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Some Inequalities of Jensen Type and Lazhar Type for the Class of Harmonically and Strongly Reciprocally Convex Functions

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Abstract: In recent years, new classes of convex functions have been introduced in order to generalize the results and to obtain new estimations, in [12], Iscan gave the definition of harmonically convex functions and in [2] Bracamonte et al. extended the concept of harmonically convex functions due to Iscan, and obtained the concept of strongly reciprocally convex functions. We give some inequalities of Jensen type and Lazhar type for the class of harmonically and strongly reciprocally convex functions.

Keywords: Harmonically convex functions, Jensen type inequalities, Lazhar type inequalities

1 Introduction

As noted by Polyak [25], “Inequalities play a role in most branches of mathematics and have different applications”. Inequalities are one of the most important instruments in many branches of mathematics such as functional analysis, theory of differential and integral equations, interpolation theory, harmonic analysis, probability theory, etc. They are also useful in mechanics, physics and other sciences. A systematic study of inequalities was started in the classical book [10] and continued in[21, 16]. In the eighties and nineties of the last century an impetuous increase of interest in inequalities took place. One result of this fact was a great number of published books on inequalities (see e.g. [3, 5, 19, 20, 18]) and on their applications (see e.g. [6]). Nowadays the theory of inequalities is still being intensively developed. This fact is comented by a great number of recent published books (see e.g. [4, 23]) and a huge number of articles on inequalities. Thus, the theory of inequalities may be regarded as an independent area of mathematics.One of the most important convex function inequalities is Jensen’s inequality; in fact it is almost no exaggeration to say that all known inequalities are particular cases of this famous result. It has been the object of much research full details of which can be found in the various references;

Jensen’s Inequality, was an inequality discovered by the Danish mathematician and engineer Johan Ludwig William Valdemar Jensen (1859 – 1925), relates the value of a convex function of an integral to the integral of the convex function. The concept of convexity and its various generalizations is very important in various fields of mathematics as well as the area of applied mathematics. The origen of interest in convexity arises from areas of application related to fixed point theory and optimization theory.

The convexity of functions plays a significant role in many fields, for example, in biological system, economy, optimization, and so on [9, 27]. And many important inequalities are established for the class of convex functions. For example, Jensen’s inequality and Hermite-Hadamard’s inequality are the best known results in the literature .

The main aim of this paper is to introduce new inequalities of the type of Jensen and Lazhar type for newly introduced notions of strongly reciprocally convex functions and for harmonically convex functions. Our paper is structured as follows. Section 2 provides some

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definitions and preliminaries results concerning with harmonically convex functions and strongly reciprocally convex functions. We recall important facts about the Jensen and Lazhar inequalities. In Section 3 we state and prove our main result: We present Jensen type inequalities (discrete and integral) for strongly reciprocally convex functions. We also obtained Lazhar type inequality for strongly reciprocally convex functions and for harmonically convex functions.

2 Preliminaries,terminology and auxiliary results

This section is devoted to recall some notations ,definitions and results that will be needed in the following. We present some definitions and recall known results concerning the Jensen inequality and Lazhar inequality.

Definition 1. *Let f be a real valued function defined on the interval $I=[a,b]$. f is said to be convex if*

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)$$

for all $x,y\in I$ and all $t\in[0,1]$.

The inequality in the next theorem is known as the discrete version of Jensen's inequality. The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as that arithmetic mean-geometric mean (AM-GM), Holder and Minkowski inequalities,Ky Fan's inequality etc. can be obtained as particular cases of it.

Theorem 1(Discrete Jensen's inequality, see [14]). *Let f be a convex function defined on an interval I. If $x_1,x_2,\cdots,x_n\in I$ and $\lambda_1,\lambda_2,\cdots,\lambda_n\geq 0$ with $\sum_{i=1}^{n}\lambda_i=1$, then*

$$f\left(\sum_{i=1}^{n}\lambda_i x_i\right)\leq\sum_{i=1}^{n}\lambda_i f(x_i).$$

The measure theoretical setting of Jensen's inequality:

Theorem 2(Classical Jensen's inequality, see [13]). *Let g be an integrable function on a probability space $(X,\mathfrak{A},\mu)$ taking values in an interval $I\subseteq\mathbb{R}$. Then $\int_X g d\mu$ lies in I. If f is a convex function on I such that $f\circ g$ is integrable, then*

$$f\left(\int_X g d\mu\right)\leq\int_X (f\circ g)d\mu.$$

By using the Jensen's inequality and convexity of the function f, Lazhar Bougoffa proved the following inequality.

Theorem 3([1]). *If f is a convex function and $x_1,x_2,\cdots,x_n$ lie in its domain, then*

$$\sum_{i=1}^{n}f(x_i)-f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)$$
$$\geq\frac{n-1}{n}\left[f\left(\frac{x_1+x_2}{2}\right)+\cdots+f\left(\frac{x_{n-1}+x_n}{2}\right)+f\left(\frac{x_n+x_1}{2}\right)\right].$$

For more information related to theorem 2.4 (see [24, 28]).

Next, we cite the following results related to some type of generalized convexity:strong convexity, harmonically convexity and strongly reciprocally convexity. In 1966 Polyak [25] introduced the notions of strongly convex and strongly quasi-convex functions. They play an important role in optimization theory and mathematical economics. Nikodem et al. have obtained some interesting properties of strongly convex functions (see [8,11,15]).

Definition 2(See [11,17,26]). *Let D be a convex subset of $\mathbb{R}$ and let $c>0$. A function $f:D\to\mathbb{R}$ is called strongly convex with module c if*

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)-ct(1-t)(x-y)^2 \quad (1)$$

for all $x,y\in D$ and $t\in[0,1]$.

The classical notion of convex function correspond to the case $c=0$.

For instance, if f is strongly convex, then it is bounded from below, its level sets $\{x\in I: f(x)\leq\lambda\}$ are bounded for each λ and f has a unique minimum on every closed subinterval of I ([22, p. 268]). Any strongly convex function defined on a real interval admits a quadratic support at every interior point of its domain.

Theorem 4. *Let D be a convex subset of $\mathbb{R}$ and c be a positive constant. A function $f:D\to\mathbb{R}$ is strongly convex with module c if and only if the function $g(x)=f(x)-cx^2$ is convex.*

Lemma 1([26]). *The following are equivalent:*

1. *$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)-t(1-t)c(x-y)^2$, for all $x,y\in(a,b)$ and $t\in[0,1]$.*
2. *For each $x_0\in(a,b)$, there is a linear function T such that $f(x)\geq f(x_0)+T(x-x_0)+c(x-x_0)^2$ for all $x,y\in(a,b)$.*
3. *For differentiable f, for each $x_0\in(a,b)$: $f(x)\geq f(x_0)+f'(x_0)(x-x_0)+c(x-x_0)^2$, for all $x,y\in(a,b)$.*
4. *For twice differentiable f, $f''(x)\geq 2c$, for all $x,y\in(a,b)$.*

In [12], Imdat Iscan gave the definition of harmonically convex functions:

Definition 3. *[12] Let I be an interval in $\mathbb{R}\setminus\{0\}$. A function $f:I\to\mathbb{R}$ is said to be harmonically convex on I if the inequality*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq t f(y) + (1-t) f(x), \tag{2}$$

holds, for all $x,y \in I$ *and* $t \in [0,1]$.

If the inequality in (2) is reversed, then f is said to be harmonically concave.

Dragomir [7] proved the following Jensen type inequality (discrete version) for harmonically convex functions:

Theorem 5. *If* $f : [a,b] \subset (0,+\infty) \to \mathbb{R}$ *is harmonically convex function, then*

$$f\left(\frac{1}{\sum_{i=1}^{n}\frac{t_i}{x_i}}\right) \leq \sum_{i=1}^{n} t_i f(x_i), \tag{3}$$

for all $x_1,\cdots,x_n \in [a,b]$, $t_1,\cdots,t_n \geq 0$ *with* $t_1+\cdots+t_n = 1$.

In [2] Bracamonte et al. introduced a new class of convex functions, they extended the concept of harmonically convex functions due to Iscan [12], and obtained the concept of strongly reciprocally convex functions and some results for this new class.

Definition 4([2]). *Let* I *be an interval in* $\mathbb{R}\setminus\{0\}$ *and let* $c \in (0,\infty)$. *A function* $f : I \to \mathbb{R}$ *is said to be* strongly reciprocally convex *with module* c *on* I, *if the inequality*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq t f(y) + (1-t) f(x) - ct(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^2, \tag{4}$$

holds, for all $x,y \in I$ *and* $t \in [0,1]$.

The symbol $\mathrm{SRC}_{(I,c)}$ will denote the class of functions that satisfy the inequality (4).

Theorem 6(Properties of strongly reciprocally convex. See [2]). *Let* $I \subset \mathbb{R}\setminus\{0\}$ *be a real interval and* $c \in (0,\infty)$. *Let* $f : I \to \mathbb{R}$ *be a function.*

1.If $f \in SRC_{(I,c)}$, *then* f *is harmonically convex.*

2. $f \in SRC_{(I,c)}$ *if and only if the function* $g : I \to \mathbb{R}$, *defined by* $g(x) := f(x) - \frac{c}{x^2}$ *is harmonically convex.*

3.The function $g : \left[\frac{1}{b},\frac{1}{a}\right] \to \mathbb{R}$, *defined by* $g(t) = f\left(\frac{1}{t}\right)$ *is strongly convex in* $\left[\frac{1}{b},\frac{1}{a}\right]$ *if and only if* $f \in SRC_{([a,b],c)}$.

It is easy to see that the theorem is also valid for intervals $(a,b) \subset \mathbb{R}\setminus\{0\}$.

Theorem 7([2]). *The following are equivalent:*

(i) $f \in SRC_{((a,b),c)}$.

(ii)For each $x_0 \in (a,b)$, *there is a linear function* T *such that*

$$f\left(\frac{1}{x}\right) \geq c(x-x_0)^2 + T(x-x_0) + f\left(\frac{1}{x_0}\right), \tag{5}$$

for all $x \in \left(\frac{1}{b},\frac{1}{a}\right)$.

(iii)For differentiable f *and* $x_0 \in (a,b)$,

$$f\left(\frac{1}{x}\right) \geq f\left(\frac{1}{x_0}\right) - f\left(\frac{1}{x_0}\right)\frac{x-x_0}{x_2} + c(x-x_0)^2, \tag{6}$$

for all $x,y \in (a,b)$.

(iv)For twice differentiable f,

$$\frac{1}{x^4}\left[f''\left(\frac{1}{x}\right) + 2xf'\left(\frac{1}{x}\right)\right] \geq 2c, \tag{7}$$

for all $x \in \left(\frac{1}{b},\frac{1}{a}\right)$.

3 Main results

In this section, we will state our main results and give their proofs as follows.

Using Theorem 5, we obtain the two following results.

Theorem 8(Lazhar type inequality). *If* $f : [a,b] \subseteq (0,+\infty) \to \mathbb{R}$ *is a harmonically convex function and* $x_1,\cdots,x_n \in [a,b]$, *then*

$$\frac{n}{n-1}\left[\sum_{i=1}^{n} f(x_i) - f\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}\right)\right] \tag{8}$$

$$\geq f\left(\frac{2x_1x_2}{x_1+x_2}\right) + \cdots + f\left(\frac{2x_{n-1}x_n}{x_{n-1}+x_n}\right) + f\left(\frac{2x_nx_1}{x_n+x_1}\right).$$

Proof. Using (2) with $t = \frac{1}{2}$, we get

$$f\left(\frac{2x_1x_2}{x_1+x_2}\right) + \cdots + f\left(\frac{2x_{n-1}x_n}{x_{n-1}+x_n}\right) + f\left(\frac{2x_nx_1}{x_n+x_1}\right)$$
$$\leq \frac{1}{2}[f(x_1)+f(x_2)] + \cdots + \frac{1}{2}[f(x_{n-1})+f(x_n)] + \frac{1}{2}[f(x_n)+f(x_1)]$$
$$= f(x_1) + \cdots + f(x_n) = \sum_{i=1}^{n} f(x_i). \tag{9}$$

On the other hand,

$$\sum_{i=1}^{n} f(x_i) = \frac{n}{n-1}\sum_{i=1}^{n} f(x_i) - \frac{1}{n-1}\sum_{i=1}^{n} f(x_i)$$
$$= \frac{n}{n-1}\left[\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n}\frac{1}{n} f(x_i)\right]. \tag{10}$$

Using (10) in (9), we get (8) which completes the proof.

Theorem 9. *If* $f : [a,b] \subseteq (0,+\infty) \to \mathbb{R}$ *is a harmonically convex function and* $a_1,\cdots,a_n \in [a,b]$, *then*

$$\sum_{i=1}^{n} f(b_i) \leq \frac{n}{n-1}\left[\sum_{i=1}^{n} f(a_i) - f\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{a_i}}\right)\right], \tag{11}$$

where $b_i = \dfrac{n-1}{n\alpha^{-1} - a_i^{-1}}$ $i = 1,\cdots,n$ *and* $\alpha = \dfrac{n}{a_1^{-1} + \cdots + a_n^{-1}}$.

Proof.

$$\begin{aligned}
&\sum_{i=1}^{n} f(b_i)\\
&= f(b_1)+\cdots+f(b_n)\\
&= f\left(\frac{n-1}{n\alpha^{-1}-a_1^{-1}}\right)+\cdots+f\left(\frac{n-1}{n\alpha^{-1}-a_n^{-1}}\right)\\
&= f\left(\frac{n-1}{a_1^{-1}+\cdots+a_n^{-1}-a_1^{-1}}\right)+\cdots+f\left(\frac{n-1}{a_1^{-1}+\cdots+a_n^{-1}-a_n^{-1}}\right)\\
&\le \frac{1}{n-1}\left[f(a_2)+\cdots+f(a_n)\right]+\cdots+\frac{1}{n-1}\left[f(a_1)+\cdots+f(a_{n-1})\right],\\
&= f(a_1)+\cdots+f(a_n)\\
&= \frac{n}{n-1}\left[f(a_1)+\cdots+f(a_n)\right]-\frac{1}{n-1}\left[f(a_1)+\cdots+f(a_n)\right]\\
&= \frac{n}{n-1}\left\{f(a_1)+\cdots+f(a_n)-\left[\frac{1}{n}f(a_1)+\cdots+\frac{1}{n}f(a_n)\right]\right\}\\
&\le \frac{n}{n-1}\left[\sum_{i=1}^{n} f(a_i) - f\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{a_i}}\right)\right].
\end{aligned}$$

This completes the proof.

Next we obtain two Jensen type inequalities for strongly reciprocally convex functions.

Theorem 10(Version discrete). *If $f \in SRC_{([a,b],c)}$, then*

$$f\left(\frac{1}{\overline{x}}\right) \le \sum_{i=1}^{n} t_i f\left(\frac{1}{x_i}\right) - c\sum_{i=1}^{n} t_i (x_i - \overline{x})^2,$$

for all $x_1,\cdots,x_n \in [a,b]$, $t_1,\cdots,t_n > 0$ with $t_1+\cdots+t_n = 1$ and $\overline{x} = t_1x_1+\cdots+t_nx_n$.

*Proof.*Fix $x_1,x_2,\cdots,x_n \in [a,b]$ and $t_1,t_2,\cdots,t_n > 0$ such that $t_1+t_2+\cdots+t_n = 1$. Put $\overline{x} = t_1x_1+t_2x_2+\cdots+t_nx_n$. By Theorem 6, the function $g : \left[\frac{1}{b},\frac{1}{a}\right] \to \mathbb{R}$ defined by $g(x) = f\left(\frac{1}{x}\right)$ is strongly convex on $\left[\frac{1}{b},\frac{1}{a}\right]$. So by [17, Theorem 4], we have that

$$g\left(\sum_{i=1}^{n} t_i x_i\right) \le \sum_{i=1}^{n} t_i g(x_i) - c\sum_{i=1}^{n} t_i (x_i - \overline{x})^2.$$

or, equivalently, that

$$f\left(\frac{1}{\overline{x}}\right) \le \sum_{i=1}^{n} t_i f\left(\frac{1}{x_i}\right) - c\sum_{i=1}^{n} t_i (x_i - \overline{x})^2.$$

In a similar way we can prove a counterpart of the integral Jensen inequality for strongly reciprocally convex functions.

Theorem 11(Version integral). *Let (X,Σ,μ) be a probability measure space and $\varphi : X \to (a,b)$ be a Lebesgue integrable function. If $f : (a,b) \to \mathbb{R}$ is strongly reciprocally convex functions with module c, then*

$$f\left(\frac{1}{\int_X \varphi(x)d\mu}\right) \le \int_X f\left(\frac{1}{\varphi(x)}\right)d\mu - c\int_X (\varphi(x)-m)^2 d\mu,$$

where $m = \int_X \varphi(x)d\mu$.

*Proof.*By Theorem 6, the function $g : \left(\frac{1}{b},\frac{1}{a}\right) \to \mathbb{R}$ defined by $g(x) = f\left(\frac{1}{x}\right)$ is strongly convex on $\left(\frac{1}{b},\frac{1}{a}\right)$. So by [17, Theorem 5], we have that

$$g\left(\int_X \varphi(x)\right)d\mu \le \int_X g(\varphi(x))\,d\mu - c\int_X (\varphi(x)-m)^2 d\mu,$$

where $m = \int_X \varphi(x)d\mu$. Or, equivalently, that

$$f\left(\frac{1}{\int_X \varphi(x)}\right)d\mu \le \int_X f\left(\frac{1}{\varphi(x)}\right)d\mu - c\int_X (\varphi(x)-m)^2 d\mu.$$

In view of the interest in this type of inequalities, we obtained the following Lazhar type inequality for strongly reciprocally convex functions, with module c on I.

Theorem 12(Lazhar type inequality in SRC$_{(I,c)}$). *Suppose $f : [a,b] \subseteq (0,+\infty) \to \mathbb{R}$ is a function. If $f \in SRC_{([a,b],c)}$ and $x_1,\cdots,x_n \in [a,b]$, then*

$$\frac{n}{n-1}\left\{\sum_{i=1}^{n} f(x_i) - f\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}\right) - c\left[\sum_{i=1}^{n}\frac{1}{x_i^2} - \frac{1}{n^2}\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)^2\right]\right\} \tag{12}$$

$$\begin{aligned}
\ge\ & f\left(\frac{2x_1x_2}{x_1+x_2}\right) - c\left(\frac{x_1+x_2}{2x_1x_2}\right)^2 + \cdots + f\left(\frac{2x_{n-1}x_n}{x_{n-1}+x_n}\right)\\
& - c\left(\frac{x_{n-1}+x_n}{2x_{n-1}x_n}\right)^2 + f\left(\frac{2x_nx_1}{x_n+x_1}\right) - c\left(\frac{x_n+x_1}{2x_nx_1}\right)^2.
\end{aligned} \tag{13}$$

*Proof.*By Theorem 6 we have that the function $g(x) := f(x) - \frac{c}{x^2}$ is harmonically convex. Now making use of Theorem 8, we get

$$\frac{n}{n-1}\left[\sum_{i=1}^{n} g(x_i) - g\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}\right)\right]$$
$$\ge g\left(\frac{2x_1x_2}{x_1+x_2}\right) + \cdots + g\left(\frac{2x_{n-1}x_n}{x_{n-1}+x_n}\right) + g\left(\frac{2x_nx_1}{x_n+x_1}\right).$$

Then we have the following inequalities:

$$\frac{n}{n-1}\left\{\sum_{i=1}^{n}\left[f(x_i)-\frac{c}{x_i^2}\right]-f\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}\right)+c\left(\frac{\sum_{i=1}^{n}\frac{1}{x_i}}{n}\right)^2\right\}$$
$$\geq f\left(\frac{2x_1x_2}{x_1+x_2}\right)-c\left(\frac{x_1+x_2}{2x_1x_2}\right)^2+\cdots+f\left(\frac{2x_{n-1}x_n}{x_{n-1}+x_n}\right)$$
$$-c\left(\frac{x_{n-1}+x_n}{2x_{n-1}x_n}\right)^2+f\left(\frac{2x_nx_1}{x_n+x_1}\right)-c\left(\frac{x_n+x_1}{2x_nx_1}\right)^2.$$

and

$$\frac{n}{n-1}\left\{\sum_{i=1}^{n}f(x_i)-f\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}\right)-c\left[\sum_{i=1}^{n}\frac{1}{x_i^2}-\frac{1}{n^2}\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)^2\right]\right\}$$
$$\geq f\left(\frac{2x_1x_2}{x_1+x_2}\right)-c\left(\frac{x_1+x_2}{2x_1x_2}\right)^2+\cdots+f\left(\frac{2x_{n-1}x_n}{x_{n-1}+x_n}\right)$$
$$-c\left(\frac{x_{n-1}+x_n}{2x_{n-1}x_n}\right)^2+f\left(\frac{2x_nx_1}{x_n+x_1}\right)-c\left(\frac{x_n+x_1}{2x_nx_1}\right)^2.$$

The proof of Theorem (12) is completed.

*Remark.*Letting $c\to 0^+$, in the inequalities (12), we obtain (8), which is the Lazhar type inequalities for harmonically convex functions.

Theorem 13.*Suppose $f:[a,b]\subseteq(0,+\infty)\to\mathbb{R}$ is a function. If $f\in SRC_{([a,b],c)}$ and $a_1,\cdots,a_n\in[a,b]$, then*

$$\sum_{i=1}^{n}\left[f(b_i)-\frac{c}{b_i^2}\right]\leq\frac{n}{n-1}\left\{\left[\sum_{i=1}^{n}f(a_i)-f\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{a_i}}\right)\right]-c\left[\sum_{i=1}^{n}\frac{1}{a_i^2}-\frac{1}{n^2}\left(\sum_{i=1}^{n}\frac{1}{a_i}\right)^2\right]\right\}, \tag{14}$$

where $\alpha=\dfrac{n}{a_1^{-1}+\cdots+a_n^{-1}}$ *and* $b_i=\dfrac{n-1}{n\alpha^{-1}-a_i^{-1}}$, $i=1,\cdots,n$.

*Proof.*By Theorem 6 we have that the function $g(x):=f(x)-\dfrac{c}{x^2}$ is harmonically convex. Now making use of Theorem 9, we get

$$(n-1)\sum_{i=1}^{n}g(b_i)\leq n\left[\sum_{i=1}^{n}g(a_i)-g\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{a_i}}\right)\right].$$

Then we have the following inequalities:

$$(n-1)\sum_{i=1}^{n}\left[f(b_i)-\frac{c}{b_i^2}\right]$$
$$\leq n\left\{\sum_{i=1}^{n}\left[f(a_i)-\frac{c}{a_i^2}\right]-f\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{a_i}}\right)+c\left(\frac{\sum_{i=1}^{n}\frac{1}{a_i}}{n}\right)^2\right\}$$

and

$$\sum_{i=1}^{n}\left[f(b_i)-\frac{c}{b_i^2}\right]\leq\frac{n}{n-1}\left\{\left[\sum_{i=1}^{n}f(a_i)-f\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{a_i}}\right)\right]-c\left[\sum_{i=1}^{n}\frac{1}{a_i^2}-\frac{1}{n^2}\left(\sum_{i=1}^{n}\frac{1}{a_i}\right)^2\right]\right\},$$

which is the required inequality. Theorem (13) is proved

*Remark.*By letting $c\to 0^+$ in (14), we obtain (11).

4 Conclusions

The main contributions of this paper has been some inequalities of Jensen type and Lazhar type for the class of harmonically and strongly reciprocally convex functions, concepts introduced by Iscan and Bracamonte et al., respectively. We hope that the ideas and techniques used in this paper may are useful for other readers interested in exploring some new applications of these kinds of functions, both in applied and pure sciences.

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