

Some Features of Quantum Fisher Information and Entanglement of Two Atoms Based on Atomic State Estimation

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Abstract: We investigate the dynamics of the QFI (quantum Fisher information) of two two-level atoms and cavity field mode when the time-dependent coupling is neglected and considered. We use the concurrence as a quantifier of the atom-atom entanglement while the nonlocal correlation between the two atoms and optical field is measured by the von Neumann entropy. We analyze the impact of the number of photon transition and initial atomic state setting on the entanglement measures and QFI for two parameters of the atomic state estimation. Also, the effect of the time dependent coupling in terms of the atomic speed on the evolution of the nonlocal correlation and QFI is discussed. It is shown that the number of photon transition and atomic speed have a significant role on the dynamical properties of QFI and entanglement. Also, it is found that there a clear relationship between the entanglement and QFI.

Keywords: Estimator parameter, quantum Fisher information, two atoms, nonlocal correlation

1 Introduction

QPTs (Quantum phase transitions) of a quantum system is an external parameter appeared when abrupt changes in the ground state properties or a tiny change, e.g., the strength of an external magnetic field [1]. Unlike thermal phase transitions, which occur at a finite temperature, a QPT refers to a process of the ground state, which is related to zero temperature. In order to experimentally detect a QPT, one needs to work in a regime in which thermal fluctuations do not suppress the quantum fluctuations. The critical points of QPTs can be detected by many theoretical tools, such as making use of the quantum fidelity [2,3], entanglement [4,5], and quantum correlation [6,7]. In particular, by investigating the external values of these quantities or their derivatives, one is able to highlight the critical points. The QFI can be used for a similar purpose [8,9,10], which is a basic concept in the QET (quantum estimation theory) giving a theoretical bound for the variance of an estimator [11,12]. QFI can be used to estimate the value of an unknown parameter. So, it is a measure of how fast a given state changes under a given evolution. It can also be used as a

pointer that how useful a quantum state is for quantum metrology [13,14,15]. A larger QFI means that the estimated parameter involves high precision.

Recently, many devices such as beam splitter [16,17], cavity QED [18,19], nanoresonators [20,21] and NMR systems [22] have been realized and proposed experiments for generating quantum entanglement. So the investigation of nonlocal correlation outdated its context of bipartite and multipartite pure quantum states, considering the case of mixed states as well as investigates the entanglement in large scale many particle quantum systems. Whereas a complete investigation of entanglement in complex systems is still absent until present. The quantification of this kind correlation in both discrete and continuous variables in the framework of theoretical approaches [23,24,25,26].

The properties of the TCM (Tavis-Cummings model) under the presence of time dependent coupling has been attracting much attention. So a lot of theoretical attempts have been motivated, by experimental progress in QED (cavity quantum electrodynamics). Moreover the experimental drive, there also exists a theoretical motivation to involve the influence time-dependent

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coupling to TCM. The influence of the time dependent coupling on the TCM has been investigated [27] and the phenomena of ESD (entanglement sudden death) and ESB (entanglement sudden birth) have been explored [28]. Also, the dynamical properties of entanglement and geometric phase of moving TLAs within the atomic speed and acceleration have been investigated [29]. It is demonstrated that the atomic motion plays a significant role in the dynamical behavior of entanglement and geometric phase.

The link between QFI and nonlocal correlation of the composite quantum system has not been fully studied. However, there are some investigations for quantifying the degree of entanglement in pure state based on the evolution of the QFI. Recently, QFI is found to be a sensitive probe to the QPTs [13,30]. It has also been proposed as a measure for macroscopicity of quantum systems [31,32]. The effects phase-damped on the dynamical properties of the QFI compared with the entanglement measured by the negativity has been studied [33]. Moreover, the QFI has been used to estimate the phase of a qubit influenced by a non-Markovian environment [34]. The main objective of this article is to explore the link between QFI and two kinds of correlation: the EAA-F (Entanglement between two atoms and Field) and the EA-A (Atom-Atom Entanglement). Also examining the influence of the initial atomic state estimator parameter, the number of photons transition and time-dependent coupling on the evolution of EAA-F, EA-A and QFI will be examined.

The rest of this paper as follows: In Sec. 2, we present the model of the interaction between field mode and the moving TLAs and obtain the atomic density matrix. In Sec. 3, the basic concepts of the entanglement quantifiers and QFI will be presented. In Sec. 4, the discussion of the numerical results of the QFI dynamics and entanglement under various conditions will be demonstrated. Finally, the main results will be summarized in Sec. 5.

2 Model and its dynamics

Here, we consider the optical field mode F donated by $\hat{a}(\hat{a}^\dagger)$ the annihilation (creation) operators interacting with symmetric two TLAs donated A,B with energy levels $|e_j\rangle$ and $|g_j\rangle$, where $|g\rangle$ is the lower level and $|e\rangle$ is the upper level of (TLA i.e. $J = A, B$ atom). The interaction Hamiltonian of the system in the RWA (rotating-wave approximation) can be written as [29]

$$H_I(t) = \Omega(t) \sum_{j=1}^2 \left(\hat{a}^k |e_j\rangle \langle g_j| + \hat{a}^{\dagger k} |g_j\rangle \langle e_j| \right), \quad (1)$$

where k is the number of photon transition and $\Omega(t)$ is the time-dependent coupling between the TLAs and field [35, 36].

Let us assume that the initial state of the TLAs $|\psi_{AB}(0)\rangle$ is a Bell state (maximally entangled state). The field is initially prepared in the CS (coherent state) (i.e. $|\psi_F(0)\rangle = |\alpha\rangle$). Then the initial state can be written as

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} (|e_1 e_2\rangle + |g_1 g_2\rangle) \otimes \sum_{n=0}^{\infty} \alpha^n \sqrt{\frac{\exp(-\langle n \rangle)}{n!}} |n\rangle,$$

where $\langle n \rangle = |\alpha|^2$, is the mean photon number of the field.

The final state vector at any time can be evaluated as follows

$$|\psi(t)\rangle = \exp \left\{ -i \int_0^t H_I(T) dT \right\} |\psi(0)\rangle. \quad (2)$$

Now, the atomic density matrix can be written as

$$\rho_{AB}(t) = \text{Tr}_{Field} \{ |\psi(t)\rangle \langle \psi(t)| \} = \sum_{\ell=1}^4 \sum_{s=1}^4 \hat{\rho}_{\ell s}(t) |\ell\rangle \langle s|. \quad (3)$$

In the next section we present the concept of the QFI, the NE as a measure of EAA-F and concurrence as a quantifier of EA-A.

3 Entanglement and quantum Fisher information

As known von Neumann entropy (NE) is the main measure of entanglement when the quantum system in the pure state, so we use NE as a measure of entanglement between the field and two atoms. The NE in terms of the eigenvalues of the atomic density matrix is given by [37, 38]

$$S_{AB}(t) = -\text{Tr}(\rho_{AB}(t) \ln \rho_{AB}(t)) = -\sum_{j=1}^4 \{ \eta_j \ln \eta_j \}, \quad (4)$$

where η_j are the eigenvalues of the atomic density matrix ρ_{AB} .

The EA-A can be measured by the concurrence [23,29]. In the case of an arbitrary joint state of two atoms, Wootters derived the formula for concurrence, which is very useful for analysis and calculation. The explicit form of concurrence C_{AB} can be formulated as

$$C_{AB}(t) = \max \left\{ 0, \sqrt{\xi_1} - \sqrt{\xi_2} - \sqrt{\xi_3} - \sqrt{\xi_4} \right\},$$

where ξ_j are the eigenvalues of the non-Hermitian matrix $\rho_{AB} \tilde{\rho}_{AB}$ where $\tilde{\rho}_{AB} = (\hat{\sigma}_y \otimes \hat{\sigma}_y) \rho_{AB}^* (\hat{\sigma}_y \otimes \hat{\sigma}_y)$, which decreasing order of $\rho_{AB} \tilde{\rho}_{AB}$, while $\tilde{\rho}_{AB}$ is the spin-flipped state of density operator ρ_{AB} . Also, ρ_{AB}^* is conjugate of in the standard basis of two atoms and $\hat{\sigma}_y$ is the Pauli spin operator. The concurrence $C_{AB} = 1$ for the Bell state and zero value for separable state,

The, QFI depends on the proper choice of the estimated parameter. If the phase shift of the initial atomic state $\hat{\vartheta} = \theta, \phi$ used as an estimator parameter hence the unitary operation with the phase shift estimator given be $u(\varphi) = |e_1 e_2\rangle \langle e_1 e_2| + \exp(i\phi) |g_1 g_2\rangle \langle g_1 g_2|$.

The QFI gives the maximum information about the estimated parameter and defined in terms of the atomic density operator $\rho_{AB}(\hat{\vartheta}, t)$ and SLD (symmetric logarithmic derivative) as [33,34,39,40]

$$I_{QF}(t) = \text{Tr} [\rho_{AB}(\hat{\vartheta}, t) L^2(\hat{\vartheta}, t)], \quad (5)$$

where $L(\hat{\vartheta}, t)$ is the quantum score given by [39].

$$2 \frac{\partial(\hat{\vartheta}, t)}{\partial \hat{\vartheta}} = L(\hat{\vartheta}, t) \rho_{AB}(\hat{\vartheta}, t) + \rho_{AB}(\hat{\vartheta}, t) L(\hat{\vartheta}, t). \quad (6)$$

The QFI is related to the uncertainty of the measurement via the Gramer-Rao bound, $\Delta\phi \geq 1/\sqrt{I_{QF}}$.

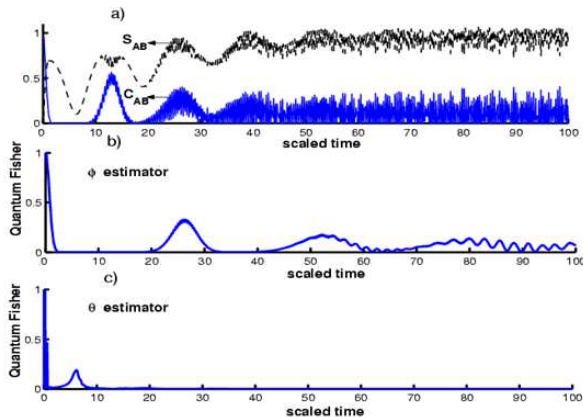


Fig. 1: The dynamics of the: (a) concurrence C_{AB} (solid line), and the VN S_{AB} (dashed line). The QFI I_{QF} for ϕ estimator (b) and for θ estimator (c) of a TLAs interacting with field initially in the CS with the mean photon number $\langle n \rangle = 16$. The case of one-photon transition ($k = 1$) and coupling constant between the field mode and TLAs (i.e. $\Omega(t) = \lambda$).

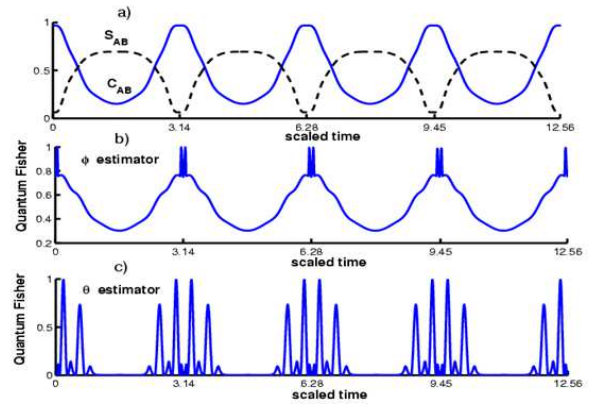


Fig. 2: The dynamics of the: (a) concurrence C_{AB} (solid line), and the VN S_{AB} (dashed line). The QFI I_{QF} for ϕ estimator (b) and for θ estimator (c) of a TLAs interacting with field initially in the CS with the mean photon number $\langle n \rangle = 16$ and $k = 1$. The time dependent coupling between the field and TLAs (i.e. $\Omega(t) = \lambda \cos(t)$).

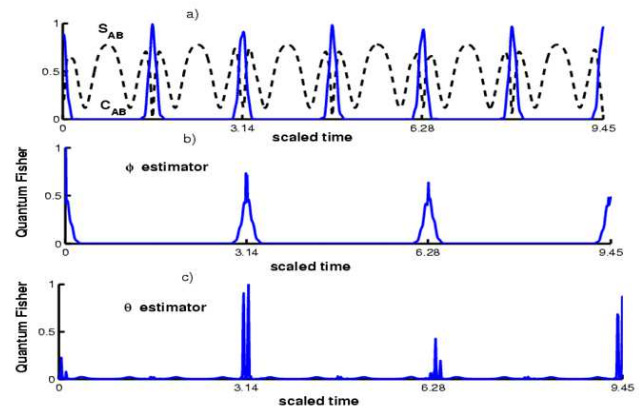


Fig. 3: Effect of two photon transition (i.e. $k = 2$) on the dynamics of the: (a) concurrence C_{AB} (solid line), and the VN S_{AB} (dashed line). The QFI I_{QF} for ϕ estimator (b) and for θ estimator (c) of a stationary TLAs interacting with field initially in the CS with the mean photon number $\langle n \rangle = 16$.

4 Discussion of numerical results

Here, we discuss the relationship between the QFI compared with EA-A, and EAA-F for two types of the initial atomic state estimator parameters. In figures 1-4, we have used the VN as a measure of EAA-F, concurrence as a quantifier of the EA-A and discuss their relationship with the QFI. The QFI, indicates the maximum information the initial atomic state position θ and phase shift parameter ϕ . Furthermore, we check the

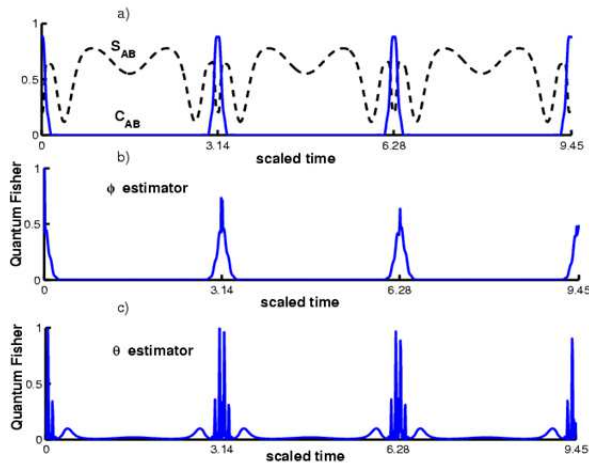


Fig. 4: Effect of time dependent coupling or atomic speed (i.e. $\Omega(t) = \lambda \cos(t)$) on the time evolution of the: (a) concurrence C_{AB} (solid line), and the VN S_{AB} (dashed line). (b) The QFI I_{QF} for ϕ estimator and (c) for estimator of a moving TLAs interacting with input field initially in the CS of the mean photon number $\langle n \rangle = 16$ and the case of two photons process ($k = 2$).

influence of the number of photon transition and time dependent coupling on the dynamical behavior of different quantities. The quantities QFI, C_{AB} and S_{AB} are plotted against the scaled time λt (one unit of time is defined by $1/\lambda$).

As seen all the quantities exhibit a chaotic behavior in the case of one-photon transition and when the time dependent coupling is ignored. On the other hand the quantities have a periodic and regular behavior when the atomic speed is considered. The EA-A and QFI based on the phase shift estimator ϕ involve a monotonic oscillatory behavior during the time dynamics (see Fig. 2). Moreover it is noticed that the maximum value of C_{AB} , and I_{QF} can be obtained at the periodic time $\lambda t = m\pi$ ($m = 0, 1, 2, \dots$) for the estimator ϕ , while it drops to zero in the case of θ estimator (see Fig.1). Moreover, C_{AB} , and I_{QF} involving sudden death followed by sudden birth in a periodic manner at $\lambda t = m\pi$ (see Fig. 4). Finally the effect of the time dependent coupling within the atomic speed on the dynamical behavior of the C_{AB} , S_{AB} , and I_{QF} is presented in Fig. 4.

Figs. 3(a) and 4(b) present the dynamical behavior of the QFI compared with the evolution of the concurrence and VN. The dynamics of the QFI and concurrence are represented by the dashed, dotted, and solid lines. The plots clearly show that for this initial atomic state the negativity keeps the dynamical behavior as the previous example with a decrease of entanglement amplitude during the time evolution, contrary to what happens to the

QFI is very sensitive to the parameter estimator with long time of ESD, and such a behavior is different from the previous one. In this case, the QFI can indicate an important physical aspects of the system under consideration.

5 Conclusion

In summary, within the atomic state estimation, we have discussed in detail the dynamical behavior of two kinds of entanglement when the time dependent coupling is ignored and considered. The maximal QFI was obtained by choosing the phase shift of atomic initial state position as estimating parameters. The nonlocal correlation between the TLAs and field is by the VN while the atom-atom entanglement is quantified by the concurrence. It is clarified that the number of transmitting photons from the field mode and atomic speed play a significant role in the dynamics of QFI and entanglement. Also, it's shown that the evolution of QFI and E A-A depicts a monotonic behavior during the time evolution. Finally, the optimal values of the QFI can be achieved by a which depends on the proper choose of the atomic state estimator parameters.

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