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Connections between Ulam-Hyers Stability and Uniform Exponential Stability of Time Varying Linear Dynamic Systems Over Time Scales

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Abstract: In this paper, we prove that the regressive time varying linear dynamic equation $x^{\Delta}(r) = G(r)x(r),\ r \in \overline{T}$ is Ulam–Hyers stable if and only if it is uniformly exponentially stable. Furthermore, the Ulam–Hyers stability and uniform exponential stability of the system $x^{\Delta}(r) = G(r)x(r),\ r \in \overline{T}$ is proved in terms of bounded-ness of solution of the following Cauchy problem:

$$\begin{cases} W^{\Delta}(r) = G(r)W(r) + \omega(r),\ 0 \le r \in \overline{T}, \\ W(0) = v_0, \end{cases}$$

where $\overline{T}$ denotes time scale, $G(r)$ is a matrix valued function, $\omega(r)$ is a bounded function on $\overline{T}$ and $v_0 \in \mathbb{C}^m$.

Keywords: Uniform exponential stability, Ulam-Hyers stability, time scale

1 Introduction

In 1940, Ulam posed an open problem about the stability of a certain functional equation in [24,25]. In 1941, the first partial answer to Ulam's question was given by Hyers [12]. Later on, it was named as Ulam-Hyers problem. Obłoza proved a result related to the Ulam-Hyers stability of differential equations in [16]. Alsina and Ger; see [1], studied Ulam-Hyers stability of first order linear differential equations and soon after their results were generalized by Takahasi, Takagi, Miura and Miyajima in [23], Jung, in [13,14,15] and Wang, Zhou and Sun in [26].

In 1978, Rassias [20] provided an extension of the Ulam-Hyers stability by introducing new function variables and he introduced another new stability concept named as Ulam–Hyers–Rassias stability. For more details on Ulam–Hyers stability, we recommend [18,21].

The theory of dynamical equations on time scales was introduced by Hilger [11] in 1988 with in order to unify the continuous and discrete calculus. For more details, see [2,3,4,5,6,7,9,10,17,19,22,27].

2 Preliminaries

The non-empty arbitrary closed subset of real numbers is called Time Scale denoted by $\overline{T}$. The forward and backward jump operators denoted by $\theta : \overline{T} \to \overline{T}$, $\rho : \overline{T} \to \overline{T}$, graininess function $\nu : \overline{T} \to \overline{T}$ are respectively defined as:

$$\theta(r) = \inf\{v \in \overline{T} : v > r\},$$

$$\rho(r) = \sup\{v \in \overline{T} : v < r\},\ \nu(r) = \theta(r) - r.$$

A point $r \in \overline{T}$ is said to be left-scattered and left-dense if $r > \rho(r)$ and $\rho(r) = r$, respectively. If $r < \theta(r)$ and $\theta(r) = r$, then such a point $r \in \overline{T}$ will be called right-scattered and right-dense, respectively. The set known as derived form of time scale $\overline{T}$ denoted by $\overline{T}^z$ is defined as follows:

$$\overline{T}^z = \begin{cases} \overline{T} \backslash (\rho(\sup \overline{T}), \sup \overline{T}], & \text{if } \sup \overline{T} < \infty \\ \overline{T}, & \text{if } \sup \overline{T} = \infty. \end{cases}$$

A function $\alpha : \overline{T} \to \overline{R}$ is said to be right-dense continuous if it is continuous at all right-dense points in $\overline{T}$ and its

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left-sided limits exist at all left-dense points in $\overline{T}$, where $\overline{R}$ denotes the set of real numbers. A function $\alpha : \overline{T} \to \overline{R}$ is said to be regressive if $1+\nu(r)\alpha(r) \neq 0$ for all $r \in \overline{T}^z$ and if $1+\nu(r)\alpha(r) > 0$, then the function α is said to be positively regressive. The set of all right-dense regressive, and right-dense positively regressive functions respectively will be denoted by $\mathbb{REG}(\overline{T})$ and $\mathbb{REG}(\overline{T})^+$. A function $w^\theta : \overline{T} \to \overline{R}$ is defined as $w^\theta(r) = w(\theta(r))$, $\forall\, r \in \overline{T}$.

Definition 1. *If $G \in \mathbb{REG}(\overline{T})$, then generalized exponential function $e_G(r,u)$ on $\overline{T}$ is defined as*

$$e_G(r,u) = exp\left(\int_u^r \chi_{\nu(v)} G(v)\Delta v\right)\ \forall\, r,u \in \overline{T},$$

with cylindrical transformation

$$\chi_{\nu(r)}G(r) = \begin{cases} \dfrac{Log(1+\nu(r)G(r))}{\nu(r)}, & \text{if } \nu(r) \neq 0,\\ G(r), & \text{if } \nu(r) = 0.\end{cases}$$

Lemma 1. *[2] Let $G,H \in \mathbb{REG}(\overline{T})$, then:*

1. $e_0(r,u) = 1$ *and* $e_G(r,r) = 1$.
2. $e_G(\theta(r),u) = (1+\nu(r)G(t))e_G(r,u)$.
3. $(e_G(r,u))^\Delta = (G(r))e_G(r,u)$.
4. $e_G(r,u) = \frac{1}{e_G(u,r)}$.
5. $e_G(r,u)e_G(u,v) = e_G(r,v)$.
6. *If $r,u,v \in \overline{T}$, then*

$$\int_r^u G(\eta)e_G(v,\theta(\eta))\Delta\eta = e_G(v,r) - e_G(v,u).$$

Definition 2. *Let G be $m \times m$ matrix-valued function on a time scale $(\overline{T})$. Then G is said to be rd-continuous on $\overline{T}$ if each entry of G is rd-continuous and G is regressive if the eigenvalues $\kappa_i(r)$ of $G(r)$ are regressive for all $1 \leq i \leq m$.*

Remark. Let G^* be the conjugate transpose of $m \times m$ matrix valued function G. If $G \in (\overline{R})^{m\times m}$, then $G^* = G^T$ and $G^* \in \mathbb{REG}(\overline{T})$. Moreover, the function defined by $\ominus G(r) = \frac{-G(r)}{1+\nu(r)G(r)}$ is also regressive.

Consider the matrix-valued IVP,

$$W^\Delta(r) = G(r)W(r),\ W(r_0) = \mathtt{I}_m, \qquad (1)$$

where $\mathtt{I}_m$ is $m \times m$ identity matrix.

Definition 3. *The fundamental matrix is defined to be the general solution to the matrix dynamic equation* (1) *and is denoted by $\Phi_G(r,r_0)$.*

Keep in mind that Φ_G as a transition matrix can be replaced with e_G in the following lemma. The next lemma lists some properties of the matrix exponential function.

Lemma 2. *[8] Let $G \in \mathbb{REG}(\overline{T})$ be the matrix-valued function on $\overline{T}$, then the family $\boldsymbol{G} = \{\Phi_G(r,u) :\ r,u \in \overline{T}\}$ has the following properties:*

1. $\Phi_0(r,u) = 1$ *and* $\phi_G(r,r) = 1$.
2. $\Phi_G(\theta(r),u) = (1+\nu(r)G(r))\Phi_G(r,u)$.
3. $\Phi^{-1}{}_G(r,u) = \Phi^*{}_{\ominus G^*}(u,r)$.
4. $\Phi_G(r,u) = \Phi^{-1}{}_G(r,u) = \Phi^*{}_{\ominus G^*}(u,r)$.
5. $\Phi_G(r,u)\phi_G(u,v) = \Phi_G(r,v)$.
6. $\Phi_G^\Delta(r,u) = G(r)\Phi_G(r,u)$.

Now the next theorem guarantees a unique solution to the regressive $m \times 1$ vector-valued dynamic IVP

$$W^\Delta(r) = G(r)W(r) + \omega(r),\ W(r_0) = v_0. \qquad (2)$$

Theorem 1. *[8] Let $r_0 \in \overline{T}$ and $v_0 \in \overline{R}^m$. Then the regressive IVP* (2) *has a unique solution $W : \overline{T} \to \overline{R}^m$ given by*

$$W(r) = \Phi_G(r,r_0)v_0 + \int_{r_0}^r \Phi_G(r,\theta(\eta))\omega(\eta)\Delta\eta.$$

Consider the regressive time varying linear dynamic system

$$x^\Delta(r) = G(r)x(r);\ x(r_0) = v_0,\ r \in \overline{T},\ v_0 \in \mathbb{C}^m, \quad (G(r))$$

Theorem 2. *[8] The time varying linear dynamic system $(G(r))$ is uniformly exponentially stable if and only if there exists $\eta,\varepsilon > 0$ with $-\eta \in \mathbb{REG}(\overline{T})^+$ such that the transition matrix Φ_G satisfies*

$$||\Phi_G(r,r_0)|| \leq \varepsilon e_{-\eta}(r,r_0),\ \forall\, r \geq r_0,\ \text{with } r,r_0 \in \overline{T}.$$

Theorem 3. *[8] Suppose that there exists a constant γ such that for all $r \in \overline{T}$, $||G(r)|| \leq \gamma$. Then the time varying linear dynamic system $(G(r))$ is uniformly exponentially stable if and only if there exists a constant $\beta > 0$ such that*

$$\int_{r_0}^r ||\Phi_G(r,\theta(\eta))||\Delta\eta \leq \beta,\ \forall\, r \geq \theta(\eta),\ \text{with } r,\eta \in \overline{T}.$$

Concerning the uniform exponential stability of the system $(G(r))$, we state the following result:

Theorem 4. *[22] The system $(G(r))$ is uniformly exponentially stable if and only if for each $v_0 \in \mathbb{C}^m$ and each bounded function $\omega(r)$, the unique solution of the following Cauchy problem*

$$\begin{cases} W^\Delta(r) = G(r)W(r) + \omega(r),\ 0 \leq r \in \overline{T},\\ W(0) = v_0,\end{cases} \quad (G(r),\omega,v_0)$$

is bounded.

Proof. **Necessity:** Let the system $(G(r))$ is uniformly exponentially stable, then by Theorem 2, we have

$$||\Phi_G(r,r_0)|| \leq \gamma e_{-\eta}(r,r_0),\ \forall\, r \geq r_0,\ \text{with } r,\ r_0 \in \overline{T}.$$

Consider the solution of the Cauchy problem $(G(r),\omega,v_0)$,

$$\begin{aligned} W(r) &= \Phi_G(r,0)v_0 + \int_0^r \Phi_G(r,\theta(\eta))\omega(\eta)\Delta\eta \\ ||W(r)|| &\le ||\Phi_G(r,0)||v_0 + \int_0^r ||\Phi_G(r,\theta(\eta))||||\omega(\eta)||\Delta\eta \\ &\le e_{-\eta}(r,0) + C\int_0^r ||\Phi_G(r,\theta(\eta))||\Delta\eta \\ &\le e_{-\eta}(r,0) + C\beta. \end{aligned}$$

Hence, the unique solution of $(G(r),\omega,v_0)$ is bounded.
Sufficiency: Suppose on contrary the system $(G(r))$ is not uniformly exponentially stable and set $\omega(r) = \frac{\Phi_G(r,0)}{1+\nu(r)\ominus G^*(r)}$. Obviously $\omega(r)$ is bounded function. Now consider the solution of $(G(r),\omega,v_0)$,

$$\begin{aligned} W(r) &= \Phi_G(r,0)v_0 + \int_0^r \Phi_G(r,\theta(\eta))\omega(\eta)\Delta\eta \\ &= \Phi_G(r,0)v_0 + \int_0^r \Phi^*{}_{\ominus G^*}(\theta(\eta)),r)\frac{\Phi_G(\eta,0)}{1+\nu(\eta)\ominus G^*(\eta)}\Delta\eta \\ &= \Phi_G(r,0)v_0 + \int_0^r \Phi^*{}_{\ominus G^*}(\eta,r)(1+\nu(\eta)\ominus G^*(\eta))\frac{\Phi_G(\eta,0)}{1+\nu(\eta)\ominus G^*(\eta)}\Delta\eta \\ &= \Phi_G(r,0)v_0 + \int_0^r \Phi_G(r,\eta)\Phi_G(\eta,0)\Delta\eta \\ &= \Phi_G(r,0)v_0 + \Phi_G(r,0)r. \end{aligned}$$

If we take $v_0 = 0$, then we have a contradiction because the map

$$r \mapsto \Phi_G(r,0)r$$

is unbounded. But since $z_0 \neq 0$, then as the system $G(r)$ is not uniformly exponentially stable, so by using Theorem 2 we can find $-\eta,\gamma > 0$ with $\eta \in \mathbb{REG}(\overline{T})^+$ such that

$$||\Phi_G(r,0)|| \ge \gamma e_\eta(r,0),$$

i.e. in this case again the solution will be unbounded and thus we arrived at a contradiction. So the system $(G(r))$ is uniformly exponentially stable.

3 Main Results

We can see an ε-approximate solution of the system $(G(r))$ as an exact solution of the Cauchy problem $(G(r),\omega,v_0)$ corresponding to $\omega(\cdot)$ bounded by ε. Thus, we state the definition of Ulam-Hyers stability as:

Definition 4. *Let $\varepsilon \in \overline{R}^+$. The system $(G(r))$ is Ulam-Hyers stable if and only if there exists a non-negative constant K such that for every $\mathbb{C}^m$-valued right-dense continuous map $\omega = \omega(r)$ bounded by ε on $\overline{T}$, and every $x \in \mathbb{C}^m$ there exists $v_0 \in \mathbb{C}^m$ such that*

$$\sup_{r\ge 0} ||\Phi_G(r,0)v_0 + \int_0^r \Phi_G(r,\theta(u))\omega(u)\Delta u|| \le K\varepsilon.$$

Theorem 5. *The system $(G(r))$ is Ulam–Hyers stable if and only if it is uniformly exponentially stable.*

Proof. **Necessity:** Let the system $(G(r))$ is Ulam–Hyers stable. Then for every $\mathbb{C}^m$-valued right-dense continuous map $\omega = \omega(.)$ bounded by ε on $\overline{T}$, the solution of the Cauchy problem $(G(r),\omega,v_0)$ is bounded by $K\varepsilon$. We need to prove that the system $(G(r))$ is uniformly exponentially stable. To show that $(G(r))$ is uniformly exponentially stable, suppose on contrary $(G(r))$ is not uniformly exponentially stable. Then by using the proof of converse part of Theorem 4, we can show that the solution of $(G(r),\omega,v_0)$ is unbounded, which is a contradiction. Thus $(G(r))$ must be uniformly exponentially stable.
Sufficiency: Let the system $(G(r))$ is uniformly exponentially stable and $\omega : \overline{T} \to \mathbb{C}^m$ be a right–dense continuous function, with $\|\omega\|_\infty \le \varepsilon$, then by Theorem 2

$$||\Phi_G(r,r_0)|| \le \varepsilon e_{-\eta}(r,r_0),\ \forall\ r \ge r_0,\ \text{with } r,\ r_0 \in \overline{T}.$$

Consider,

$$\begin{aligned} W(r) &= \Phi_G(r,0)v_0 + \int_0^r \Phi_G(r,\theta(\eta))\omega(\eta)\Delta\eta \\ ||W(r)|| &\le ||\Phi_G(r,0)||v_0 + \int_0^r ||\Phi_G(r,\theta(\eta))||||\omega(\eta)||\Delta\eta \\ &\le \varepsilon e_{-\eta}(r,0) + \varepsilon\int_0^r ||\Phi_G(r,\theta(\eta))||\Delta\eta \\ &\le \varepsilon e_{-\eta}(r,0) + \varepsilon\beta. \\ &\le (e_{-\eta}(r,0) + \beta)\varepsilon. \end{aligned}$$

The desired assertion follows by choosing $K = e_{-\eta}(r,0) + C\beta$.

4 Conclusion

In this paper, we establish a connection between Ulam–Hyers Stability and Uniform Exponential Stability of Time Varying Linear Dynamic Systems Over Time Scales. Moreover, this connection is proved in terms of boundedness of the Cauchy problem $(G(r),\omega,v_0)$.

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