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Semilocal Convergence of Fourth Order Iterative Method in Banach Spaces by Using Recurrence Relations

Deepak Kumar and Janak Raj Sharma*

Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal-148106, Sangrur, India

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Abstract: We present the semilocal convergence of fourth order iterative method for solving nonlinear equation by using the recurrence relation. In this paper we derived the recurrence relation for the method, and then we prove the an existence-uniqueness theorem, along with the priori error bounds which shows the *R*-order of convergence. Furthermore, with help of numerical example, we are presented the applicability and efficiency of the convergence results.

Keywords: Nonlinear equation, Semilocal convergence, Banach space, R-order of convergence, Hammerstein integral equation

1 Introduction

Let consider the nonlinear equation in Banach space is

$$F(x) = 0, \tag{1}$$

where $F : D \subseteq X \to Y$ is a nonlinear operator on an open convex subset of D of a Banach space X with the values in Banach space Y. Finding the solution of such a operator equation is very interesting task which is widely used in both theoretical and applied areas of mathematics. Newton's method is well known method for solving nonlinear equation with quadratic convergence.

$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k), \qquad k = 0,1,2,\ldots \tag{2}$$

One of the most important results on Newton's method is the well known Newton-Kantorovich theorem [13]. Recently many progress in Newton's method is referred to [4,5,6]. Also Rall [14], has proposed a different approach for the semilocal convergence of same scheme with the help of recurrence relations. In literature, many researcher have developed a number of higher order of convergence method based on Newton's method [1,2,3,4, 7,10,17]. In this paper, we consider the semilocal convergence for the fourth order method presented in [9]. We first extend this method in Banach spaces and write it as

$$\begin{aligned} y_k &= x_k - \Gamma_k F(x_k) \\ x_{k+1} &= y_k - (2[y_k, x_k; F]^{-1} - \Gamma_k)F(y_k), \end{aligned} \tag{3}$$

where $\Gamma_k = F'(x_k)^{-1}$. The approach of recurrence relations has been popularly and successfully used in establishing the convergence of Newton's method [2,3,7, 8] and various high order method, for example, [1,11,15]. We will use the recurrence relations to establish the semilocal convergence for this method. we construct the system of recurrence relations and prove the convergence of method, along with the error estimates. Finally, some numerical results are presented to demonstrate our approach.

2 Preliminary results

Let $x_0 \in D$ and the nonlinear operator $F : D \subset X \to Y$ be continuously third-order Fréchet differentiable where D is an open set and X and Y are Banach spaces. We assume that

$(A1)$ $\|\Gamma_0 F(x_0)\| \leq \eta$,
$(A2)$ $\|\Gamma_0\| \leq \beta$,
$(A3)$ $\|F''(x_0)\| \leq M, \ x \in D$,
$(A4)$ $\|F'''(x_0)\| \leq N, \ x \in D$,
$(A5)$ there exist a positive real number L such that

$$\|F'''(x_k) - F'''(y_k)\| \leq L\|x - y\|, \ \forall \ x, y \in D. \tag{4}$$

Now, we first give some approximation of the operator F in the following lemma and which will be used in later results.

* Corresponding author e-mail: deepak.babbi@gmail.com

Lemma 2.1 *[10] Assume that the nonlinear operator $F : D \subset X \to Y$ is continuously third-order Fréchet differentiable where D is an open set and X and Y are Banach spaces. then we have*

$$\begin{aligned} F(x_{n+1}) &= \frac{1}{2}F''(x_k)(y_k - x_k)(x_{(k+1)} - z_k) \\ &+ \frac{1}{2}\int_0^1 \Big(F''(x_k + \frac{t}{2}(y_k - x_k)) - F''(x_k)\Big)dt(y_k - x_k) \\ &\times (x_{(k+1)} - z_k) \\ &- \int_0^1 \Big(F'(x_k + th) - F'(y_k + t(z_k - y_k))\Big)\Gamma F(z_k) \\ &+ \int_0^1 \Big(F'(z_k + t(x_{k+1} - z_k)) - F'(y_k)\Big)dt \\ &\times (x_{(k+1)} - z_k). \end{aligned} \tag{5}$$

We now define the following some scalars functions that will be used in the later developments. Let

$$p(t) = \frac{t(3-t)}{2(1-t)}, \tag{6}$$

$$\begin{aligned} g(t) &= 1 + p(t) \\ &= 1 + \frac{t(3-t)}{2(1-t)}, \end{aligned} \tag{7}$$

$$h(t) = \frac{1}{1 - tg(t)}, \tag{8}$$

$$\begin{aligned} \varphi(t,u,v) &= \frac{(t-3)^2 t^3}{8(t-1)^2} + \frac{(t-3)(t^2-t-2)^2}{48(t-1)^3}u \\ &+ \frac{(t^2-t-2)^2(20-4t-t^2-6t^3+3t^4)}{1152(t-1)^4}v. \end{aligned} \tag{9}$$

Let $q(t) = g(t)t - 1$, then $q(t)$ has at least one zero in $(0, \frac{1}{2})$ since $q(0) = -1$ and $q(1/2) = \frac{1}{8} > 0$. Some properties of the functions defined above are given in the following lemma.

Lemma 2.2 *Suppose the real functions g, h and φ be given in (7), (8) and (9) and s be smallest positive zero of the scalar function $g(t)t - 1$. Then*

(1) $g(t)$ and $h(t)$ are the increasing and $g(t) > 1$, $h(t) > 1$ for $t \in (0,s)$,

(2) $\varphi(t,u,v)$ is increasing for $t \in (0,s), u > 0, v > 0$.

Assume that the conditions (A1)-(A5) hold. We now denote $\eta_0 = \eta$, $\beta_0 = \beta$, $a = M\beta\eta$, $b = M\beta\eta^2$, $c = M\beta\eta^3$ and $d_0 = h(a_0)p(a_0,b_0,c_0)$. Let $a_0 < s$ and $h(a_0)d_0 < 1$ where s is the smallest positive zero of the scalar function $g(t)t - 1$.

Furthermore, we can define the following sequences for $n \geq 0$

$$\eta_{n+1} = d_n \eta_n, \tag{10}$$

$$\beta_{n+1} = h(a_n)\beta_n, \tag{11}$$

$$a_{n+1} = M\beta_n \eta_{n+1}, \tag{12}$$

$$b_{n+1} = N\beta_n \eta_{n+1}^2, \tag{13}$$

$$c_{n+1} = L\beta_n \eta_{n+1}^3, \tag{14}$$

$$d_{n+1} = h(a_{n+1}p(a_{n+1}, b_{n+1}, c_{n+1}). \tag{15}$$

From the definitions of $a_{n+1}, b_{n+1}, c_{n+1}$ and (12)–(14), we also have

$$a_{n+1} = h(a_n)d_n a_n, \tag{16}$$

$$b_{n+1} = h(a_n)d_n^2 a_n, \tag{17}$$

$$c_{n+1} = h(a_n)d_n^3 a_n. \tag{18}$$

Next, we shall study the some more properties of the previous scalars sequences. Later developments will require the following lemma.

Lemma 2.3 *Let the real functions g, h and φ be given in (7)–(9) and s be the smallest zero of the scalar functiong$(t)t - 1$. If*

$$a_0 < s \text{ and } h(a_0)d_0 < 1, \tag{19}$$

then we have

(1) $h(a_n) > 1$ and $d_n < 1$ for $n \geq 0$,

(2) the sequences $\{\eta_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ are decreasing,

(3) $g(a_n)a_n < 1$ and $h(a_n)d_n < 1$ for $n \geq 0$.

Proof. By Lemma (2.2) and (19), $h(a_0) > 1$ and $d_0 < 1$ hold. It follows from the definitions that $\eta_1 < \eta_0$, $a_1 < a_0$, $b_1 < b_0$, $c_1 < c_0$. Moreover, by Lemma (2.2), we have $h(a_1) < h(a_0)$ and $p(a_1,b_1,c_1) < p(a_0,b_0,c_0)$. This yields $d_1 < d_0$ and item (2). Based on these results we obtain $g(a_1)a_1 < g(a_0)a_0 < 1$ and $h(a_1)d_1 < h(a_0)d_0 < 1$ and item (3) holds. By induction we can derive that items (1), (2) and (3) hold.

Lemma 2.4 *Assume that the real functions g, h and φ in (7)–(9) . Let $\theta \in (0,1)$, then $g(\theta t) < g(t)$, $h(\theta t) < h(t)$ and $\varphi(\theta t, \theta^2 u, \theta^3 v) < \theta^4 \varphi(t,u,v)$ for $t \in (0,\rho)$, where s is the smallest zero of the scalar function $g(t)t - 1$.*

Proof. For $\theta \in (0,1)$ and $t \in (0,\rho)$, by using the (7)–(9), the lemma can be easily proved.

Lemma 2.5 *Let the assumptions of Lemma (2.3), let $\gamma = h(a_0)d_0$, then*

$$d_n \geq \lambda \gamma^{4^n}, \; n \geq 0, \tag{20}$$

where $\lambda = \frac{1}{h(a_0)}$. *Also for* $n \geq 0$, *we have*

$$\prod_{i=0}^{n} d_i \geq \lambda^{n+1}\gamma \frac{4^{n+1}-1}{3}. \quad (21)$$

Proof Since $a_1 = \gamma a_0$, $b_1 = h(a_0)d_0^2 b_0 < \gamma^2 b_0$, $c_1 = h(a_0)d_0^3 c_0 < \gamma^3 c_0$, by Lemma we have

$$d_1 = h(\gamma a_0)\varphi(\gamma a_0, \gamma^2 b_0, \gamma^3 c_0) < \gamma^3 d_0 = \gamma^{4^1-1} d_0 = \lambda \gamma^{4^1}.$$

Suppose $d_n \leq \lambda\gamma^{4^n}$, $n \geq 1$. Then by Lemma, we have $a_{n+1} < a_n$ and $h(a_n)d_k < 1$.
Thus

$$\begin{aligned} d_{n+1} &< h(a_n)\varphi\big(h(a_n)d_n a_n, h(a_n)d_n^2 b_n, h(a_n)d_n^3 c_n\big) \\ &< h(a_n)\varphi\big(h(a_n)d_n a_n, h(a_n)^2 d_n^2 b_n, h(a_n)^3 d_n^3 c_n\big) \\ &< h(a_n)^3 d_n^4 \\ &< \lambda\gamma^{4^{n+1}}. \end{aligned} \quad (22)$$

Therefore it holds that $d_n \leq \lambda\gamma^{4^n}$, $n \geq 0$.
By (20), we get

$$\prod_{i=0}^{n} d_i \leq \prod_{i=0}^{n} \lambda\gamma^{4^i} = \lambda^{n+1}\gamma^{\Sigma_{i=0}^{n} 4^i} = \lambda^{n+1}\gamma^{\frac{4^{n+1}-1}{3}}, \; n \geq 0.$$

This shows (21) holds. The proof is completed. □

Lemma 2.6 *Under the suppositions of Lemma* 2.5. *Let* $\gamma = h(a_0)d_0$ *and* $\gamma = \frac{1}{h(a_0)}$. *The sequence* $\{\eta_n\}$ *satisfies*

$$\eta_n \leq \eta\lambda\gamma^{\frac{4^n-1}{3}}, n \geq 0. \quad (23)$$

Hence the sequence $\{\eta_n\}$ *converges to zero. Moreover, for any* $n \geq 0$, $m \geq 1$, *it holds*

$$\sum_{i=n}^{n+m} \leq \eta\lambda^n\gamma^{\frac{4^n-1}{3}} \frac{1-\lambda^{m+1}\gamma^{\frac{4^n(4^m+2)}{3}}}{1-\lambda\gamma^{5^n}}. \quad (24)$$

Proof.From (10) and (20), we have

$$\eta_n = d_{n-1}\eta_{n-1} = d_{n-1}d_{n-2}\eta_{n-2} = \cdots = \eta\Big(\prod_{i=0}^{n-1} d_i\Big)$$
$$\leq \eta\lambda^n\gamma^{\frac{4^n-1}{3}}, \; n \geq 0.$$

Because $\lambda < 1$ and $\gamma < 1$, it follows that $\eta_n \to 0$ as $n \to \infty$.
Let

$$\rho = \sum_{i=n}^{n+m} \lambda^i \gamma^{\frac{4^i}{3}},$$

where $n \geq 0$, $m \geq 1$. Since

$$\rho \leq \lambda^n\gamma^{\frac{4^n}{3}} + \gamma^{4^n}\Big(\sum_{i=n}^{n+m} \lambda^i\gamma^{\frac{4^{i-1}}{3}},\Big)$$

$$= \lambda^n\gamma^{\frac{4^n}{3}} + \lambda\gamma^{4^n}\Big(\sum_{i=n}^{n+m-1} \lambda^i\gamma^{\frac{4^i}{3}},\Big)$$

$$= \lambda^n\gamma^{\frac{4^n}{3}} + \lambda\gamma^{4^n}\Big(\rho - \lambda^{n+m}\gamma^{\frac{4^{n+m}}{3}}\Big),$$

we can obtain

$$\rho \leq \lambda^n\gamma^{\frac{4^n}{3}} \frac{1-\lambda^{m+1}\gamma^{\frac{4^n(4^n+2)}{3}}}{1-\lambda\gamma^{4^n}}.$$

Furthermore, we have

$$\sum_{i=n}^{n+m} \eta_i \leq \eta\Big(\sum_{i=n}^{n+m} \lambda^i\gamma^{\frac{4^i-1}{3}}\Big)$$

$$= \eta\gamma^{-\frac{1}{3}}\rho$$

$$\leq \eta\lambda^n\gamma^{\frac{4^n-1}{3}} \frac{1-\lambda^{m+1}\gamma^{\frac{4^n(4^m+2)}{3}}}{1-\lambda\gamma^{4^n}}.$$

.

Therefore $\sum_{n=0}^{\infty} \eta_n$ exists. □

3 Recurrence relations for the method

Here we denote $B(x,r) = \{y \in X : \|y-x\| < r\}$ and $\overline{B(x,r)} = \{y \in X : \|y-x\| < r\}$. Now we will derive in the following the recurrence relations for the method mentioned in (3) under the assumptions considered in the previous section.

For $n = 0$, the existence of Γ_0 implies the existence y_0, x_0 and x_1. And we have

$$\|y_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leq \eta_0.$$

and

$$\|F(y_0)\| \leq \frac{1}{2}M\eta_0^2$$

Thus $y_0 \in B(x_0, R\eta)$, where $R = \frac{g(a_0)}{1-h(a_0)\varphi(a_0,b_0,c_0)}$. Let us suppose $H(x_0) = \Gamma_0([y_n,x_n;F] - F'(x_0))$, then

$$\|H(x_0)\| \leq a_0.$$

Since $a_0 < 1$, hence by Banach Lemma it shows that $H(x_0)^{-1}$ exists and satisfies

$$\|[I + H(x_0)]^{-1}\| \leq \frac{1}{1-a_0}.$$

Consequently, we derive

$$[y_n, x_n; F]^{-1} \leq \frac{\beta_0}{1-a_0}.$$

Similarly

$$\|x_1 - y_0\| \leq \frac{a_0(3-a_0)}{2(1-a_0)}\eta_0 = p(a_0)\eta_0$$

and

$$\|x_1 - x_0\| \leq \|x_1 - y_0\| + \|y_0 - x_0\|,$$

$$\leq \left(\frac{a_0(3-a_0)}{2(1-a_0)} + 1\right)\eta_0,$$

$$\leq g(a_0)\eta_0.$$

From the assumption $d_0 < \frac{1}{h(a_0)} < 1$, it follows that $x_1 \in B(x_0, R\eta)$.

By $a_0 < s$ and $g(a_0) < g(s)$, we have

$$\|I - \Gamma_0 F'(x_1)\| \leq \|\Gamma_0\|\|F'(x_0) - F'(x_1)\|$$

$$\leq M\beta_0\|x_1 - x_0\|$$

$$\leq a_0 g(a_0) < 1.$$

It follows by Banach Lemma that $\Gamma_1 = [F'(x_1)]^{-1}$ exists and

$$\|\Gamma_1\| \leq \frac{\beta_0}{1 - a_0 g(a_0)} = h(a_0)\beta_0 = \beta_1.$$

Using the required relations which are obtained above, we can derived that

$$\|F(y_0)\| \leq \frac{1}{2}M\eta_0^2,$$

$$\|F(x_1)\| \leq \frac{1}{2}M\|x_1 - y_0\|^2 + \frac{1}{6}N\|x_1 - y_0\|\|x_1 - x_0\|^2 + \frac{1}{36}L\|y_0 - x_0\|\|x_1 - x_0\|^2 + \frac{1}{24}L\|x_1 - x_0\|^4,$$

$$\|F(x_1)\| \leq \left(\frac{p(a_0)^2}{2}M\eta_0 + \frac{p(a_0)g^2(a_0)}{6}N\eta^2 + \frac{g^2(a_0)(2 + 3g^2(a_0))}{72}L\eta^3\right)\eta_0.$$

In addition, we have

$$M\|\Gamma_1\|\|\Gamma_1 F(x_1)\| \leq h(a_0)d_0 a_0 = a_1,$$

$$N\|\Gamma_1\|\|\Gamma_1 F(x_1)\|^2 \leq h(a_0)d_0^2 b_0 = b_1,$$

$$M\|\Gamma_1\|\|\Gamma_1 F(x_1)\| \leq h^2(a_0)d_0^3 c_0 = c_1.$$

Repeating the above derivation, we can obtain the system of recurrence relations given in next lemma.

Lemma 2.7 *Let the assumptions of Lemma and the conditions (A1)-(A5) hold. Then the following items are true for all $n \geq 0$:*

(I) There exist $\Gamma_n = [F'(x_n)]^{-1}$ and $\|\Gamma_n\| \leq \beta_n$,
(II) $\|\Gamma_n F(x_n)\| \leq \eta_n$,
(III) $M\|\Gamma_n\|\|\Gamma F(x_n)\| \leq a_n$ and $\|K_F(x_n)\| \leq a_n$,
(IV) $N\|\Gamma_n\|\|\Gamma F(x_n)\|^2 \leq b_n$,
(V) $L\|\Gamma_n\|\|\Gamma F(x_n)\|^3 \leq c_n$,
(VI) $\|x_{n+1} - x_n\| \leq g(a_n)\eta_n$,
(VII) $\|x_{n+1} - x_n\| \leq R\eta$, Where $R = \frac{g(a_0)}{1-d_0}$.

Proof The proof of (I)-(VI) follows by using the above-mentioned way and invoking the induction hypothesis. We only consider (VI). By (VI) and Lemma (2.6) we obtain

$$\|x_{n+1} - x_0\| \leq \sum_{i=0}^{n} \|x_{i+1} - xi\|$$

$$\leq \sum_{i=0}^{n} g(a_i)\eta_i$$

$$\leq g(a_0)\sum_{i=0}^{n} \eta_i$$

$$\leq g(a_0)\eta\frac{1 - \lambda^{n+1}\gamma^{\frac{4^n+2}{3}}}{1 - d_0} < R\eta,$$

since $\gamma < 1$, $\lambda < 1$ and $\lambda\gamma = d_0$. The lemma is proved.

4 Semilocal convergence

Lemma 2.8 *Let $R = \frac{g(a_0)}{1-d_0}$. If $h(a_0)d_0 < 1$, then $R < 1$.*

Proof The proof is immediate by the definitions.

Now we give a theorem to establish the semilocal convergence of (3), the existence and uniqueness of the solution and the domain in which it is located, along with a priori error bounds, which lead to the R-order of convergence at least four of iteration (3).

Theorem 1 *Let X and Y be two Banach spaces and $F : D \subseteq X \to Y$ be a three times Fréchet differentiable on a non-empty open convex subset D. Assume that $x_0 \in D$ and all the conditions (A1)-(A5) hold.Let $a_0 = M\beta\eta$, $b_0 = N\beta\eta^2$, $c_0 = L\beta\eta^3$ and $d_0 = h(a_0)\varphi(a_0, b_0, c_0)$ satisfy $a_0 < s$ and g, h, φ are defined by (7)– (9). Let $\overline{B(x_0, R\eta)} \subseteq D$ where $R = \frac{g(a_0)}{1-d_0}$, then starting from x_0, the sequence $\{x_n\}$ generated by the method (3) converges to a solution x^* of F(x) with x_n, x^* belong to $\overline{B(x_0, R\eta)}$ and x^* is the unique solution of $F(x)$ in $B(x_0, \frac{2}{M\beta} - R\eta)\bigcap D$. Moreover, a priori error estimate is given by*

$$\|x_n - x^*\| \leq g(a_0)\eta\lambda^n\gamma^{\frac{4^n-1}{3}}\frac{1}{1 - \lambda\gamma^{4^n}}, \qquad (25)$$

where $\gamma = h(a_0)d_0$ and $\lambda = \frac{1}{h(a_0)}$.

Proof. By Lemma (2.7), the sequence $\{x_n\}$ is well defined in $\overline{B(x_0, R\eta)}$. Next we prove that $\{x_n\}$ is a cauchy sequence. Since

$$
\begin{aligned}
\|x_{n+m}-x_n\| &\le \sum_{i=n}^{n+m-1}\|x_{i+1}-x_i\| \\
&\le \sum_{i=n}^{n+m-1} g(a_i)\eta_i \\
&\le g(a_0)\sum_{i=n}^{n+m-1}\eta_i \\
&\le g(a_0)\eta\lambda^n\gamma^{\frac{4^n-1}{3}}\frac{1-\lambda^m\gamma^{\frac{4^n(4^{m-1}+2)}{3}}}{1-\lambda\gamma^{4^n}}. \quad (26)
\end{aligned}
$$

It follows that $\{x_n\}$ is a cauchy sequence. So there exist a x^* such that $\lim_{n\to\infty}x_n = x^*$.

By letting $n=0$, $m\to\infty$ in (26), we obtain

$$\|x^*-x_0\| \le R\eta. \quad (27)$$

This shows $x^* \in \overline{B(x_0,R\eta)}$.

Now we prove that x^* is a solution of $F(x)=0$. Since

$$\|\Gamma_0\|\|F(x_n)\| \le \|\Gamma_n\|\|F(x_n)\| \le \eta_n. \quad (28)$$

By letting $n\to\infty$ in (28), we obtain $\|F(x_n)\|\to 0$ since $g(a_n)<g(a_0)$ and $\eta_n\to 0$. Hence, by the continuity of F in D, we obtain $F(x^*)=0$.

We prove the uniqueness of x^* in $B(x_0,\frac{2}{M\beta}-R\eta)\bigcap D$, since it follows by Lemma

$$\frac{2}{M\beta}-R\eta = \left(\frac{2}{a_0}-R\right)\eta > \frac{1}{a_0}\eta > R\eta, \quad (29)$$

and then $\overline{B(x_0,R\eta)} \subseteq B(x_0,\frac{2}{M\beta}-R\eta)\bigcap D$.

Let x^{**} be another zero of $F(x)$ in $B(x_0,\frac{2}{M\beta}-R\eta)\bigcap D$. By Taylor theorem, we have

$$0 = F(x^{**})-F(x^*) = \int_0^1 F'((1-t)x^*+tx^{**})dt(x^{**}-x^*). \quad (30)$$

Since

$$
\begin{aligned}
&\|\Gamma_0\|\|\int_0^1 [F'((1-t)x^*+tx^{**})-F'(x_0)]dt\| \quad (31)\\
&\le M\beta\int_0^1 [(1-t)\|x^*-x_0\|+t\|x^{**}-x_0\|dt \quad (32)\\
&< \frac{M\beta}{2}\left[R\eta+\frac{2}{M\beta}-R\eta\right]=1, \quad (33)
\end{aligned}
$$

it follows by Banach lemma that $\int_0^1 F'((1-t)x^*+tx^{**})dt$ is invertible and hence $x^{**}=x^*$.

Finally, by letting $m\to\infty$ in (26), we obtain (25) and furthermore

$$\|x_n-x^*\| \le \frac{g(a_0)\eta}{\gamma^{\frac{1}{3}}(1-d_0)}\left(\gamma^{\frac{1}{3}}\right)^{4^n}. \quad (34)$$

This means that the method given by (3) is of R-order of convergence at least four.

5 Numerical results

We consider the nonlinear integral equation $F(x)=0$, where

$$F(x)(s) = x(s)-1+\frac{1}{2}\int_0^1 s\cos(x(t))dt, \quad (35)$$

where $s\in[0,1]$ and $x\in D=B(0,2)\subset X$. Here, $X=C[0,1]$ is the space of continuous functions on $[0,1]$ with the max-norm,

$$\|x\| = \max_{s\in[0,1]}|x(s)|.$$

We can obtain the derivatives of given by

$$F'(x)y(s) = y(s)-\frac{1}{2}\int_0^1 s\sin(x(t))y(t)dt, y\in D,$$

$$F''(x)yz(s) = -\frac{1}{2}\int_0^1 s\cos(x(t))y(t)z(t)dt, y,z\in D.$$

Furthermore, we have

$$\|F''(x)\| \le \frac{1}{2} \equiv M,\ x\in D,$$

$$\|F'''(x)\| \le \frac{1}{2} \equiv N,\ x\in D$$

and the Lipschitz condition with $L=\frac{1}{2}$

$$\|F'''(x)-F'''(y)\| \le \frac{1}{2}\|x-y\|,\ x,y\in D.$$

A constant function, i.e. $x_0(t)=\frac{4}{3}$, is chosen as the initial approximate solution. It follows that

$$F(x_0)\| \le \frac{1}{2}\cos\frac{4}{3}.$$

In this case, we have

$$\|I-F(x_0)\| \le \frac{1}{2}\sin\frac{4}{3}$$

and then by Banach lemma we include that Γ_0 exists and satisfies

$$\|\Gamma_0\| \le \frac{2}{2-\sin\frac{4}{3}} \equiv \beta.$$

It follows that

$$\|\Gamma_0 F(x_0)\| \le \frac{\cos\frac{4}{3}}{2-\sin\frac{4}{3}} \equiv \eta.$$

Therefore, we obtain

$$a_0 = M\beta\eta = \frac{\cos\frac{4}{3}}{(2-\sin\frac{4}{3})^2},$$

$$b_0 = N\beta\eta^2 = \frac{\cos\frac{4}{3}}{(2-\sin\frac{4}{3})^3},$$

$$c_0 = L\beta\eta^3 = \frac{\cos\frac{4}{3}}{(2-\sin\frac{4}{3})^4}.$$

As a result, we compute

$$q(a_0) = a_0 g(a_0) - 1 \simeq -0.689073 < 0$$

Table 1: Results of recurrence relations

n	η	β	a_n	b_n	c_n	d_n
0	0.2288	1.9454	0.2225	0.0509	0.0116	0.0348
1	2.66e-3	2.82083	3.75e-3	1.00e-5	2.67e-8	7.173e-8
2	7.12e-11	2.83155	1.00e-10	7.17e-21	5.10e-31	1.36e-30
3	3.63e-41	2.83155	5.14e-41	1.87e-81	6.80e-122	1.82e-121
4	2.47e-162	2.83155	3.50e-162	8.65e-324	2.14e-485	5.73e-485
5	5.29e-647	2.83155	7.49e-647	3.96e-1293	2.10e-1939	5.62e-1939

and

$$d_0 h(a_0) \simeq 0.0505026 < 1.$$

This means that the hypotheses of Theorem 1 is satisfied. Hence the recurrence relations for the method given by (3) is demonstrated in Table 1. Besides, the solution x^* belongs to $\overline{B(x_0,R\eta)} = \overline{B(4/3,0.331255\cdots)} \subset D$ and it is unique in $B(4/3,1.72408\cdots)\bigcap D$.

6 Conclusion

A system of recurrence relations is developed to establish the semilocal convergence of a modified Newton's method (3) used to solving $F(x) = 0$ in Banach spaces. Based on the recurrence relations, an existence uniqueness theorem is also established to show the R-order convergence of the method to be four. A numerical example is worked out to demonstrate our approach and show that our method can be of practical interest.

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Deepak Kumar received his Bachelor degree in Science (2005-2008) and the Master of Science (2008-2010) from the Panjab University, Chandigarh, India. He served as Assistant Professor in S.P.N. college, Mukerian, Punjab, India during 2010-2015. Currently, he is pursuing Doctorate degree in Mathematics from Department of Mathematics, Sant Longowal Institute of Engineering and Technology,Longowal, India. His research interests include: Numerical analysis and Numerical Functional analysis.

Janak Raj Sharma received his Bachelor degree in Science from Himachal Pradesh University, Shimla, India (1985-1988) and the Master of Science and PhD degrees in Mathematics (1989-1995) from the University of Roorkee (now Indian Institute of Technology), Roorkee, India. He worked as Assistant Professorin National Institute of Technology, Hamirpur (India) and Kumaon Engineering College, Dwarahat (India) during 1996-2000. Currently, he works as a Professor of Mathematics at Sant Longowal Institute of Engineering and Technology, Department of Mathematics, Longowal, Punjab, India. His research interests include: Numerical analysis and Theory of General Relativity. He has published around hundred articles in conferences and peer reviewed journals.