2} + 3\hbar^2t^2 \\ -3\hbar^2t + \frac{3\hbar t^2}{2} - 3\hbar t + 1 \end{pmatrix}$$

A comparison between the absolute error for HAM ($\hbar = -1.07$), HPM and the exact solution at $t = 1$ and various values of $x$ is presented in Table 2.
Suppose that $N : R \times [0,1] \to R$ and taking $v_0(x,t) = w_0(x,t) = w(x,0)$ and using Eq. (22) we have

$$\|v_0 - w\| = \|1 - exp(t - \tfrac{t^2}{2})\|$$

$$\|V_1 - w\| = \left\|1 + \tfrac{1}{2}\hbar t(t-2) - exp(t - \tfrac{t^2}{2})\right\|$$

$$\leqslant \left\|1 - exp(t - \tfrac{t^2}{2})\right\| \left\|1 + \frac{\hbar t(t-2)}{2 - 2exp(t - \frac{t^2}{2})}\right\|$$

Since, for all $t \in [0,1]$ we have

$$\left\|1 + \frac{\hbar t(t-2)}{2 - 2exp(t - \frac{t^2}{2})}\right\| \leqslant \gamma < 1$$

where $\gamma = 0.0020$ for HAM ($\hbar = -1.3$) and $\gamma = 0.2293$ for HPM ($\hbar = -1$).
Therefore,

$$\|V_1 - w\| \leqslant \gamma\left\|1 - exp(t - \tfrac{t^2}{2})\right\| = \gamma\|v_0 - w\|$$

$$\|V_2 - w\| = \left\|\begin{matrix} 1 + \frac{1}{2}\hbar t(t-2) - exp(t - \frac{t^2}{2}) \\ +\frac{1}{8}\hbar t(t-2)(\hbar t^2 - 2\hbar t - 4\hbar + 4) \end{matrix}\right\|$$

$$\forall t \in [0,1], \quad \left\|1 + \frac{\hbar t(t-2)(\hbar t^2 - 2\hbar t - 4\hbar + 4)}{8 + 4\hbar t(t-2)}\right\| \leqslant \gamma_1 < \gamma$$

where $\gamma_1 = 0.0099$ for HAM ($\hbar = -1.12$) and $\gamma_1 = 0.1595$ for HPM ($\hbar = -1$).
Thus,

$$\|V_2 - w\| \leqslant \gamma^2\|v_0 - w\|$$

$$\|V_3 - w\| =$$

$$\left\|\begin{matrix} 1 + \frac{1}{2}\hbar t(t-2) \\ +\frac{1}{8}\hbar t(t-2)(\hbar t^2 - 2\hbar t - 4\hbar + 4) - exp(t - \frac{t^2}{2}) \\ +\frac{1}{48}\hbar t(t-2)\begin{pmatrix} \hbar^2t^4 - 4\hbar^2t^3 + 16\hbar^2t^2 \\ -24\hbar^2t + 24\hbar^2 + 12\hbar t^2 \\ -24\hbar t + 48\hbar + 24 \end{pmatrix} \end{matrix}\right\|$$

$$\forall t \in [0,1], \quad \left\|1 + \frac{\hbar t(t-2)\begin{pmatrix} \hbar^2t^4 - 4\hbar^2t^3 + 16\hbar^2t^2 \\ -24\hbar^2t + 24\hbar^2 + 12\hbar t^2 \\ -24\hbar t + 48\hbar + 24 \end{pmatrix}}{48 + 24\hbar t(t-2) + 6\hbar t(t-2)(\hbar t^2 - 2\hbar t - 4\hbar + 4)}\right\| \leqslant \gamma_2 < \gamma$$

where $\gamma_2 = 0.0060$ for HAM ($\hbar = -1.07$) and $\gamma_2 = 0.1217$ for HPM ($\hbar = -1$).
Thus,

$$\|V_3 - w\| \leqslant \gamma^3\|v_0 - w\|$$

Therefore, $\|V_n - w\| \leqslant \gamma^n\|v_0 - w\|$ that is $lim_{n\to\infty}V_n = w = sin(\frac{\pi}{2}x)exp(x + t - \frac{t^2}{2})$ which is an exact solution.

**Table 1:** Comparison of the results of the HAM ($\hbar = -1.09$) and the HPM ($\hbar = -1$) at $t = 1$.

| $x$ | Exact | HAM | HPM | Error | Error |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.2152579 | 0.2152434 | 0.2142121 | 0.0000145 | 0.0010458 |
| 0.2 | 0.4757935 | 0.4757615 | 0.4734820 | 0.0000320 | 0.0023115 |
| 0.3 | 0.7887498 | 0.7886967 | 0.7849179 | 0.0000531 | 0.0038319 |
| 0.4 | 1.1622711 | 1.1621928 | 1.1566246 | 0.0000783 | 0.0056465 |
| 0.5 | 1.6056353 | 1.6055272 | 1.5978348 | 0.0001081 | 0.0078005 |
| 0.6 | 2.1294017 | 2.1292584 | 2.1190567 | 0.0001433 | 0.0103450 |
| 0.7 | 2.7455783 | 2.7453935 | 2.7322397 | 0.0001848 | 0.0133386 |
| 0.8 | 3.4678094 | 3.4675760 | 3.4509623 | 0.0002334 | 0.0168471 |
| 0.9 | 4.3115873 | 4.3112974 | 4.2906408 | 0.0002899 | 0.0209465 |
| 1 | 5.2944899 | 5.2941337 | 5.2687683 | 0.0003562 | 0.0257216 |

**Table 2:** Comparison of the results of the HAM ($\hbar = -1.07$) and the HPM ($\hbar = -1$) at $t = 1$.

| $x$ | Exact | HAM | HPM | Error | Error |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.2850422 | 0.2850506 | 0.2845429 | 0.0000084 | 0.0004993 |
| 0.2 | 0.6222838 | 0.6223022 | 0.6211938 | 0.0000184 | 0.0010900 |
| 0.3 | 1.0103744 | 1.0104042 | 1.0086046 | 0.0000298 | 0.0017698 |
| 0.4 | 1.4457184 | 1.4457612 | 1.4431860 | 0.0000428 | 0.0025324 |
| 0.5 | 1.9221156 | 1.9221723 | 1.9187487 | 0.0000567 | 0.0033669 |
| 0.6 | 2.4304214 | 2.4304931 | 2.4261642 | 0.0000717 | 0.0042572 |
| 0.7 | 2.9582458 | 2.9583333 | 2.9530642 | 0.0000875 | 0.0051816 |
| 0.8 | 3.4897084 | 3.4898117 | 3.4835958 | 0.0001033 | 0.0061126 |
| 0.9 | 4.0052738 | 4.0053921 | 3.9982581 | 0.0001183 | 0.0070157 |

## 5 Conclusion

In this paper, homotopy methods have been analyzed in an inverse parabolic equation. Comparison of homotopy analysis method (HAM) and homotopy perturbation method (HPM) has been given with the convergence control parameter. The theorem outlined in the paper have demonstrated that if specific values are assigned to the auxiliary parameters in the homotopy analysis method, then the approximate homotopy results successfully converge to the exact solution. HAM has been shown to be a better tool with several examples.

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**Mine Aylin Bayrak** is Assistant Professor of Mathematics Department at University Kocaeli,Turkey, in 2003. She received B.Sc. and Ph.D. degrees in applied mathematics from Kocaeli University. Her research interests include partial differential equations, fuzzy set and systems, numerical methods.

**Emine Can** received her Ph.D degree in Physics from Yildiz Technical University, Turkey, in 2000. Her research interests include the areas of mathematical modelling, numerical methods for inverse problems, fuzzy set and systems and finite difference methods for PDES. E. Can is an Associate Professor of Mathematical Physics at Kocaeli University, Kocaeli, Turkey.