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Arcsine-G Family of Distributions

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Abstract: This article introduces a family of distributions. The moments and simultaneous estimation for the model parameters of the family are discussed. A special case of this family, called the arcsine-exponential distribution is proposed. The distributional properties of the arcsine-exponential distribution are discussed along with the distributions of the several order statistics. Maximum likelihood and method of moment estimation techniques are discussed for the model parameter of the distribution. An extensive simulation study is conducted to asses the performance of the maximum likelihood estimation technique. Two real-life data sets are considered to observe the applicability of the proposed arcsine-exponential distribution.

Keywords: Arcsine distribution, exponential distribution, maximum likelihood estimation, order statistics, reliability function.

1 Introduction

In order to capture the real world phenomena, statistical distributions are commonly used. The theory of statistical distributions are widely studied along with new developments for their usefulness. To describe different real world phenomena, several family of distributions are developed. Actually, this new development in the area of distribution theory is ongoing process. Karl Pearson (1857-1936) developed a family of continuous probability distributions [1], for which every density function $y = f(x)$ is defined to be any valid solution to the differential equation, see [1,2], as

$$\frac{f'(x)}{f(x)} = \frac{c+x}{c_0+c_1x+c_2x^2},$$

where $c,\ c_0,\ c_1$ and c_2 are the parameters. Eugene et al. [3], introduced a beta-generated family of distributions by using logit of the beta random variable. The proposed family has the cdf as

$$F(x) = \int_0^{G(x)} r(t)\mathrm{d}t, \tag{1}$$

where r(t) is the density function of the beta random variable and $G(x)$ is the distribution function of any random variable. Jones [4], and Cordeiro and de Castro [5], extended the beta-generated family of distributions by replacing the beta distribution in (1) with the Kumaraswamy distribution. The limited support, $(0,1)$ exist, for both of the beta and Kumaraswamy distributions. The main target of this article is to use the density $r(t)$ as another limited support arcsine distribution described as

$$r(t) = \frac{1}{\pi\sqrt{t(1-t)}},\ t\in(0,1), \tag{2}$$

and introduces a new family called Arcsine-G family. The layout plan of the article is organized as follows: In Section 2, Arcsine-G family of distributions is developed along with its general moments and simultaneous estimation of the model parameters. Some special cases of the proposed family are listed in Section 3. Section 4 presents the arcsine-exponential distribution in detail. Section 5 describes some of its distributional properties. The distributions of several order statistics are presented in Section 6. Two estimation techniques for the model parameter are given in Section 7. In Section 8, an extensive simulation study is conducted to assess the performance of the maximum likelihood estimation technique. Two real-life applications are described in Section 9. Some concluding remarks are listed at the end.

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2 Arcsine-G Family of Distributions

In order to introduce the cumulative distribution function of the Arcsine-G family of distributions, setting probability density function of the arcsine distribution (2) in (1) and further proceed as

$$F(x) = \int_0^{G(x)} \frac{1}{\pi\sqrt{t(1-t)}} \mathrm{d}t,$$

which can be further obtained as

$$F(x) = \frac{2}{\pi} \operatorname{arcsine}\left[\sqrt{G(x)}\right], \quad x \in \mathbb{R}, \tag{3}$$

and the density function corresponding to (3) is

$$f(x) = \frac{g(x)}{\pi\sqrt{G(x)\left[1-G(x)\right]}}, \quad x \in \mathbb{R}. \tag{4}$$

Hence researcher can take the advantage of introducing new members of the family, by using standard distribution function $G(x)$ in (3) for any suitable distribution. Some special cases of this family are described in the following section.

2.1 Moments of the Family

The expression for the moments of Arcsine-G family of distributions has been obtained in terms of probability weighted moments of base distribution $G(x)$. For this, consider the expression for the probability weighted moments of a distribution $G(x)$ and is given as

$$M_{r,s,t} = \int_{-\infty}^{\infty} x^r \{G(x)\}^s \{1-G(x)\}^t g(x)\, dx. \tag{5}$$

The rth moment for the Arcsine-G family of distributions is given as

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx. \tag{6}$$

Now using (4) in (6), the expression for rth moment is given as

$$\mu_r' = \frac{1}{\pi}\int_{-\infty}^{\infty} x^r \frac{g(x)}{\sqrt{G(x)\left[1-G(x)\right]}} dx.$$

Using (5) the rth moment for the Arcsine-G family of distributions is

$$\mu_r' = \frac{M_{r,-\frac{1}{2},-\frac{1}{2}}}{\pi}. \tag{7}$$

From (7) it can be easily observed that the moments for the Arcsine-G family of distributions can be expressed as probability weighted moments of the base distribution $g(x)$.

2.2 Maximum Likelihood Estimation

In order to discussion of the general case, in which more than one model parameter are to be estimated simultaneously, let $x_1, x_2, \cdots, x_n$ be a random sample drawn from the Arcsine-G family $\left\{f_\xi(x) : \xi \in \Xi\right\}$, where ξ is a vector of parameters. The set of admissible values of the parameters $\xi_1, \xi_2, \cdots, \xi_k$ which makes the likelihood function an absolute maximum. The necessary condition for a local turning-point in the likelihood function, see [6], is

$$\frac{\partial}{\partial \xi_r} \mathrm{Log}\left[L(\mathbf{x}|\xi_1, \xi_2, \ldots, \xi_k)\right] = 0, \quad r = 1, 2, \cdots, k, \tag{8}$$

and a sufficient condition that this be a maximum is that the matrix

$$\frac{\partial^2 \mathrm{Log} L}{\partial \xi_r \cdot \partial \xi_s};\ r, s = 1, 2, \cdots, k,$$

be negative definite. The k equations (8) are to be solved for the k maximum likelihood estimators $\hat{\xi}_1, \hat{\xi}_2, \cdots, \hat{\xi}_k$. For some models, these equations can be explicitly solved for $\hat{\xi}$, but in general no closed-form solution to the maximization problem is known or available, and the maximum likelihood estimators can only be found via numerical optimization.

3 Special Cases of the Arcsine-G Family

Some special cases of the proposed Arcsine-G family of distributions are described in the Table 1, by using different base distribution functions $G(x)$ in (3).

Table 1: Special cases of the Arcsine-G family of distributions

Distribution	Cumulative Distribution Function	Support
Arcsine – Normal	$\frac{2}{\pi}\sin^{-1}\left[\sqrt{\Phi(x)}\right]$	$x\in\mathbb{R}$
Arcsine – Rayleigh	$\frac{2}{\pi}\sin^{-1}\left[\sqrt{1-e^{-\frac{x^2}{2\sigma^2}}}\right]$	$x\in[0,\infty)$
Arcsine – Weibull	$\frac{2}{\pi}\sin^{-1}\left[\sqrt{1-e^{-\left(\frac{x}{\lambda}\right)^k}}\right]$	$x\in[0,\infty)$
Arcsine – Gompertz	$\frac{2}{\pi}\cos^{-1}\left[\sqrt{e^{\eta-\eta e^{bx}}}\right]$	$x\in[0,\infty)$
Arcsine – Kumaraswamy	$\frac{2}{\pi}\sin^{-1}\left[\sqrt{1-(1-x^a)^b}\right]$	$x\in[0,1]$
Arcsine – Log-logistic	$\frac{2}{\pi}\sin^{-1}\left[\sqrt{\frac{x^\beta}{\alpha^\beta+x^\beta}}\right]$	$x\in[0,\infty)$
Arcsine – Pareto	$\frac{2}{\pi}\sin^{-1}\left[\sqrt{1-\left(\frac{k}{x}\right)^\theta}\right]$	$x\in[k,\infty)$
Arcsine – Dagum	$\frac{2}{\pi}\sin^{-1}\left[\sqrt{\left(\left(\frac{x}{b}\right)^{-a}+1\right)^{-p}}\right]$	$x\in\mathbb{R}^+$
Arcsine – Burr XII	$\frac{2}{\pi}\sin^{-1}\left[\sqrt{1-(x^c+1)^{-k}}\right]$	$x\in\mathbb{R}^+$

With view to illustration, choose distribution function $G(x)$ of exponential distribution in case of alternative parametrization and introduce arcsine-exponential distribution as follow.

4 Arcsine-Exponential Distribution

The distribution function of the exponential distribution in case of alternative parametrization is

$$G(x)=1-e^{-\frac{x}{\xi}},\quad x\in[0,\infty),\tag{9}$$

where $\xi\in[0,\infty)$ is the scale parameter of the distribution. By setting (9) in (3), we have the distribution function of the arcsine-exponential distribution as

$$F(x)=\frac{2}{\pi}\,\text{arcsine}\left(\sqrt{1-e^{-\frac{x}{\xi}}}\right),\quad x\in[0,\infty),\tag{10}$$

where $\xi\in[0,\infty)$.

Definition. A continuous random variable X is said to have an arcsine-exponential distribution if its probability density function can be written as follow

$$f(x)=\frac{\sqrt{e^{-\frac{x}{\xi}}}}{\pi\xi\sqrt{1-e^{-\frac{x}{\xi}}}},\quad x\in[0,\infty),\tag{11}$$

where $\xi\in[0,\infty)$.

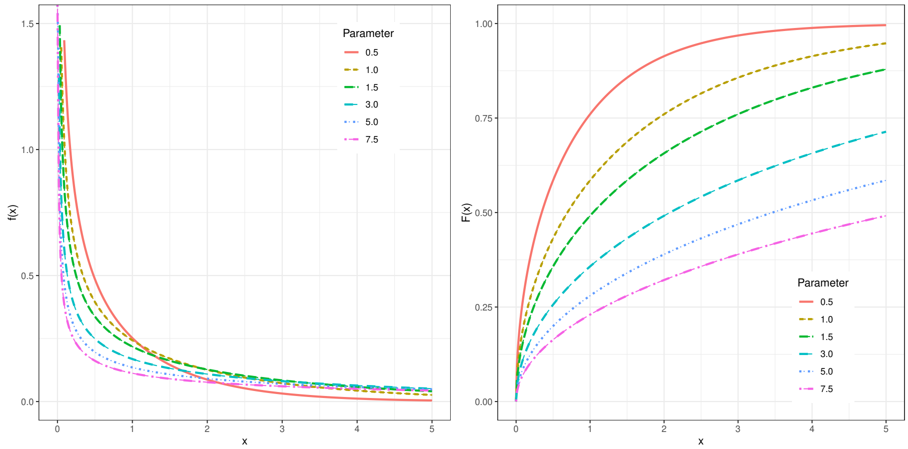

Fig. 1: The density function $f(x,\xi)$ and distribution function $F(x,\xi)$ of the proposed distribution

By differentiating distribution function of arcsine-exponential distribution given in (10) with respect to x, we have the density function of arcsine-exponential distribution defined in (11). Figure 1, describes some of the possible shapes for the density and distribution functions of the arcsine-exponential distribution, and observed that it has the capability of capturing the related complexity of the random variables.

Several distributional properties of the arcsine-exponential distribution are described in the following section.

5 Distributional Properties

Some of the essential distributional properties for the proposed arcsine-exponential distribution are discussed by the following subsections.

5.1 Moments

Moment is the specific quantitative measure of the shape of a function. Actually, it is very much essential to know the shape characteristic of a specific probability distribution. The moment equation for the arcsine-exponential distribution is obtained as follow

$$\mu_r' = \int_0^\infty x^r \frac{\sqrt{e^{-\frac{x}{\xi}}}}{\pi\xi\sqrt{1-e^{-\frac{x}{\xi}}}}\mathrm{d}x,$$

and hence by setting $r = 1,2$ we have the mean and variance respectively as

$$\text{Mean} = \mu_1' = 1.39\xi, \tag{12}$$

and

$$\text{Variance} = \mu_2' - (\mu_1')^2 = \frac{\pi^2\xi^2}{3}.$$

5.2 Moment Generating Function

Theorem 1. *Let X be a continuous random variable follows arcsine-exponential distribution, then moment generating function, $M_X(t)$, can be obtained as*

$$M_X(t) = \frac{\Gamma\left(\frac{1}{2}-t\xi\right)}{\sqrt{\pi}\,\Gamma(1-t\xi)}, \quad \Re(t) \le 0,$$

where $\xi \in [0,\infty)$ and $\Re(t)$ gives the real part of the complex number t.

5.3 Characteristic Generating Function

Theorem 2. *Let X be a continuous random variable follows arcsine-exponential distribution, then characteristic function,* $\phi_X(t)$*, can be obtained as*

$$\phi_X(t) = \frac{\Gamma\left(\frac{1}{2} - it\xi\right)}{\sqrt{\pi}\Gamma(1 - it\xi)}, \quad \Im(t) \geq 0,$$

where $\xi \in [0, \infty)$ *and* $\Im(t)$ *gives the imaginary part of the complex number* t*.*

5.4 Quantile Function and Median

The quantile function is linked with the probability distribution of a random variable. It specifies the value of the random variable such that the probability of the variable being less than or equal to that value equals the given probability. It is also called the inverse cumulative distribution function. The quantile function for the arcsine-exponential distribution is obtained and given, see for example [7,8], as

$$x_q = \xi \log\left[\frac{1}{1 - \sin^2\left(\frac{\pi q}{2}\right)}\right]. \tag{13}$$

By using (13), one can easily obtain its first quartile (Q_1) by setting $q = \frac{1}{4}$ as

$$\text{First quartile} = Q_1 = \xi \log\left[\frac{1}{1 - \sin^2\left(\frac{\pi}{8}\right)}\right],$$

median or second quartile (Q_2) by setting $q = \frac{2}{4}$ as

$$\text{Median} = Q_2 = \xi \log\left[\frac{1}{1 - \sin^2\left(\frac{2\pi}{8}\right)}\right],$$

and third quartile (Q_3) by setting $q = \frac{3}{4}$ as

$$\text{Third quartile} = Q_3 = \xi \log\left[\frac{1}{1 - \sin^2\left(\frac{3\pi}{8}\right)}\right].$$

5.5 Generating Random Sample

According to the probability integral transformation, a continuous random variable X with its distribution function $F(x)$, the random variable $Y = F(x) \sim U[0,1]$. The inverse probability integral transform states that if $Y \sim U[0,1]$ and if X has a cumulative distribution $F(x)$, then the random variable $F^{-1}(Y)$ has the same distribution as X. Using this concept, the random sample from the proposed arcsine-exponential distribution can be obtained, see for example [9,10], as

$$X = \xi \log\left[\frac{1}{1 - \sin^2\left(\frac{\pi u}{2}\right)}\right]. \tag{14}$$

Hence, one can easily obtain the random sample from the proposed arcsine-exponential distribution using (14) and use it for further analysis.

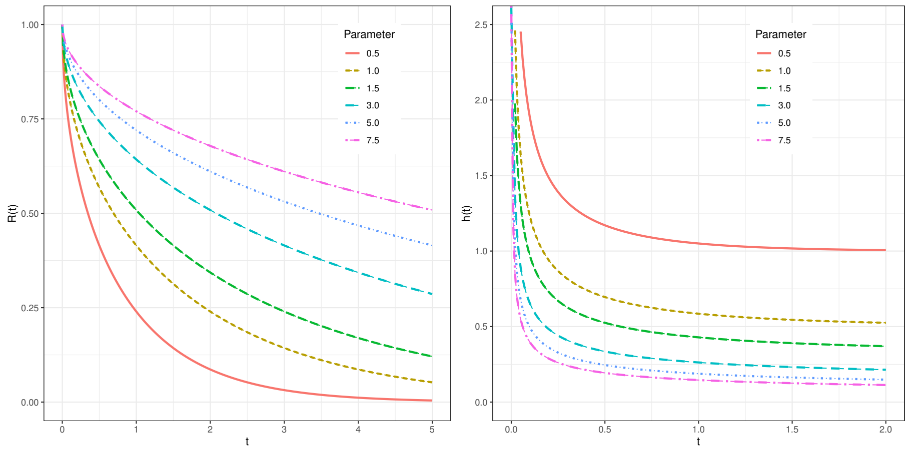

Fig. 2: The reliability function $R(t,\xi)$ and hazard function $h(t,\xi)$ of the proposed distribution

5.6 Reliability Function

The reliability function gives the probability that a device, patient, or other object of interest will survive beyond any specified time t, see [11]. This function is also known as the survivor function or survival function. The term reliability function is very much common in engineering while the term survival function is used in human mortality. Another name for the this function is the complementary distribution function, see for example [12], can further expressed as

$$R(t) = 1 - \frac{2}{\pi}\,\text{arcsine}\left(\sqrt{1-e^{-\frac{t}{\xi}}}\right), \quad t \in [0,\infty).$$

The hazard function is a synonym of force of mortality which is used particularly in actuarial science and demography. The hazard function for the proposed arcsine-exponential distribution is

$$h(t) = \frac{\left[e^{\frac{t}{\xi}}\left(1-e^{-\frac{t}{\xi}}\right)\right]^{-\frac{1}{2}}}{\pi\xi\left[1-\frac{2}{\pi}\,\text{arcsine}\left(\sqrt{1-e^{-\frac{t}{\xi}}}\right)\right]}, \quad t \in [0,\infty).$$

Some possible shapes for the reliability and hazard functions are presented in Figure 2. Hence, critically observed that the hazard function shows sharply decreasing then almost constant hazard rates.

6 Order Statistics

The probability distribution function of the rth order statistic for the proposed arcsine-exponential distribution can be determined as follow

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!}\left[\frac{2}{\pi}\,\text{arcsine}\left(\sqrt{1-e^{-\frac{x}{\xi}}}\right)\right]^{r-1}\left[1-\frac{2}{\pi}\,\text{arcsine}\left(\sqrt{1-e^{-\frac{x}{\xi}}}\right)\right]^{n-r}\left\{\frac{\sqrt{e^{-\frac{x}{\xi}}}}{\pi\xi\sqrt{1-e^{-\frac{x}{\xi}}}}\right\},$$

where $r = 1,2,\cdots,n$. Therefor, for $r = 1$, we have the pdf of the lowest order statistic $X_{(1)}$, and is given as

$$f_{X_{(1)}}(x) = n\left[1-\frac{2}{\pi}\,\text{arcsine}\left(\sqrt{1-e^{-\frac{x}{\xi}}}\right)\right]^{n-1}\left\{\frac{\sqrt{e^{-\frac{x}{\xi}}}}{\pi\xi\sqrt{1-e^{-\frac{x}{\xi}}}}\right\},$$

and for $r = n$, the pdf of the highest order statistic $X_{(n)}$, is obtain by

$$f_{X_{(n)}}(x) = n\left[\frac{2}{\pi}\ \text{arcsine}\left(\sqrt{1-e^{-\frac{x}{\xi}}}\right)\right]^{n-1}\left\{\frac{\sqrt{e^{-\frac{x}{\xi}}}}{\pi\xi\sqrt{1-e^{-\frac{x}{\xi}}}}\right\}.$$

7 Estimation of the Model Parameter

Let $x_1, x_2, \cdots, x_n$ be a random sample draw from arcsine-exponential distribution. Now the model parameter estimation techniques are discussed by the following subsections.

7.1 Maximum Likelihood Estimation (MLE)

The likelihood function for the proposed arcsine-exponential distribution is

$$L = \frac{1}{\pi^n \xi^n} \cdot e^{-\frac{1}{2\xi}\sum_{i=1}^{n} x_i} \cdot \prod_{i=1}^{n}\left[1-e^{-\frac{x_i}{\xi}}\right]^{-\frac{1}{2}},$$

which has an equivalent log-likelihood function as

$$l = -n\log(\pi) - n\log(\xi) - \frac{\sum_{i=1}^{n} x_i}{2\xi} - \frac{1}{2}\sum_{i=1}^{n}\log\left(1-e^{-\frac{x_i}{\xi}}\right). \tag{15}$$

The maximum likelihood estimate of ξ is obtained by maximizing the log-likelihood function (15). For doing this, the derivatives with respect to unknown parameter ξ is given as

$$\frac{\delta l}{\delta \xi} = \frac{\sum_{i=1}^{n} x_i}{2\xi^2} - \frac{1}{2\xi}\left[2n - \frac{1}{\xi}\sum_{i=1}^{n}\frac{x_i e^{-\frac{x_i}{\xi}}}{\left(1-e^{-\frac{x_i}{\xi}}\right)}\right].$$

By setting $\frac{\delta l}{\delta \xi} = 0$, the theoretical solution of $\hat{\xi}$ is not so easy, even sometimes impossible, for this nonlinear equation. In order to get the numerical solution of $\hat{\xi}$, apply R-package "bbmle", for more details see [13].

7.2 Method of Moment Estimation (MME)

The method of moment estimation is based upon equating the population moments with the sample moments and then solving for unknown parameters, see [14], as follow

$$\int_0^\infty x^r f(x)\mathrm{d}x = \frac{1}{n}\sum_{i=0}^{n} x_i^r,\ r = 1, 2, \cdots, k. \tag{16}$$

By setting $r = 1$ we have $\int_0^\infty x^r f(x)\mathrm{d}x = 1.39\xi$ as obtained in (12). Hence using (16) further obtained as

$$\hat{\xi} = \frac{1}{1.39n}\sum_{i=1}^{n} x_i.$$

This can also be done for higher values of r. The MME is fairly simple and gives consistent estimators (under very weak assumptions), though these estimators are frequently biased.

7.3 Estimate Parameter using MLE and MME

The MME can be computed quickly and easily whereas in some cases the likelihood equations may be uncontrollable without computers. In order to estimate parameter of the proposed model and make a comparative study, two real-life data sets (used in Section 9) are considered here. The method of moment estimates for the respective data sets are 6.738 and 8.780, which are very close to the maximum likelihood estimates given in Table 4 and Table 5 respectively.

Table 2: Average estimate of the parameter and corresponding *MSE* for the arcsine-exponential distribution

Sample Size	Estimate of ξ	MSE of ξ
50	2.460	14.628
100	2.212	13.690
200	1.872	5.119
500	1.460	0.517

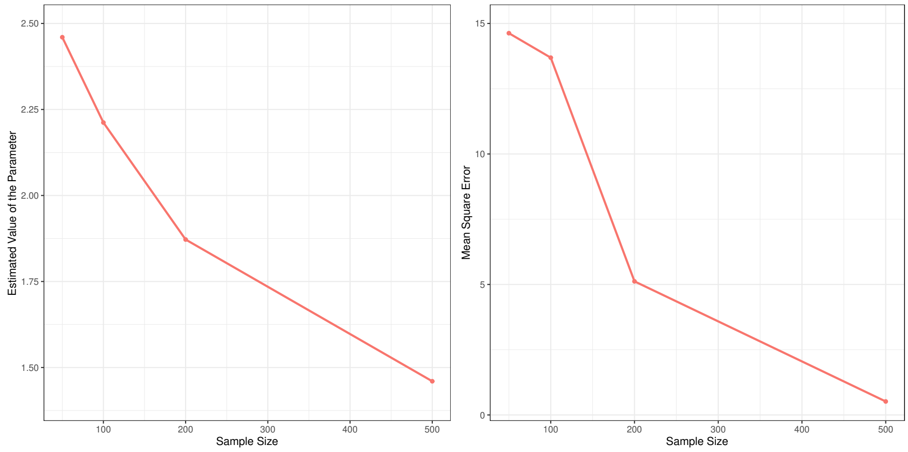

Fig. 3: Average estimate and corresponding mean square error of the model parameter

8 Simulation Study

In this section, we have described a simulation study to asses the performance of the estimation technique. In order to conduct this simulation study, we have drawn several samples of sizes 50, 100, 200, and 500 from arcsine-exponential distribution by setting $\xi = 1.5$ initially. Maximum likelihood estimate of ξ is obtained for each setup. The procedure is repeated for 1000 times and taken the average value of the estimate along with respective mean square error. The results are described in Table 2 and plotted in Figure 3 and observed that true value of the estimate is very close to the estimated value, for the sample size 500. It has been also observed that, mean square error of the estimate consistently decreases with increasing in the sample sizes. This shows the adequacy of the estimation method.

9 Real-life Applications

In order to observe the applicability of the proposed arcsine-exponential distribution, consider the exponentiated exponential (EE) and base exponential distributions. To make a comparative study among the distributions, two real-life applications are presented in the following subsections.

Table 3: Summary statistics for the selected data sets

Data Set	Min.	Q_1	Median	Mean	Q_3	Max.	Skew.
Bladder Cancer	0.080	3.348	6.395	9.366	11.838	79.050	3.248
River Flood	0.100	2.125	9.500	12.204	20.125	64.000	1.442

9.1 Bladder Cancer Data

The data set consist of a set of remission times collected from cancer patients in a bladder cancer study, see [15]. The data set considered here only for the illustrative purposes. Summary statistics for the bladder cancer data are given in Table 3, and observed that the given data is positively skewed. We estimate the model parameter of the arcsine-exponential distribution along with selected distributions using restricted maximum likelihood estimation technique. The estimate of the model parameters alongside the standard errors are presented in Table 4.

Table 4: The estimate of the parameters and respective standerd errors along with model selection criteria values

Distribution	Parameter	Estimate	Standard Error	KS	AD	C-vM
Arcsine-Exponential	ξ	6.482	0.748	0.229	9.218	4.156
Exponentiated Exponential	θ,λ	0.010, 11.209	0.001, 7.2e-07	0.999	74.398	—
Exponential	λ	0.107	0.009	0.959	—	18.055

Several model selection criteria like KS (Kolmogorow-Smirnow statistic), AD (Anderson-Darling statistic), and C-vM (Cramér-von Mises statistic) are considered for the data set to make a comparative study among the proposed and selected models. The better distribution corresponds to the smaller KS, AD, and C-vM values:

$$KS = \sup_x |F_n(x) - F(x)|,$$

$$AD = n\int_{-\infty}^{\infty} [F_n(x) - F(x)]^2\, w(x) dF(x),$$

and

$$C-vM = n\int_{-\infty}^{\infty} [F_n(x) - F(x)]^2\, dF(x),$$

where k is the number of parameters in the statistical model, n is the sample size, $\sup_x$ is the supremum of the set of distances, $F_n(x)$ is the empirical distribution function, $F(x)$ is the cumulation distribution function and $w(x) = [F(x)\{1-F(x)\}]^{-1}$ is the weight function.

The computed values of the various model selection criterion's are described in Table 4, and observed that the proposed arcsine-exponential distribution is the appropriate model as compared with the other selected models. The estimated cdf of the arcsine-exponential distribution along with the selected models are plotted over the empirical cdf plot for the bladder cancer data in the left of Figure 4. Hence observed that the arcsine-exponential distribution fitted well as compared with the other selected models.

9.2 River Flood Data

The Wheaton River data set provides 72 exceedances of flood peaks in m^3/s for the year 1958-1984. It has been previously used by [16], and [17]. The summary statistics of the data is presented in Table 3, and observed that the data set is skewed to the right. Hence, again estimate the model parameter of the arcsine-exponential distribution alongside the selected models applying the restricted maximum likelihood estimation technique. The estimated results for the model parameters along with standard errors are described in Table 5.

Table 5: The estimate of the parameters and respective standerd errors along with model selection criteria values

Distribution	Parameter	Estimate	Standard Error	KS	AD	C-vM
Arcsine-Exponential	ξ	8.481	1.292	0.114	3.105	1.194
Exponentiated Exponential	θ,λ	0.006, 14.793	0.001, 2.4e-07	1.000	44.733	—
Exponential	λ	0.082	0.010	0.951	—	12.532

The model selection criteria KS, AD, and K-vM values are estimated and described in Table 5. Hence, observed that the results shows confirmation in favor of the proposed arcsine-exponential distribution among the selected models

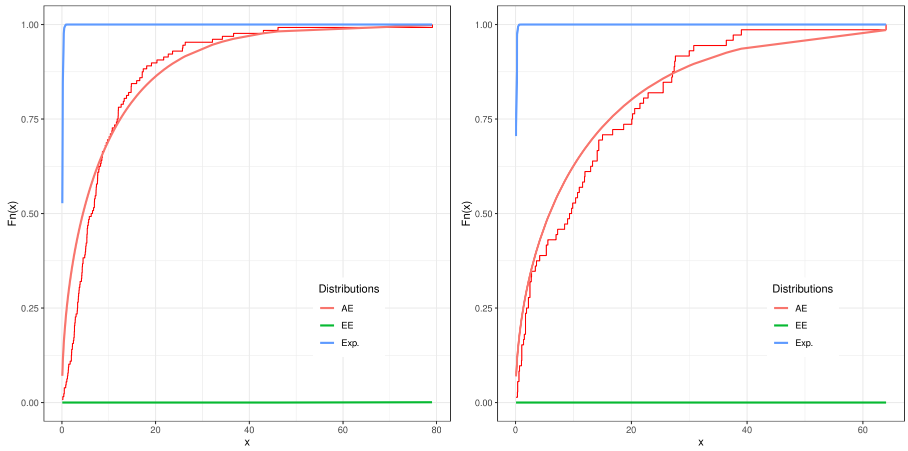

Fig. 4: Estimated distribution functions are plotted over the empirical distribution function

consider in this study. The estimated cdf for the selected models are plotted over the empirical cdf plot which are given in right of Figure 4. It has been observed that the proposed arcsine-exponential distribution shows better fit than any other selected models.

10 Concluding Remarks

In this article, a new Arcsine-G family of distributions is introduced. In order to clarify the concept of this family, we consider an exponential distribution in case of alternative parameterization, and developed arcsine-exponential distribution. Some of its distributional properties including moment, characteristic function, quantile function along with median, generation of random sample, reliability function alongside the distributions of order statistics are described. An exclusive simulation study is conducted, to asses the performance of the maximum likelihood estimation procedure, and observed that estimated values of the model parameters are very much close to the true values, and it ensure the adequacy of the estimation technique. At the end, two real-life applications are considered to investigate the applicability of the proposed distribution.

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