Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/200501

A System of Generalized Operator Quasi-Equilibrium Problems in Hausdorff Topological Vector Spaces

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Received: 17 Jun. 2026, Revised: 8 Aug. 2026, Accepted: 27 Aug. 2026
Published online: 1 Sep. 2026

Abstract: The purpose of this paper is to investigate a system of generalized operator quasi-equilibrium problems (SGOQEP) in Hausdorff topological vector spaces. Such coupled systems arise naturally in two-player operator games and coupled optimisation problems (for instance duopoly-type competition over pricing or production operators, and network-flow or resource-allocation settings with operator-valued capacities or tolls), where each player's feasible set and payoff depend on the other player's current position. Under suitable continuity, convexity, and compactness assumptions, existence theorems for solutions are established. Our results generalize and improve various corresponding results for generalized equilibrium and variational inequality problems reported in earlier studies. The problem asks for a pair (f_1°, f_2°) in a product of operator spaces such that each component satisfies a quasi-equilibrium condition with a constraint map that depends on both components simultaneously. For the compact case, existence is proved using $C(f)$-quasiconvexity and the Ding–Kim–Tan fixed-point theorem applied to a product auxiliary map. When the operator domains are non-compact, we replace compactness by a ϕ-condensing assumption on the constraint maps and invoke the maximal-element theorem of Lin, Park and Yu together with an escaping-sequence condition. Earlier results of Raouf–Gupta–Sharma [21], Sharma–Gupta–Raouf [22], Kazmi–Raouf [11,12] and Kim–Raouf [17] follow as special cases. We also obtain existence of solutions for a system of operator quasi-variational-like inequality problems and derive an operator quasi-saddle point result.

Keywords: $C(f)$-quasiconvexity; coupled constraint maps; Hausdorff topological vector space; ϕ-condensing maps; quasi-variational-like inequality

1 Introduction and Motivation

The study of equilibrium problems in the sense of Blum and Oettli [4] has attracted considerable attention since 1994. Their formulation, which asks for a point f in a convex set K such that $F(f,g) \geq 0$ for all $g \in K$, subsumes variational inequalities, saddle-point problems, complementarity problems and convex minimisation as special cases. Various extensions and generalisations have since been studied in topological and geometric settings; see [1,2,4,13,14,16,15]. This line of inquiry remains active: existence, convergence, and sharp-minima results for quasi-equilibrium and quasi-variational-inequality formulations continue to appear, both in the classical Banach-space setting and under weaker local-reproducibility-type conditions [19,3]. More recently, the KKM condition itself has been weakened for generalized quasi-equilibrium problems [6], and mixed variational-like inequalities have been studied under generalized quasimonotonicity [20].

Kazmi and Raouf [11] moved the solution domain from a topological vector space to $L(W,Z)$, the space of all continuous linear operators from W to Z equipped with the topology of point-wise convergence. When $W = \mathbb{R}$ and $Z = \mathbb{R}$, one has $L(\mathbb{R},\mathbb{R}) \cong \mathbb{R}$ and the framework reduces to scalar equilibrium in the sense of [4]. When $W = \mathbb{R}$, $L(\mathbb{R},Z) \cong Z$ and the setting subsumes vector equilibrium problems. Subsequent work addressed perturbed versions [12], quasi-equilibrium formulations with $C(f)$-quasiconvexity [21], generalized operator equilibrium [17], and ϕ-condensing settings [22].

All of the above results concern a single operator quasi-equilibrium condition of the form: find $f \in \mathrm{cl}^D S(f)$ with $F(f,g) \nsubseteq -C(f)$ for all $g \in S(f)$. In contrast,

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two-player operator games and coupled optimisation problems in function spaces require a pair (f_1°, f_2°) where each component satisfies its own equilibrium condition and the feasible set of each player depends on the other player's current position. No existence theorem for such a coupled system appears in the operator equilibrium literature.

We fill this gap by introducing and studying the system of generalized operator quasi-equilibrium problems (SGOQEP): find $(f_1^\circ, f_2^\circ) \in D_1 \times D_2$ such that

$$f_1^\circ \in \mathrm{cl}_1^D T_1(f_1^\circ, f_2^\circ) \text{ and } F_1(f_1^\circ, f_2^\circ, g_1) \nsubseteq -C(f_1^\circ, f_2^\circ), \quad \forall g_1 \in T_1(f_1^\circ, f_2^\circ),$$

$$f_2^\circ \in \mathrm{cl}_2^D T_2(f_1^\circ, f_2^\circ) \text{ and } F_2(f_1^\circ, f_2^\circ, g_2) \nsubseteq -C(f_1^\circ, f_2^\circ), \quad \forall g_2 \in T_2(f_1^\circ, f_2^\circ).$$

In (SGOQEP), T_1 and T_2 are set-valued maps from $D_1 \times D_2$ to $\Pi(D_1)$ and $\Pi(D_2)$ respectively, so that $T_1(f_1, f_2)$ and $T_2(f_1, f_2)$ each depend on both components of the pair. F_1 and F_2 are operator-valued bifunctions. When $T_1(f_1, f_2) \equiv T_1(f_1)$ and $T_2(f_1, f_2) \equiv T_2(f_2)$ the two conditions decouple, and one is simply solving two independent OQEPs. The genuine coupling in both constraint maps is therefore what makes (SGOQEP) a new problem.

We prove two existence theorems. In Theorem 4, both D_1 and D_2 are compact and convex. We define a product auxiliary map $P(f) = P_1(f) \times P_2(f)$ on $D = D_1 \times D_2$ and verify that $f \notin \mathrm{co}\, P(f)$ using $C(f)$-quasiconvexity. We then show that $P^{-1}(g)$ is open via the closed-graph hypothesis on $G_i(f) = W \setminus (-C(f))$, and apply the Ding–Kim–Tan theorem [8,9] to obtain the solution. In Theorem 5, each D_i is written as a countable increasing union of compact convex sets; the constraint maps T_1, T_2 are assumed ϕ-condensing. Theorem 4 is applied on each compact level set to produce a sequence of approximate solutions. An escaping-sequence condition rules out the sequence leaving every compact piece, and the limit point is verified to solve (SGOQEP) using the Lin–Park–Yu theorem [18]. The product structure of the auxiliary map must be tracked carefully in both arguments.

The remainder of the paper is laid out as follows. Section 2 fixes notation and recalls the definitions and fixed-point theorems needed. Section 3 states (SGOQEP) and contains the proofs of Theorems 4 and 5. Section 4 shows how the results of [11,12,17,21,22] are recovered as corollaries. Section 5 gives the application to the system of operator quasi-variational-like inequality problems and the quasi-saddle point corollary.

Figure 1 summarises how the three special cases discussed later in the paper sit relative to (SGOQEP).

Every existing operator quasi-equilibrium result, including [11,12,17,21,22], proves existence for a single condition $f \in \mathrm{cl}^D S(f)$ with $F(f,g) \nsubseteq -C(f)$; none allows the constraint map or the equilibrium bifunction of one variable to depend on a second, jointly varying

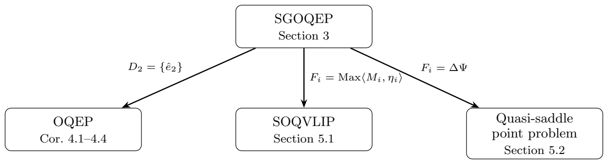

Fig. 1 Each arrow names the substitution into (SGOQEP) that produces the special case, and the section where it is carried out.

variable. (SGOQEP) is therefore the first coupled formulation of this kind in the operator setting. Theorems 4 and 5, proved below, handle the coupling by building a single product auxiliary map on $D = D_1 \times D_2$ and applying a fixed-point argument once, on the product space, instead of solving each player's condition separately. Setting $T_i(f_1, f_2) \equiv T_i(f_1)$ recovers a single OQEP as the special case $D_2 = \{\hat{e}_2\}$ (Remark 3.2, Corollary 4.1), so the earlier single-condition results remain available as corollaries, and the coupled case that [11,12,17,21,22] cannot express is covered as well.

2 Preliminaries

The notation and setting below hold throughout. Table 1 collects every symbol used in the paper, grouped by role, for quick reference.

Let W, Z_1, Z_2 be Hausdorff topological vector spaces. For $i = 1,2$, let $L(Z_i, W)$ denote the space of all continuous linear operators from Z_i to W, endowed with the topology of point-wise convergence (w.r.t.p.c.). Let $D_i \subseteq L(Z_i, W)$ be nonempty convex sets and set $D = D_1 \times D_2$. We denote by $\Pi(W)$ the family of all subsets of W. Let $C : D \to \Pi(W)$ be a multi-valued map such that for each $f = (f_1, f_2) \in D$, the value $C(f)$ is a solid, convex and open cone with $0 \notin C(f)$; set $C_0(f) = C(f) \cup \{0\}$. Let W be ordered by $C(f)$, i.e., $(W, C(f))$ is an ordered topological vector space with $C(f) \neq W$.

For $i = 1,2$, let $T_i : D \to \Pi(D_i)$ be set-valued maps. Each $T_i(f_1, f_2)$ depends on both f_1 and f_2, which is the coupling condition. For $i = 1,2$, let $\mathscr{F}_i(D)$ denote the family of all set-valued maps from $D \times D_i$ to $\Pi(W) \setminus \{\varnothing\}$. Let $F_i \in \mathscr{F}_i(D)$.

The following definitions are used in the proofs.

Definition 1(Graph, inverse, upper semicontinuity). *Let $Q : B \subseteq L(Z, W) \to \Pi(W)$ be a multi-valued map.*

(i)The graph *of Q is $G(Q) = \{(f, w) \in B \times W : w \in Q(f)\}$.*
(ii)The inverse Q^{-1} is defined by $f \in Q^{-1}(w)$ if and only if $w \in Q(f)$.
(iii)Q is called upper semicontinuous *(u.s.c.) on B if for each $f \in B$ and each open set $V \subseteq W$ with $Q(f) \subseteq V$, there exists an open neighbourhood U of f in B such that $Q(f') \subseteq V$ for all $f' \in U$.*

Table 1 Notation used throughout the paper.

Symbol	**Meaning**
W, Z_1, Z_2	Hausdorff topological vector spaces fixed throughout the paper
Y	A Hausdorff topological vector space fixed for the Section 5 application
$L(Z_i, W)$	Space of continuous linear operators $Z_i \to W$, topology of pointwise convergence
$D_i \subseteq L(Z_i, W)$; $D = D_1 \times D_2$	Nonempty convex operator domains ($i = 1,2$); D is their product
D_i^n; $D^n = D_1^n \times D_2^n$	Increasing compact exhaustion of D_i, and of D, used in Theorem 5
$\Pi(\cdot)$	Family of all subsets of the space named in parentheses
$C(f), C_0(f)$	Solid, convex, open cone with $0 \notin C(f)$; $C_0(f) = C(f) \cup \{0\}$
$T_i : D \to \Pi(D_i)$; T	Set-valued constraint (feasible-set) maps and their product $T = T_1 \times T_2$
$F_i \in \mathscr{F}_i(D)$	The equilibrium map F_i and the family of set-valued maps $D \times D_i \to \Pi(W) \setminus \{\varnothing\}$ it belongs to
$G_i(f)$	$G_i(f) := W \setminus (-C(f))$; used to test $h \notin -C(f)$ via a closed-graph argument
$P_i(f)$; P	Auxiliary map $P_i(f) = \{g_i \in D_i : F_i(f, g_i) \subseteq -C(f)\}$ and its product $P = P_1 \times P_2$ (proof of Theorem 4)
ϕ; Ω	A measure of non-compactness $\phi : \Pi(E) \to \Omega$, and its codomain lattice; "ϕ-condensing" describes a map compatible with it (Definition 2.3)
$\mathscr{X}, \mathscr{A}, \mathscr{P}, \mathscr{T}, \mathscr{S}$	Generic space and maps in the abstract fixed-point results quoted from the literature (Theorems 2.1–2.3); deliberately distinct from the paper's own D, P, T_i so the two are never confused
S	Constraint map of a single OQEP (Introduction; Remark 3.2; Corollary 4.1) – unrelated to the generic $\mathscr{S}$ of Theorem 2.2
B	The fixed feasible set in Corollaries 4.3–4.4 (Kazmi–Raouf, Kim–Raouf) – unrelated to the generic compact set B of Theorem 2.1
$f^\circ, f_i^\circ, (f_1^\circ, f_2^\circ)$	A solution (pair) of (SGOQEP)
$\hat{e}_2$	The fixed singleton point used to collapse (SGOQEP) to a single OQEP (Remark 3.2, Corollary 4.1)
M_i, η_i, R_i	Operator map, bifunction, and constraint map used to build F_i in the SOQVLIP application (Section 5.1)
Ψ	The bifunction whose saddle points are sought in Section 5.2 – a different object from ϕ; deliberately a different symbol to avoid confusion
$\bar{f}, (\bar{f}_1, \bar{f}_2)$	A $C(\bar{f})$-quasi-saddle point of Ψ (Section 5.2)
$\mathrm{cl}_{(\cdot)}^{(\cdot)}$	Closure; the superscript names the ambient space the closure is taken in, the subscript (when present) names the player index, e.g. $\mathrm{cl}_i^D T_i$
co	Convex hull (not closed); $\overline{\mathrm{co}}$ denotes the closed convex hull, used only in Definition 2.3

Definition 2(Escaping sequence). *Let $D = \bigcup_{n=1}^{\infty} D^n$ where $\{D^n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact sets with $D^n \subseteq D^{n+1}$ for all $n \in \mathbb{N}$. A sequence $\{f^n\}_{n=1}^{\infty}$ in D is called* escaping from D relative to *$\{D^n\}$ if for each $n \in \mathbb{N}$ there exists $m \geq n$ such that $f_m \notin D^n$.*

Definition 3(Measure of non-compactness; ϕ-condensing map). *Let E be a Hausdorff topological vector space and Ω a lattice with least element 0. A map $\phi : \Pi(E) \to \Omega$ is called a* measure of non-compactness *on E if for all $X, Y \in \Pi(E)$: (i) $\phi(X) = 0$ if and only if X is pre-compact; (ii) $\phi(\overline{\mathrm{co}} X) = \phi(X)$; (iii) $\phi(X \cup Y) = \max\{\phi(X), \phi(Y)\}$. A multi-valued map $S : J \to \Pi(E)$ is called ϕ*-condensing *if whenever $X \subseteq J$ and $\phi(S(X)) \geq \phi(X)$, the set X is relatively compact.*

Definition 4($C(f)$-quasiconvexity). *Let $C : D \to \Pi(W)$ be as above. A multi-valued bifunction $F_i \in \mathscr{F}_i(D)$ is called* $C(f)$-quasiconvex *(in the second variable) if for all $f \in D$, $g_1, g_2 \in D_i$ and $\lambda \in [0,1]$, setting $g_\lambda = \lambda g_1 + (1-\lambda) g_2$, we have*

$$F_i(f, g_\lambda) \subseteq F_i(f, g_1) - C(f) \quad \text{or}$$
$$F_i(f, g_\lambda) \subseteq F_i(f, g_2) - C(f).$$

Lemma 1.*Let $F_i \in \mathscr{F}_i(D)$ be $C(f)$-quasiconvex in the second variable. Let $f \in D$, $n \geq 1$, $g_1, \ldots, g_n \in D_i$, and $\alpha_1, \ldots, \alpha_n \geq 0$ with $\sum_{j=1}^{n} \alpha_j = 1$. Set $g_0 = \sum_{j=1}^{n} \alpha_j g_j$. If $F_i(f, g_j) \subseteq -C(f)$ for all $j = 1, \ldots, n$, then there exists $j^* \in \{1, \ldots, n\}$ such that*

$$\begin{aligned} F_i(f, g_0) &\subseteq F_i(f, g_{j^*}) - C(f) \\ &\subseteq -C(f) - C(f) \subseteq -C(f). \end{aligned}$$

*Proof.*We argue by induction on n. For $n = 1$ the result is trivial with $j^* = 1$. For $n = 2$, set $g_\lambda = \alpha_1 g_1 + \alpha_2 g_2$. $C(f)$-quasiconvexity gives $F_i(f, g_\lambda) \subseteq F_i(f, g_1) - C(f)$ or $F_i(f, g_\lambda) \subseteq F_i(f, g_2) - C(f)$; since $F_i(f, g_j) \subseteq -C(f)$ for both j, either branch gives $F_i(f, g_\lambda) \subseteq -C(f) - C(f) = -C(f)$ (since $C(f)$ is a convex cone, $C(f) + C(f) = C(f)$).

For the inductive step with $n \geq 3$: if $\alpha_n = 1$ then $g_0 = g_n$ and $j^* = n$ gives the result trivially. Otherwise $0 \leq \alpha_n < 1$; set $\beta = \alpha_n$ and

$$h = \sum_{j=1}^{n-1} \frac{\alpha_j}{1-\beta} g_j \in D_i$$

(a convex combination, since $\sum_{j=1}^{n-1} \alpha_j/(1-\beta) = 1$). By the inductive hypothesis applied to $g_1, \ldots, g_{n-1}$ and h, $F_i(f,h) \subseteq -C(f)$. Applying the two-point case to g_n and h with $g_0 = \beta g_n + (1-\beta)h$, and using $F_i(f,g_n) \subseteq -C(f)$ and $F_i(f,h) \subseteq -C(f)$, gives $F_i(f,g_0) \subseteq -C(f)$ by the base case.

The proofs rely on the following two fixed-point results.

Theorem 1(Ding–Kim–Tan [8]). *Let B be a nonempty, compact and convex subset of a Hausdorff topological vector space $\mathscr{X}$. Suppose $\mathscr{A}, \mathrm{cl}^{\mathscr{X}} \mathscr{A}, \mathscr{P} : B \to \Pi(B)$ satisfy:*

(a)$\mathscr{A}(x)$ is nonempty and convex for each x;
(b)$\mathscr{A}^{-1}(y)$ is open for each y;
(c)$\mathrm{cl}^{\mathscr{X}} \mathscr{A}$ is u.s.c.;
(d)$x \notin \mathrm{co}\, \mathscr{P}(x)$ for each x;
(e)$\mathscr{P}^{-1}(y)$ is open for each y.

Then there exists $x^* \in B$ such that $x^* \in \mathrm{cl}^{\mathscr{X}} \mathscr{A}(x^*)$ and $\mathscr{A}(x^*) \cap \mathscr{P}(x^*) = \varnothing$.

Theorem 2(Lin–Park–Yu [18] — ϕ-condensing version). *Let E be a locally convex Hausdorff topological vector space and $\mathscr{X} \subseteq E$ a nonempty closed convex set. Let $\mathscr{T} : \mathscr{X} \to \Pi(\mathscr{X})$ and $\mathscr{S} : \mathscr{X} \to \Pi(\mathscr{X})$ be set-valued maps satisfying:*

(a)$\mathscr{X} = \bigcup_{y \in \mathscr{X}} \mathrm{int}^{\mathscr{X}} \mathscr{T}^{-1}(y)$;
(b)$\mathscr{T}(x)$ is convex for each $x \in \mathscr{X}$;
(c)$\mathscr{S}(x) \cap \mathscr{T}(x) \neq \varnothing$ for each $x \in \mathscr{X}$;
(d)$\mathscr{S}$ is ϕ-condensing.

Then there exists $x^* \in \mathscr{X}$ such that $x^* \in \mathscr{S}(x^*)$.

Theorem 3(Non-compact KKM-type, see [7]). *Let $\mathscr{X} = \bigcup^n \mathscr{X}^n$ with $\{\mathscr{X}^n\}$ an increasing sequence of nonempty, compact and convex subsets of a topological vector space E. Let $\mathscr{T} : \mathscr{X} \to \Pi(\mathscr{X})$ satisfy:*

(a)$\mathscr{T}^{-1}(x) \cap \mathscr{X}^n$ is open in $\mathscr{X}^n$ for each x and each n;
(b)$x \notin \mathrm{co}\, \mathscr{T}(x)$ for each x;
(c)for every escaping sequence $\{x^n\}$ with $x^n \in \mathscr{X}^n$ there exists $n \in \mathbb{N}$ and $y^n \in \mathscr{X}^n$ such that $y^n \in \mathrm{co}\, \mathscr{T}(x^n) \cap \mathscr{X}^n$.

Then there exists $x_0 \in \mathscr{X}$ with $\mathscr{T}(x_0) = \varnothing$.

3 System of Generalized Operator Quasi-Equilibrium Problems

3.1 Problem Formulation

The central problem of this paper is stated below.

Problem (SGOQEP). Find $(f_1^\circ, f_2^\circ) \in D_1 \times D_2$ such that, for $i = 1,2$:

$$f_i^\circ \in \mathrm{cl}_i^D T_i(f_1^\circ, f_2^\circ) \text{ and } F_i(f_1^\circ, f_2^\circ, g_i) \nsubseteq -C(f_1^\circ, f_2^\circ), \quad \text{for all } g_i \in T_i(f_1^\circ, f_2^\circ). \tag{SGOQEP}$$

Three observations clarify the scope of (SGOQEP).

Remark 31(Coupling). Both $T_1(f_1, f_2)$ and $T_2(f_1, f_2)$ vary with f_1 and f_2 jointly. When T_i depends only on f_i, the two equilibrium conditions become independent, and (SGOQEP) is simply two separate OQEPs without interaction.

Remark 32(Single-problem reduction). Setting $D_2 = \{\hat{e}_2\}$ (a singleton) and taking $T_2 \equiv \{\hat{e}_2\}$ with F_2 identically satisfied, (SGOQEP) collapses to a single OQEP in D_1. With $T_1(f_1, \hat{e}_2) = S(f_1)$ and $F_1 = F$ one obtains exactly the problem of [21, Theorem 2.1].

Remark 33(Special cases). Particular choices of W, Z_1, Z_2, and C yield systems of vector quasi-equilibrium problems as in [1]. When the operator domains are subsets of W directly, systems of vector equilibrium problems [14,16] are recovered.

3.2 Existence Theorem: Compact Case

Theorem 4. *Let $D_i \subseteq L(Z_i, W)$ $(i = 1,2)$ be nonempty, compact and convex. Let $C : D \to \Pi(W)$ be such that $C(f)$ is a solid, convex and open cone with $0 \notin C(f)$ for each $f \in D$. Let T_i, $\mathrm{cl}_i^D T_i : D \to \Pi(D_i)$ be set-valued maps such that $T_i(f)$ is nonempty and convex for each $f \in D$, $T_i^{-1}(g_i)$ is open in D for each $g_i \in D_i$, and $\mathrm{cl}_i^D T_i$ is upper semicontinuous. For $i = 1,2$, let $F_i \in \mathscr{F}_i(D)$ satisfy:*

(i)$F_i(f, f_i) \subseteq C_0(f)$ for all $f \in D$;
(ii)F_i is $C(f)$-quasiconvex in the second variable;
(iii)the set $G_i(f) := W \setminus (-C(f))$ has a closed graph in $D \times W$ for each $f \in D$;
(iv)for each $g_i \in D_i$, the map $f \mapsto F_i(f, g_i)$ is upper semicontinuous with compact values on D.

Then the SGOQEP has a solution $(f_1^\circ, f_2^\circ) \in D_1 \times D_2$.

Proof. We work on the product space $D = D_1 \times D_2$, which is compact and convex as the product of compact convex sets.

Step 1 (Auxiliary maps). For each $f = (f_1, f_2) \in D$ and $i = 1,2$, define

$$P_i(f) = \{g_i \in D_i : F_i(f, g_i) \subseteq -C(f)\}.$$

Define the product auxiliary map $P : D \to \Pi(D)$ by $P(f) = P_1(f) \times P_2(f)$. Also define the product constraint map $T : D \to \Pi(D)$ by $T(f) = T_1(f) \times T_2(f)$.

Step 2 ($f \notin \operatorname{co} P(f)$). Suppose for contradiction that $f_0 = (f_{1,0}, f_{2,0}) \in \operatorname{co} P(f_0)$ for some $f_0 \in D$. Then for each i, $f_{i,0} = \sum_j \alpha_j g_{j,i}$ with $g_{j,i} \in P_i(f_0)$, so $F_i(f_0, g_{j,i}) \subseteq -C(f_0)$ for all j. By Lemma 1 (applied with $f = f_0$ and the n-point convex combination $f_{i,0} = \sum_j \alpha_j g_{j,i}$), there exists j^* such that

$$\begin{aligned} F_i(f_0, f_{i,0}) &\subseteq F_i(f_0, g_{j^*,i}) - C(f_0) \subseteq -C(f_0) - C(f_0) \\ &= -C(f_0) \quad (\text{by } C(f_0) + C(f_0) = C(f_0)). \end{aligned}$$

But condition (i) gives $F_i(f_0, f_{i,0}) \subseteq C_0(f_0) = C(f_0) \cup \{0\}$. Since $C(f_0) \cap (-C(f_0)) = \varnothing$ (because $C(f_0)$ is a convex cone with $0 \notin C(f_0)$), we have $(C(f_0) \cup \{0\}) \cap (-C(f_0)) = \varnothing$ (since $0 \notin -C(f_0)$ as $-C(f_0)$ is also a cone not containing 0). Hence $F_i(f_0, f_{i,0}) \subseteq C_0(f_0)$ and $F_i(f_0, f_{i,0}) \subseteq -C(f_0)$ together force $F_i(f_0, f_{i,0}) \subseteq (C(f_0) \cup \{0\}) \cap (-C(f_0)) = \varnothing$. Since F_i has nonempty values (by definition of $\mathscr{F}_i(D)$), this is a contradiction. Therefore $f \notin \operatorname{co} P(f)$ for all $f \in D$.

Step 3 ($P^{-1}(g)$ is open). We show $[P_i^{-1}(g_i)]^c = \{f \in D : F_i(f, g_i) \nsubseteq -C(f)\}$ is closed in D. Let $\{f_\lambda\}$ be a net in $[P_i^{-1}(g_i)]^c$ converging to f_0. For each λ there exists $h_\lambda \in F_i(f_\lambda, g_i)$ with $h_\lambda \in G_i(f_\lambda)$. Since $F_i(\cdot, g_i)$ is u.s.c. with compact values (condition (iv)), the union $\bigcup_\lambda F_i(f_\lambda, g_i)$ has compact closure in W (the image of the compact set $\{f_\lambda\} \cup \{f_0\}$ under a u.s.c. compact-valued map is compact; see e.g. [23, Lemma 5.2]). Hence the net $\{h_\lambda\}$ has an accumulation point h_0, and by u.s.c. of $F_i(\cdot, g_i)$, $h_0 \in F_i(f_0, g_i)$. By condition (iii), G_i has a closed graph, so $h_0 \in G_i(f_0)$, i.e., $h_0 \notin -C(f_0)$. Hence $f_0 \in [P_i^{-1}(g_i)]^c$, confirming it is closed. Consequently $P_i^{-1}(g_i)$ is open in D, and $P^{-1}(g) = P_1^{-1}(g_1) \times P_2^{-1}(g_2)$ is open in D (product of open sets in the product topology) for each $g = (g_1, g_2)$.

Step 4 (Apply Theorem 1). Set $\mathscr{A} = T$ and $\mathrm{cl}^{\mathscr{X}} \mathscr{A} = \mathrm{cl}^D T$. The product constraint map T satisfies: $T^{-1}(g)$ is open in D (product of open fibres $T_i^{-1}(g_i)$); $T(f)$ is nonempty and convex for each f; and $\mathrm{cl}^D T$ is u.s.c. (the product of u.s.c. maps $\mathrm{cl}_i^D T_i$ is u.s.c. on the product space D). From Steps 2 and 3, all hypotheses of Theorem 1 hold with $\mathscr{A} = T$ and $\mathscr{P} = P$ as defined. Therefore there exists $(f_1^\circ, f_2^\circ) \in D$ such that

$$\begin{aligned} &(f_1^\circ, f_2^\circ) \in \mathrm{cl}^D T(f_1^\circ, f_2^\circ) \\ \text{and} \quad &T(f_1^\circ, f_2^\circ) \cap P(f_1^\circ, f_2^\circ) = \varnothing. \end{aligned}$$

The first condition gives $f_i^\circ \in \mathrm{cl}_i^D T_i(f_1^\circ, f_2^\circ)$ for $i = 1,2$. The second gives $P_i(f^\circ) \cap T_i(f^\circ) = \varnothing$ for $i = 1,2$, i.e., $F_i(f^\circ, g_i) \nsubseteq -C(f^\circ)$ for all $g_i \in T_i(f^\circ)$. This is exactly SGOQEP.

The idea is to combine the two players' individual auxiliary maps into a single map on the product space D and run the Ding–Kim–Tan theorem once, on that product, rather than solving each player's condition separately. Conditions (i)–(ii) rule out fixed points of the auxiliary map, conditions (iii)–(iv) make its inverse open, and the two conditions already assumed on T_i supply the rest of what Theorem 1 needs.

3.3 Existence Theorem: Non-compact Case (ϕ-Condensing)

Theorem 5. *For $i = 1,2$, let $D_i = \bigcup_n D_i^n$ where $\{D_i^n\}_{n=1}^\infty$ is an increasing sequence of nonempty compact and convex subsets of $L(Z_i, W)$. Let W be locally convex. Let $C : D \to \Pi(W)$ be as in Theorem 4. Let T_i, $\mathrm{cl}_i^D T_i : D \to \Pi(D_i)$ be ϕ-condensing set-valued maps with $T_i(f)$ nonempty and convex for each f, $T_i^{-1}(g_i)$ open in D for each g_i, and $\mathrm{cl}_i^D T_i$ upper semicontinuous. For $i = 1,2$, let $F_i \in \mathscr{F}_i(D)$ satisfy conditions (i)–(iv) of Theorem 4 and, in addition:*

- *(v) for every escaping sequence $\{(f_1^n, f_2^n)\}_{n=1}^\infty$ in D with $f_i^n \in D_i^n$ for each n, there exists $m \in \mathbb{N}$ and $(g_1^m, g_2^m) \in D_1^m \times D_2^m$ with $g_i^m \in T_i(f_1^m, f_2^m)$, such that for each $f_i^m \in \mathrm{cl}_i^D T_i(f_1^m, f_2^m)$,*

$$F_i(f_1^m, f_2^m, g_i^m) \subseteq -C(f_1^m, f_2^m);$$

- *(vi) $T_i(f) \cap D_i^n \neq \varnothing$ for every $f \in D^n = D_1^n \times D_2^n$ and every $n \in \mathbb{N}$.*

Then the SGOQEP has a solution $(f_1^\circ, f_2^\circ) \in D_1 \times D_2$.

Proof. Set $D^n = D_1^n \times D_2^n$ for each n. Since D_1^n and D_2^n are compact and convex, so is D^n. For each n and i, define the truncated map $T_i^n : D^n \to \Pi(D_i^n)$ by $T_i^n(f) = T_i(f) \cap D_i^n$ for $f \in D^n$. By condition (vi), $T_i^n(f)$ is nonempty for each $f \in D^n$. Since $T_i(f)$ is convex and D_i^n is convex, $T_i^n(f)$ is convex. The fibre $(T_i^n)^{-1}(g_i) = T_i^{-1}(g_i) \cap D^n$ is open in D^n (restriction of the open set $T_i^{-1}(g_i)$ to D^n). The closure $\mathrm{cl}_i^{D^n} T_i^n$ is u.s.c. on D^n since $\mathrm{cl}_i^D T_i$ is u.s.c. on D and D_i^n is compact. Apply Theorem 4 on D^n with constraint maps T_i^n: there exists $(f_1^n, f_2^n) \in D^n$ such that

$$\begin{aligned} &f_i^n \in \mathrm{cl}_i^D T_i(f_1^n, f_2^n) \text{ and } F_i(f_1^n, f_2^n, g_i) \nsubseteq -C(f_1^n, f_2^n) \\ &\quad \text{for all } g_i \in T_i^n(f_1^n, f_2^n) \subseteq T_i(f_1^n, f_2^n). \end{aligned} \tag{$*$}$$

Suppose the sequence $\{(f_1^n, f_2^n)\}_{n=1}^\infty$ is escaping from D relative to $\{D^n\}$. By condition (v), there exists $m \in \mathbb{N}$ and $g_i^m \in T_i(f_1^m, f_2^m) \cap D_i^m$ such that $F_i(f_1^m, f_2^m, g_i^m) \subseteq -C(f_1^m, f_2^m)$. Since $g_i^m \in T_i^m(f_1^m, f_2^m)$, this contradicts $(*)$. Therefore $\{(f_1^n, f_2^n)\}$ is not escaping.

Since $\{(f_1^n, f_2^n)\}$ is not escaping, there exists $r \in \mathbb{N}$ such that infinitely many terms f_i^n lie in D_i^r. Passing to this subsequence, still denoted $\{(f_1^n, f_2^n)\}$, each term belongs to the compact set $D^r = D_1^r \times D_2^r$. Regarding $\{(f_1^n, f_2^n)\}$ as a net indexed by $\mathbb{N}$, let $\{(f_1^\alpha, f_2^\alpha)\}_{\alpha \in \Lambda}$ be a subnet converging to $(f_1^\circ, f_2^\circ) \in D^r$. Since $\mathrm{cl}_i^D T_i$ is u.s.c. with compact values and $f_i^\alpha \in \mathrm{cl}_i^D T_i(f_1^\alpha, f_2^\alpha)$, u.s.c. gives $f_i^\circ \in \mathrm{cl}_i^D T_i(f_1^\circ, f_2^\circ)$. Now let $g_i \in T_i(f_1^\circ, f_2^\circ)$ be arbitrary. Since $D_i = \bigcup_n D_i^n$ and $g_i \in D_i$, there exists $k_0 \in \mathbb{N}$ such that $g_i \in D_i^{k_0}$. Choose $r_0 = \max(r, k_0)$; since $\{D_i^n\}$ is increasing, $g_i \in D_i^{r_0}$. Without loss of generality we may take $r \geq k_0$ (replacing r by r_0 if necessary; the subsequence argument above still works since infinitely many terms lie in $D_i^{r_0} \supseteq D_i^r$). Hence $g_i \in D_i^r$. Since $T_i^{-1}(g_i)$ is open in D and $(f_1^\alpha, f_2^\alpha) \to (f_1^\circ, f_2^\circ)$, there exists $\alpha_0 \in \Lambda$ such that $g_i \in T_i(f_1^\alpha, f_2^\alpha)$ for all $\alpha \geq \alpha_0$. Since $g_i \in D_i^r$ and $T_i^r(f) = T_i(f) \cap D_i^r$, we have $g_i \in T_i^r(f_1^\alpha, f_2^\alpha)$ for all $\alpha \geq \alpha_0$. By $(*)$, $F_i(f_1^\alpha, f_2^\alpha, g_i) \nsubseteq -C(f_1^\alpha, f_2^\alpha)$, so there exists $h_\alpha \in F_i(f_1^\alpha, f_2^\alpha, g_i)$ with $h_\alpha \in G_i(f_1^\alpha, f_2^\alpha)$. By condition (iv), $F_i(\cdot,\cdot,g_i)$ is u.s.c. with compact values; passing to a further subnet, $h_\alpha \to h_0 \in F_i(f_1^\circ, f_2^\circ, g_i)$. By the closed graph of G_i (condition (iii)), $h_0 \in G_i(f_1^\circ, f_2^\circ)$, i.e., $h_0 \notin -C(f_1^\circ, f_2^\circ)$. Therefore $F_i(f_1^\circ, f_2^\circ, g_i) \nsubseteq -C(f_1^\circ, f_2^\circ)$.

Since $g_i \in T_i(f_1^\circ, f_2^\circ)$ was arbitrary,

$$f_i^\circ \in \mathrm{cl}_i^D T_i(f_1^\circ, f_2^\circ) \text{ and } F_i(f_1^\circ, f_2^\circ, g_i) \nsubseteq -C(f_1^\circ, f_2^\circ) \quad \text{for all } g_i \in T_i(f_1^\circ, f_2^\circ).$$

This is SGOQEP.

Since D need not be compact, Theorem 4 is applied not to D itself but to each finite level D^n of the exhaustion, producing an approximate solution at every level. Condition (v) is what prevents this sequence of approximate solutions from drifting off toward infinity. Once that is ruled out, a subsequence stays inside some fixed compact D^r, and its limit is checked to solve the original problem on all of D.

4 Reduction to Known Results

The following corollaries confirm that Theorems 4 and 5 contain the earlier operator (quasi-)equilibrium results as strict particular cases.

Corollary 1(Raouf–Gupta–Sharma [21], Theorem 2.1). *Take $D_2 = \{\hat{e}_2\}$, $T_2 \equiv \{\hat{e}_2\}$, $F_2 \equiv 0$, $T_1(f_1, \hat{e}_2) = S(f_1)$, and $F_1(f, g_1) = F(f_1, g_1)$. Then (SGOQEP) asks: find $f^\circ \in D_1$ with $f^\circ \in \mathrm{cl}_1^D S(f^\circ)$ and $F(f^\circ, g) \nsubseteq -C(f^\circ)$ for all $g \in S(f^\circ)$, which is the OQEP of [21]. Theorem 4 gives [21, Theorem 2.1] and Theorem 5 gives [21, Theorem 2.2].*

Corollary 2(Sharma–Gupta–Raouf [22], Theorems 3.1–3.2). *Apply the same substitution as in the previous corollary, now with $T = T_1$ assumed ϕ-condensing. Theorem 4 yields [22, Theorem 3.1] (compact case) and Theorem 5 yields [22, Theorem 3.2] (non-compact case). Condition (v) of Theorem 5 corresponds precisely to condition (iv) of [22, Theorem 3.2].*

Corollary 3(Kazmi–Raouf [11], Theorem 3.1). *Starting from Corollary 4.1, further set $C(f) \equiv C$ (independent of f) and $S(f) \equiv B$ for all f (no quasi-constraint on the feasible set). The problem becomes: find $f^\circ \in B$ with $F(f^\circ, g) \nsubseteq -C$ for all $g \in B$, which is the operator equilibrium problem of [11]. Theorem 4 then recovers [11, Theorem 3.1].*

Corollary 4(Kim–Raouf [17], Theorem 2.1). *Set $W = \mathbb{R}$ (scalar field). Then $L(\mathbb{R}, Z) \cong Z$, so the operator domain is identified with Z itself. Taking $B \subseteq Z$ and $C(f) = C$ (a fixed ordering cone in Z), Theorem 4 gives the vector quasi-equilibrium existence result of [17, Theorem 2.1].*

5 Applications

5.1 System of Operator Quasi-Variational-Like Inequality Problems

Fix a Hausdorff topological vector space Y. For each $i = 1, 2$, let $M_i : D \times Z_i \to \Pi(L(Z_i, Y))$ be a set-valued map, let $\eta_i : D_i \times D_i \to L(Z_i, Y)$ be a given bifunction, and let $R_i : D \to \Pi(Z_i)$ be a set-valued map. Define

$$F_i(f, g_i) = \mathrm{Max}\langle M_i(f, v_i), \eta_i(g_i, f_i)\rangle,$$

where $v_i \in R_i(f)$ and $\mathrm{Max}\langle\cdot,\cdot\rangle$ collects all maximal elements under the ordering of $(Y, C(f))$.

Definition 5(SOQVLIP). *Find $(f_1^\circ, f_2^\circ) \in D$ and $v_i^\circ \in R_i(f^\circ)$ such that for $i = 1, 2$:*

$$f_i^\circ \in \mathrm{cl}_i^D T_i(f^\circ) \text{ and } \mathrm{Max}\langle M_i(f^\circ, v_i^\circ), \eta_i(g_i, f_i^\circ)\rangle \nsubseteq -C(f^\circ) \quad \text{for all } g_i \in T_i(f^\circ).$$

Theorem 6. *Suppose all conditions of Theorem 4 hold with F_i defined as above. Assume additionally: (α) $\eta_i(f_i, f_i) = 0$ for all $f \in D$ and $i = 1, 2$; (β) $\eta_i(g_i, f_i)$ is $C(f)$-quasiconvex in g_i; (γ) $M_i(\cdot,\cdot)$ is u.s.c. with compact values; (δ) η_i is continuous. Then the SOQVLIP has a solution.*

Proof. We verify that $F_i(f, g_i) = \mathrm{Max}\langle M_i(f, v_i), \eta_i(g_i, f_i)\rangle$ satisfies conditions (i)–(iv) of Theorem 4.

(i) Setting $g_i = f_i$: we assume $\eta_i(f_i, f_i) = 0$ (standard condition in variational-like inequalities; see e.g. [15]). Then $F_i(f, f_i) = \mathrm{Max}\langle M_i(f, v_i), 0\rangle \subseteq \{0\} \subseteq C_0(f)$.

(ii) $C(f)$-quasiconvexity of $F_i(f, \cdot)$ in g_i follows from the assumed $C(f)$-quasiconvexity of $\eta_i(g_i, f_i)$ in g_i and the linearity of the pairing $\langle M_i(f, v_i), \cdot\rangle$ in the second argument.

(iii) The closed-graph condition for G_i holds by the assumed continuity properties of M_i and η_i.

(iv) Upper semicontinuity with compact values of $F_i(\cdot,g_i)$ holds by the assumed u.s.c. of M_i and continuity of η_i.

Hence (SOQVLIP) is a special case of (SGOQEP) and Theorem 4 applies. The non-compact case follows from Theorem 5.

The idea is that (SOQVLIP) does not require a separate existence argument. With F_i built from M_i and η_i as above, conditions (α)–(δ) are exactly what hypotheses (i)–(iv) of Theorem 4 require, so existence for the variational-like inequality problem follows directly from the SGOQEP framework already established.

Corollary 5. *If $T_i(f^\circ) = D_i$ for all f° (no quasi-constraint), then SOQVLIP reduces to the system of generalized operator variational-like inequality problems (SGOVLIP): find $(f_1^\circ, f_2^\circ) \in D$ and $v_i^\circ \in R_i(f^\circ)$ such that* $\mathrm{Max}\langle M_i(f^\circ,v_i^\circ),\eta_i(g_i,f_i^\circ)\rangle \nsubseteq -C(f^\circ)$ *for all $g_i \in D_i$ and $i = 1,2$.*

5.2 System of Operator Quasi-Saddle Point Problems

Definition 6 **(System of operator quasi-saddle points).** *Let $\Psi : D \to W$. A pair $\bar f = (\bar f_1, \bar f_2) \in D_1 \times D_2$ is called a* $C(\bar f)$-quasi-saddle point of Ψ with respect to T_1, T_2 *if:*

(a) $\bar f_i \in \mathrm{cl}_i^D T_i(\bar f_1,\bar f_2)$ *for $i = 1,2$; and*
(b) $\Psi(h_1,\bar f_2) - \Psi(\bar f_1,h_2) \notin -C(\bar f)$ *for all $(h_1,h_2) \in T_1(\bar f)\times T_2(\bar f)$.*

Corollary 6. *Let the hypotheses of Theorem 4 hold with $\Psi : D \to W$ $C(\bar f)$-lower semicontinuous and $C(\bar f)$-quasi-convex in f_1, and $C(\bar f)$-upper semicontinuous and $C(\bar f)$-quasi-concave in f_2. Define*

$$F_1(f,h_1) = \Psi(h_1,f_2) - \Psi(f_1,f_2),$$
$$F_2(f,h_2) = \Psi(f_1,f_2) - \Psi(f_1,h_2).$$

Then the system of operator quasi-saddle point problems has a solution.

Proof. We verify conditions (i)–(iv) for $F_1(f,h_1) = \Psi(h_1,f_2) - \Psi(f_1,f_2)$.

(i) $F_1(f,f_1) = \Psi(f_1,f_2) - \Psi(f_1,f_2) = 0 \in C_0(f)$.

(ii) $C(f)$-quasiconvexity of $F_1(f,\cdot)$ in h_1 follows directly from the $C(f)$-quasi-convexity of $\Psi(\cdot,f_2)$ assumed in the hypotheses.

(iii) Closedness of G_1 follows from $C(f)$-lower semicontinuity of Ψ in f_1: if $h_\alpha \in G_1(f_{1,\alpha},f_2)$ and $(f_{1,\alpha},h_\alpha) \to (f_{1,0},h_0)$, then $h_0 = \Psi(h_0,f_2) - \Psi(f_{1,0},f_2)$ and $h_0 \notin -C(f)$ by $C(f)$-lower semicontinuity.

(iv) U.s.c. of $F_1(\cdot,h_1)$ with compact values holds by $C(f)$-continuity of Ψ.

The same verification applies to $F_2(f,h_2) = \Psi(f_1,f_2) - \Psi(f_1,h_2)$ using $C(f)$-upper semicontinuity and $C(f)$-quasi-concavity of $\Psi(f_1,\cdot)$ in f_2. Apply Theorem 4 to obtain $(\bar f_1,\bar f_2)$ satisfying (SGOQEP), which is precisely the $C(\bar f)$-quasi-saddle point condition.

As with Theorem 6, no new fixed-point argument is needed here. The two differences $F_1(f,h_1) = \Psi(h_1,f_2) - \Psi(f_1,f_2)$ and $F_2(f,h_2) = \Psi(f_1,f_2) - \Psi(f_1,h_2)$ are constructed so that the quasiconvexity/concavity and semicontinuity assumed of Ψ translate directly into hypotheses (i)–(iv) of Theorem 4, so a quasi-saddle point of Ψ is recovered as a solution of (SGOQEP).

5.3 Illustrative Scenarios

The coupling structure of (SGOQEP) mirrors situations that arise naturally outside pure operator theory. Consider a two-player game over function spaces – for instance, a duopoly in which each firm chooses a pricing or production *operator* rather than a single scalar quantity. Each player's feasible strategy set $T_i(f_1,f_2)$ can then depend on the opponent's current strategy, exactly as in a coupled-constraint (generalized) Nash game [5]. The quasi-saddle point formulation of Section 5.2 is the natural equilibrium notion for such a setting. Network-flow and resource-allocation problems in which capacities or tolls are themselves operators (e.g. frequency-dependent or time-varying) fit the same template, with $C(f)$ encoding the ordering by which congestion or cost is compared.

These are offered as motivating interpretations of the abstract framework, not as new formal results. A rigorous treatment of any specific application would still require verifying the hypotheses of Theorem 4 or 5 for the concrete maps T_i, F_i, and C arising in that setting.

6 Conclusion

This paper has studied (SGOQEP), a system of two generalized operator quasi-equilibrium conditions linked through coupled set-valued constraint maps in Hausdorff topological vector spaces. Theorem 4 covers compact operator domains, using the $C(f)$-quasiconvexity of the bifunctions and the Ding–Kim–Tan theorem on the product auxiliary map. Theorem 5 handles non-compact domains by means of ϕ-condensing constraint maps, the Lin–Park–Yu maximal-element theorem, and an escaping-sequence condition. Corollaries 4.1–4.4 recover the earlier results of [11,12,17,21,22] as special cases, and Section 5 obtains operator quasi-variational-like inequality and quasi-saddle point existence theorems.

Because [11,12,17,21,22] each treat a single condition $f \in \mathrm{cl}^D S(f)$, none of them can express a constraint map or bifunction that depends on a second, jointly varying component. Working on the product space $D = D_1 \times D_2$ with a single product auxiliary map $P = P_1 \times P_2$ lets Theorems 4 and 5 recover every one of those single-condition results as the special case $D_2 = \{\hat{e}_2\}$ (Corollary 4.1), and also cover the coupled case that motivated this paper.

These are existence theorems, consistent with the Ding–Kim–Tan framework the results build on: the solution pair (f_1°, f_2°) is shown to exist, while questions of uniqueness and computability are left for future work, as is the extension to systems of $n \geq 2$ coupled conditions. The non-compact case (Theorem 5) includes an additional compatibility condition, condition (vi), between T_i and the exhaustion $\{D_i^n\}$, which supports the escaping-sequence step of the argument.

Four directions remain open: (i) sensitivity of the solution set of (SGOQEP) under perturbations of T_i and F_i; (ii) gap-function and error-bound formulations that would support iterative and numerical solution methods (for instance auxiliary-problem or projection-type algorithms adapted to the operator setting), in the spirit of recent iterative-algorithm and well-posedness results for related quasi-equilibrium and quasi-optimization problems [24]; (iii) relaxing the convexity and continuity hypotheses placed on T_i, F_i, and C in Theorems 4 and 5, for instance by weakening $C(f)$-quasiconvexity to a generalized or approximate quasiconvexity condition; (iv) extension to G-convex spaces using a generalized KKM framework.

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