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On the Numerical Solution of Partial Differential Equation with Convection Term by using Bernoulli Wavelets

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Abstract: A numerical method for solving a partial differential equation with a convection term is presented. The proposed method is based on the Bernoulli wavelet in which Bernoulli polynomial is used. First, we use the 2-point Euler backward differentiation formula, and then we use collocation points that convert the differential equation into a system of algebraic equations. Explaining examples are added to demonstrate the validity and applicability of the method. .

Keywords: partial differential equation, 2-point Euler backward differentiation formula, Bernoulli wavelets, collocation method.

1 Introduction

Partial differential equations are used in many branches of science and engineering for simulating the physical process. These equations describe different processes in gas dynamics, viscous fluid flow, heat conduction, chemical reactions, filtration of liquids, biological species, environmental pollution, etc [1-2].

Partial differential equations have been numerically solved by different techniques [3-6]. We can convert all types of equations: parabolic, elliptic, and hyperbolic equations by using the method of finite element [7]. The finite point [8], Galerkin method [9], method of local Petrov-Galerkin [10], the method of reproducing kernel particle [11].

Wavelet theory is a relatively modern and emerging tool in the applied mathematical part. It has been used in an enormous range of engineering disciplines; especially, signal analysis for waveform representation and segmentation, time-frequency analysis, and fast algorithms for easy application. Wavelets allow the accurate exemplification of a collection of functions and factors. Further, wavelets establish a relation with quick numerical algorithms. Since 1991 the different types of wavelet methods have been used for numerical solutions of various kinds of the integral equation, elaborated research on these papers can be formed in [12].

The present paper is concerned with an efficient numerical approximation scheme to find the solution of the following linear parabolic differential equation. Given $\Omega \equiv [a,b]$ and $I=(0,T)$, find $u(x,t)$ such that

$$u_t(x,t)-u_{xx}(x,t)+\gamma u_x(x,t)=f(x,t),\ \ in\ Q\equiv \Omega\times I, \tag{1.1}$$

connected with the conditions of boundary and initial

$$u(a,t)=g_1(t),\qquad u(b,t)=g_2(t),\qquad t\in(0,T) \tag{1.2}$$

$$u(x,0)=u_0(x),\qquad \forall x\in\Omega. \tag{1.3}$$

Where the real-valued function f is defined on $\Omega\times I$ and $g_1,\ g_2$ are continuous functions in their variables.

For solving this problem, we introduce a new nonstandard feasible and the method that we propose can be obtained by Bernoulli wavelet collocation with the characteristic method in time. The stability and analysis of convergent for the approximate solution will be presented. The numerical outcomes obtained by the suggested technique display the prospective convergence to the exact solution.

Our main target is to display a fully discretized numerical method to find the solution of (1.1). We utilize the backward Euler method for the discretization in time (also called the Rothe method; (see, e.g., M.S. El-azab [13])), and method of Bernoulli wavelet collocation for space-discretization. The numerical outcomes gained by the

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suggested technique are compared with the exact solution of the problem; the numerical solution displays the expected convergence to the exact one as the mesh size is refined.

The rest of this article is orderly as follows. We stat notations, assumptions, and definitions that use in our problem in section 2. Section 3, give the semi-discretization scheme, prior estimates. In section 4, we give the description Bernoulli wavelet, then function approximation used to solve PDE, then the method of solution of our problem is described. Section 5, gives specific two examples to test the suggested method. The conclusion is present in section 6.

2 Notations, Assumptions and Definitions

We use the standard functional spaces $L_2(\Omega)$, $V \equiv W_0^{1,2}(\Omega)$, $C(I; L_2(\Omega))$, $L_2(I; L_2(\Omega))$, and (see e.g. [14, 15]). $(.,.)$We shall denote either the inner product in $L_2(\Omega)$, or the duality between V andV^* (dual of V). We denote by $|\cdot|$, $\|\cdot\|$, $\|\cdot\|_*$, $\|\cdot\|_{L_2(I;V)}$, the norms in$L_2(\Omega)$,V, $V^*$and $L_2(I;V)$, respectively. all the constants which occur in the course of this paper will be denoted by C (ε is small and $C_\varepsilon = C(\varepsilon^{-1})$).

The following elementary relations will be used in the following analysis:

$$2\sum_{i=1}^{s}(w_i, w_i - w_{i-1}) = \|w_s\|^2 - \|u_0\|^2 + \sum_{i=1}^{s}\|w_i - w_{i-1}\|^2, \tag{2.1}$$

and Young's inequality

$$ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \quad a, b \in R \tag{2.2}$$

We will assume, throughout this work, the following hypotheses on the given data.

•**(H1)** The function $f: \Omega \times I \times R \to R$, is Lipschitz continuous in the meaning of

$$\left|f(x,t,s) - f(x,t',s')\right| \le c\left\{\left|t - t'\right|\left(|s| + \left|s'\right|\right) + \left|s - s'\right|\right\}, \quad \forall t, t' \in I, \quad \forall s, s' \in R. \tag{2.3}$$

Moreover, the function satisfy the growth condition

$$|f(x,t,\Im)| \le C(1 + |\Im|), \quad \forall (x,t,\Im) \in \Omega \times I \times R \tag{2.4}$$

• **(H2)** $u_0 \in V$.

Under these assumptions, we can define the various solution of problem (1.1)-(1.3).

Problem (P): Find $u: Q \to R$ such that for all $\varphi \in V$ and a.e. $t \in I$,

$$u \in C(I; L_2(\Omega)) \cap L_2(I; \mathrm{V}) \quad \text{with } \partial_t u \in L_2(I; L_2(\Omega)), \tag{2.5}$$

$$(u_t, \varphi) + (u_x, \varphi_x) + (\gamma u_x, \varphi) = (f, \varphi), \tag{2.6}$$

$$u(x, 0) = u_0(x), \qquad \forall x \in \Omega \tag{2.7}$$

3 The Semi Discretization Scheme. Prior Estimates

Our main goal is to approximate (1.1) − (1.3) from a numerical point of view and to prove its convergence. The suggested technique is based on the combination of the characteristics and Roth methods. Using a 2-point Euler backward differentiation formula for the time derivative, and then applying the characteristic method to compensate the convection term which is discretized explicitly so that the underlying equation is converted into a linear system of algebraic equations that easily solved numerically at each subsequent time level. To this purpose, let n be a positive integer. Subdivide the time interval I by the points t_i where $t_i = i\tau,\ \tau = {}^{T}\!/_{n}, i = 0,1, \dots, n.$ The suggested discretization scheme of the problem (1.1) consists of the following problem

Problem(P_τ): Find $u_i \cong u(\cdot, t_i) \in V, i = 1,2, \dots$ such that

$$u_0(x) = u(x, 0) \tag{3.1}$$

$$(u_i - u_{i-1}^*, \varphi) + \tau(u_i', \varphi') = \tau(f_i, \varphi), \quad \forall \varphi \in V, \tag{3.2}$$

$$u_i^*(x) = \tilde{u}_i(x - \tau\gamma), \tag{3.3}$$

where $f_i = f(x, t_i, u_{i-1})$ and $\tilde{u} \in V$ is an extension of $u \in V(\Omega^*)$, $\Omega^* \supset \overline{\Omega}$ satisfying

$$\|u\|_{V(\Omega^*)} \le C\|u\|_{V(\Omega)} \tag{3.4}$$

The presence of a weak solution $u_i \in V$ is guaranteed by Lax–Milgram argument. The application of the coercivity condition requires the application of the inequality

$$2(\varphi_x, \varphi) = \int_{\partial\Omega} \frac{d}{dx}\varphi^2 = [\varphi^2(x)]_{\partial\Omega} \ge 0, \tag{3.5}$$

with the convection term.

Using $u_i, (i = 0,1, \dots, n)$ determined by the recommended scheme (3.2) in each time step t_i, we introduce the following piecewise linear functions (Rothe functions)

$$u^n(0) = u_0, \ \ u^n(t) = u_{i-1} + (t - t_{i-1})\delta u_i, \ \text{ for} t \in (t_{i-1}, t_i], \ \ i = 1,2, \dots n, \tag{3.6}$$

and the corresponding step function

$$\bar{u}^n(0), \ \ \bar{u}^n(t) = u_i, \ \text{ for} t \in (t_{i-1}, t_i] \tag{3.7}$$

Using the notation of Rothe function and its corresponding step function, a piecewise constant interpolation of equation (1.3) over I yields $\forall \varphi \in V$:

$$(\partial_t u^n, \varphi) + (\partial_x \bar{u}^n, \partial_x \varphi) + (\gamma \partial_x \bar{u}_t^n, \varphi) = (\bar{f}^n, \varphi), \tag{3.8}$$

where $\bar{u}_\tau^n(t) = \bar{u}^n(t - \tau)$, $t_{i-1} \le t \le t_i$, and $\bar{f}^n = f(\cdot, \bar{t}^n)$ with $\bar{t}^n = t_i$.

To show the stability of the separated solution and prove the convergence results, we shall derive some a priori estimates.

Lemma 3.1 Under the assumptions **(H1)** and **(H2)**, there exists a positive constant C such that

$$\sum_{i=1}^{s} \tau|\delta u_i|^2 + \max_s \|u_s\|^2 + \sum_{i=1}^{s}\|u_i - u_{i-1}\|^2 \le C, \tag{3.9}$$

for any s.

Proof. Putting $\varphi = \tau\delta u_i$ in (3.2) and sum over i for $i =$

$1,2,\dots,s$ we obtain

$$\sum_{i=1}^{s}\left(\frac{u_{i+1}(x)-u_i^*(x)}{\tau},\tau\delta u_i\right)+\sum_{i=1}^{s}(u'_{i+1},u'_{i+1}-u'_i)=\sum_{i=1}^{s}(f_{i+1},\tau\delta u_i) \tag{3.10}$$

The first term I is estimated by $I = I_1 + I_2$ where

$$I_1 = \sum_{i=1}^{s}\tau(\delta u_i,\delta u_i) = \sum_{i=1}^{s}\tau|\delta u_i|^2, \tag{3.11}$$

$$|I_2| \le \left|\sum_{i=1}^{s}\tau(u'_{i-1},\delta u_i)\right| \le C_\varepsilon\sum_{i=1}^{s}\tau\|u_{i-1}\|^2+\varepsilon\sum_{i=1}^{s}\tau|\delta u_i|^2 \tag{3.12}$$

In estimating (3.12), we have used Schwarz and Young's inequalities. Taking into consideration the elementary identity (2.1), the elliptic term of (3.10) is bounded by

$$2\sum_{i=1}^{s}(u'_i,u'_i-u'_{i-1}) = \|u_s\|^2-\|u_0\|^2+\sum_{i=1}^{s}\|u_i-u_{i-1}\|^2 \tag{3.13}$$

On collecting (3.10)–(3.13), choosing ε, ŋ sufficiently small and applying the discrete version of Gronwall's lemma, one conclude the estimates (3.9).

Lemma 3.2 There exists a constant C such that

$$\|\bar{u}^n\|_{L_2(I;V)} \le C,\quad \|u^n-\bar{u}^n\|_{L_2(I;V)} \le \frac{C}{\sqrt{n}},$$
$$\|u^n-\bar{u}^n\|_{L_2(I;L_2(\Omega))} \le \frac{C}{n},\quad \|u^n-\bar{u}^n_\tau\|_{L_2(I;L_2(\Omega))} \le \frac{C}{n} \tag{3.14}$$

Proof. The estimate (3.14)1 is a consequence of (3.9) and the estimates (3.14)2 is a consequence of the definitions of u^n and $\bar{u}^n$ the fact that

$$|u^n-\bar{u}^n| \le |u_i-u_{i-1}|, \tag{3.15}$$

and thus the proof completes.

Now we show the uniqueness of a solution to the problem (P).

Theorem 3.1. Let u_1 and u_2 be two solutions to the problem (P). Then $u_1 = u_2$.

Proof. Denoting by $e_u = u_1 - u_2$ and $e_f = f(u_1) - f(u_2)$. Since we assume that u_1 and u_2 are two solutions to the Problem (P), then from (2.6) we can write

$$(\partial_t e_u,\varphi)+(\partial_x e_u,\varphi_x)+(\gamma\partial_x e_u,\varphi)=(e_f,\varphi) \tag{3.16}$$

Put $\varphi = e_u$, we get

$$\frac{1}{2}\frac{d}{dt}|e_u|^2+\|e_u\|^2=(e_f,e_u) \tag{3.17}$$

The integration of (3.17) over $(0,t^*)$ yields

$$\frac{1}{2}|e_u(t^*)|^2+\int_0^{t^*}\|e_u\|^2dt=\int_0^{t^*}(e_f,e_u)dt \tag{3.18}$$

Due to (2.3) we arrive at

$$|e_u(t^*)|^2+\int_0^{t^*}\|e_u\|^2dt \le C\int_0^{t^*}\|e_u\|^2dt \tag{3.19}$$

Apply Gronwall lemma we conclude that $|e_u(t^*)|^2 = 0$. This is valid for an arbitrary t^* and thus $u_1 = u_2$.

We can now formulate the following convergence results.

Theorem 3.2 Let the assumptions **(H1)–(H2)** be fulfilled and let u^n and $\bar{u}^n$ be the sequences defined by (3.6) and (3.7), respectively. Then there exists $u \in L_2(I;V)$, which is the solution of **Problem (P)**, with $\partial_t u \in L_2(I;L_2(\Omega))$ such that

$$\bar{u}^n \to u,\quad \text{in } C(I;L_2(\Omega)) \cap L_2(I;V), \tag{3.20}$$

$$\partial_t\bar{u}^n \to \partial_t u,\quad \text{in} L_2(I;L_2(\Omega)) \tag{3.21}$$

(in the sense of subsequences). Moreover,

$$\|u-u^n\|^2_{L_2(I;L_2(\Omega))}+\|u-u^n\|^2_{L_2(I;V)} \le C\left(\frac{1}{n^2}+\frac{1}{n}\right) \tag{3.22}$$

Proof. From lemma 3.2, $\bar{u}^n$ is bounded in the reflexive space $L_2(I;V)$ and this implies the existence of subsequences of $\bar{u}^n$ (we denote it by $\bar{u}^n$ again) such that

$$\bar{u}^n \to u,\quad \text{in} L_2(I;V) \tag{3.23}$$

The estimate (3.14)2 implies

$$\int_Q \|u^n-\bar{u}^n\|^2 \le C\tau \tag{3.24}$$

Hence, $\{\bar{n}^n\}$ is compact in $L_2(I;L_2(\Omega))$ and also pointwise in Q. because of Kolmogorov's compactness argument (see [15]). So we can conclude that

$$u^n \xrightarrow{w} u,\quad \text{in} L_2(I;V), \tag{3.25}$$

$$u^n \longrightarrow u,\ \text{in } L_2(I;L_2(\Omega)), \tag{3.26}$$

For each $t \in I$, by lemma 3.1, $\partial_t u^n$ is uniformly bounded in the reflexive Banach space $L_2(\Omega)$ and hence has a subsequence which converges weakly to an element $\Im \in L_2(\Omega)$ (Eberlin-Smulian theorem [15]). Thus

$$\partial_t u^n \overset{w}{\to} \Im \ \text{in } L_2(\Omega)\,\forall t \in I \tag{3.27}$$

Using Fubini theorem, we get

$$(u^n-u_0,x)=\int_0^t(\partial_t u^n,x)dt \tag{3.28}$$

Taking the limit as $n \to \infty$, we obtain

$$(u-u_0,x)=\int_0^t(\Im,x)\,dt, \tag{3.29}$$

which implies

$$\left(u-u_0-\int_0^t\Im(t)\,dt,x\right)=0 \tag{3.30}$$

Therefore, we have $\partial_t u = \Im$. Due to Arzela-Ascoli theorem, the convergence

$$u^n \to u,\quad \text{in } L_2(I;L_2(\Omega)), \tag{3.31}$$

and the estimate

$$\int_I \|\partial_t u^n\|^2+\int_I\|\partial_t u\|^2 \le C, \tag{3.32}$$

mean that there is a subsequence for which

$$\bar{u}^n \to u,\quad \text{in } C(I;L_2(\Omega)), \tag{3.33}$$

Now, we have to prove that u is the solution of the problem (P).

On integrating (3.8) over the interval $(0,t^*)$ and then pass to the limit as $n \to \infty$. We will demonstrate separately this on each term of the equation

$$\int_0^{t^*}(\partial_t u^n,\varphi)dt+\int_0^{t^*}(\partial_x\bar{u}^n,\partial_x\varphi)dt+\int_0^{t^*}(\gamma\partial_x\bar{u}^n_t,\varphi)dt = \int_0^{t^*}(\bar{f}^n,\varphi)dt \tag{3.34}$$

as $n \to \infty$, (3.21) implies that

$$\int_0^{t^*}(\partial_t u^n,\varphi)dt \to \int_0^{t^*}(\partial_t u,\varphi)\,dt \tag{3.35}$$

For the second and third terms of (3.34),

$$II=\int_0^{t^*}(\partial_x u^n,\partial_x\varphi)dt+\int_0^{t^*}(\partial_x(u^n-\bar{u}^n),\partial_x\varphi)dt=(II)_1+(II)_2, \tag{3.36}$$

$$|(II)_2|\le C\|u^n-\bar{u}^n\|\|\varphi\|\le C\|\varphi\|\sqrt{\tau}$$
(3.37)

Thus, (3.20) gives

$$\int_0^{t^*}(\partial_x\bar{u}^n,\partial_x\varphi)dt\to\int_0^{t^*}(\partial_x u,\partial_x\varphi)dt \quad \text{as } n\to\infty \tag{3.38}$$

Similarly, we conclude that

$$\int_0^{t^*}(\gamma\partial_x\bar{u}_\tau^n,\varphi)dt\to\int_0^{t^*}(\gamma\partial_x u,\varphi)dt \text{ as } n\to\infty \tag{3.39}$$

For the right-hand side, we write

$$\left(\bar{f}^n,\varphi\right)=\left(f(\bar{t}^n,\bar{u}_\tau^n)-f(\bar{t}^n,\bar{u}^n)+f(\bar{t}^n,\bar{u}^n)-f(\bar{t}^n,u^n)+f(\bar{t}^n,u^n),\varphi\right) \tag{3.40}$$

$$|f(\bar{t}^n,\bar{u}_\tau^n)-f(\bar{t}^n,\bar{u}^n)|\le C\|\bar{u}^n-\bar{u}_\tau^n\|\|\varphi\|\le C\tau\|\varphi\| \tag{3.41}$$

$$|f(\bar{t}^n,\bar{u}^n)-f(\bar{t}^n,u^n)|\le C\|u^n-\bar{u}^n\|\|\varphi\|\le C\tau\|\varphi\| \tag{3.42}$$

$$(f(\bar{t}^n,u^n),\varphi)\to(f,\varphi) \quad \text{as n}\to\infty \tag{3.43}$$

Thus we have proved

$$\int_0^{t^*}\left(\bar{f}^n,\varphi\right)dt\to\int_0^{t^*}(f,\varphi)\,dt \quad \text{as n}\to\infty \tag{3.44}$$

Therefore, on passing to the limit as $n\to\infty$, and differentiate both sides of the result with respect to t^*, we see that u is the solution of problem (P).

Subtracting (3.8) from (2.6) and using $\varphi=e_u(t)=u(t)-u^n(t)$ as a test function we obtain

$$\frac{1}{2}\frac{d}{dt}|e_u|^2+\|e_u\|^2\le\varepsilon|e_f|^2+C_\varepsilon|e_u|^2+\eta\|e_u\|^2+C_\eta\|u^n-\bar{u}^n\|^2, \tag{3.45}$$

where $e_f(t)=f(t,u)-\bar{f}_\tau^n$. Now we consider

$$|e_f|^2\le\left|f(t,u)-f(t,u^n)\right|^2+\left|f(t,u^n)-f(t,\bar{u}_\tau^n)\right|^2+\left|f(t,\bar{u}_\tau^n)-f(t_i,\bar{u}_\tau^n)\right|^2$$
$$\le|e_u|^2+|u^n-\bar{u}_\tau^n|^2+\tau^2 \tag{3.46}$$

On integrating (3.45) over I, substituting from lemma 3.1, choosing ε,η sufficiently small and applying Gronwall's lemma the proof completes.

4 Wavelet

4.1Bernoulli Wavelet

Dilation and translation of a function (mother wavelet) construct a family of functions called wavelets and is defined as follows [16-19]

$$\varphi_{a,b}(t)=|a|^{\frac{-1}{2}}\varphi\left(\frac{t-b}{a}\right), \qquad a\neq0, \tag{4.1.1}$$

where $a,b\in R$ are dilation and translation parameters that vary continuously. The family of discrete wavelets that form a basis for $L^2(R)$ is defined as follows

$$\varphi_{k,n}(t)=|a_0|^{\frac{k}{2}}\varphi(a_0^k t-nb_0) \tag{4.1.2}$$

where n and k are positive integers and $a_0>1, b_0>1$. When we choose Bernoulli polynomial as a mother function we get Bernoulli wavelet. Bernoulli wavelets are $\varphi_{n,m}(t)=\varphi(k,\hat{n},m,t)$ have four arguments; $\hat{n}=n-1, n=1,2,3,\dots,2^{k-1}, k$ is any positive integer, m is the order of Bernoulli polynomials and t is the normalized time. The definition of Bernoulli wavelets [20, 21] in the interval [0,1) is given by

$$\varphi_{n,m}(t)==\begin{cases}2^{\frac{k-1}{2}}\tilde{B}_m(2^{k-1}t-\hat{n}), & \text{for}\frac{\hat{n}}{2^{k-1}}\le t<\frac{\hat{n}+1}{2^{k-1}}\\ 0, & \text{otherwise,}\end{cases} \tag{4.1.3}$$

where

$$\tilde{B}_m(t)=\begin{cases}1, & \text{for} m=0,\\ \frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}B_m(t), & \text{for} m>0\end{cases} \tag{4.1.4}$$

where $m=0,1,\dots,M-1$ order of the Bernoulli polynomial for a constant positive integer M. The coefficient $\frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}$ is used for orthonormality, and

The Bernoulli polynomial is defined over the interval [0,1] is given by

$$B_m(t)=\sum_{i=0}^{m}\binom{m}{i}\alpha_{m-i}\,t^i \tag{4.1.5}$$

where the relation of Bernoulli numbers $\alpha_i, i=0,1,\dots,m$ is given by

$$\frac{t}{e^t-1}=\sum_{i=0}^{\infty}\alpha_i\frac{t^i}{i!} \tag{4.1.6}$$

Hence we can see the first seven Bernoulli numbers as

$$\alpha_0=1,\alpha_1=\frac{-1}{2},\alpha_2=\frac{1}{6},\alpha_3=0,\dots \tag{4.1.7}$$

and hence the six-four Bernoulli polynomials can found from equation (4.1.5) as

$$B_0(t)=1, \qquad B_1(t)=t-\frac{1}{2}, \qquad B_2(t)=t^2-\mathrm{t}+\frac{1}{6},\dots \tag{4.1.8}$$

4.2Function Approximation

Bernoulli wavelet form an orthonormal basis for [0,1), so we can approximate a function $y(t)$ which is defined on [0,1) as

$$y(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}a_{n,m}\varphi_{n,m}(t), \tag{4.2.1}$$

where

$$a_{n,m}=\left(y(t),\varphi_{n,m}(t)\right) \tag{4.2.2}$$

in which $(\cdot,\cdot)$ denotes the inner product. The truncated form of the series is

$$y(x)\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}a_{n,m}\varphi_{n,m}(x)=A\emptyset(x) \tag{4.2.3}$$

where A and $\emptyset$ are $2^{k-1}M \times 1$ matrices given by

$$A = [a_{1,0}, \dots, a_{1,M-1}, a_{2,0}, \dots, a_{2,M-1}, \dots, a_{2^{k-1},0}, \dots, a_{2^{k-1},M-1}], \quad (4.2.4)$$

$$\emptyset = [\varphi_{1,0}, \dots, \varphi_{1,M-1}, \varphi_{2,0}, \dots, \varphi_{2,M-1}, \dots, \varphi_{2^{k-1},0}, \dots, \varphi_{2^{k-1},M-1}]^T. \quad (4.2.5)$$

4.3 Method of Solution

The 2-point Euler backward differentiation formula is used to approximate $u_t(x,t)$, given in equation (1.1), at the time-level t_i for $i = 1,2, \dots, l$. Therefore, we have

$$\frac{u^i(x)-u^{i-1}(x)}{\tau} - \left(u''\right)^i(x) + \gamma\left(u'\right)^{i-1}(x) = f^i(x) \quad (4.3.1)$$

where $f^i(x) = f(x,t_i)$ and $u^i(x) = u(x,t_i)$. Equivalently, we can rewrite equation (4.3.1) as

$$u^i(x) - \tau\left(u''\right)^i(x) = F^i(x), \quad (4.3.2)$$

where

$$F^i(x) = \tau f^i(x) + (u^*)^{i-1}(x). \quad (4.3.3)$$

$$(u^*)^{i-1}(x) = (\tilde{u})^{i-1}(x - \tau\gamma) \quad (4.3.4)$$

Then we approximate the unknown function $u^i(x)$ with help of Bernoulli wavelet as

$$u^i(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c^i_{n,m}\varphi_{n,m}(x) = C^i\,\emptyset(x). \quad (4.3.5)$$

Where C^i and $\emptyset$ are $2^{k-1}M \times 1$ matrices given by equation

$$C^i = \left[c^i_{1,0}, \dots, c^i_{1,M-1}, c^i_{2,0}, \dots, c^i_{2,M-1}, \dots, c^i_{2^{k-1},M-1}\right], \quad (4.3.6)$$

$$\varphi = \left[\varphi_{1,0}, \dots, \varphi_{1,M-1}, \varphi_{2,0}, \dots, \varphi_{2,M-1}, \dots, \varphi_{2^{k-1},M-1}\right], \quad (4.3.7)$$

then we can rewrite equation (4.3.2) as

$$(C^i)^T\emptyset(x) - \tau(C^i)^T\emptyset''(x) = F^i(x), \quad (4.3.8)$$

subject to the boundary conditions

$$(C^i)^T\emptyset(a) = g_1(t_i), \quad (C^i)^T\emptyset(b) = g_2(t_i) \quad (4.3.9)$$

Now collocating equation (4.3.8) at points

$$, \quad (4.3.10)\, x_i = \frac{i-0.5}{2^k M}, i = 1,2, \dots, 2^{k-1}M - 2$$

Using the collocation points in equation (4.3.10), we get

$$(C^i)^T\emptyset(x_j) - \tau(C^i)^T\emptyset''(x_j) = F^i(x_j) \quad (4.3.11)$$

From equation (4.3.5) there are $2^{k-1}M$ unknown constants. To find out the values of these unknown, we need $2^{k-1}M$ equation out of which 2 equations are obtained from boundary conditions (4.3.11) which the remaining $2^{k-1}M - 2$ equations are obtained from (4.3.11). By solving this system of equations given the unknown constants.

5 Illustrative Examples

In this section, we solve partial differential equations with boundary conditions to test the efficiency of our method. We first develop a system of the algebraic equation from boundary conditions and with the help of collocation points. We use the MATLAB program to obtain the solution.

Example 1:

Consider the following partial differential equation

$$u_t(x,t) - u_{xx}(x,t) + \gamma u_x(x,t) = e^{xt}(x - t^2 + t) + 2x - 2, \quad in \quad Q \equiv (0,1)\times(0,T), \quad (5.1)$$

with the boundary conditions

$$u(0,t) = 6, \qquad u(1,t) = e^1 + 6 \quad (5.2)$$

the exact solution of equation (5.1) is as follows

$$u(x,t) = e^{xt} + x^2 + 5 \quad (5.3)$$

Table 5.1. Comparison between exact and numerical solution.

x	$t = 0.1, \gamma = 1, M = 4,$ and $k = 1$		
	Exact solution	numerical solution	Error
0.0	6.000000	6.000000	0.000000E+00
0.1	6.020050	6.020050	5.199959E-08
0.2	6.060201	6.060201	2.485691E-08
0.3	6.120455	6.120455	3.470314E-07
0.4	6.200811	6.200812	8.197794E-07
0.5	6.301271	6.301272	1.338052E-06
0.6	6.421837	6.421838	1.786392E-06
0.7	6.562508	6.562510	2.038828E-06
0.8	6.723287	6.723289	1.958773E-06
0.9	6.904174	6.904176	1.398912E-06
1.0	7.105171	7.105171	0.000000E+00

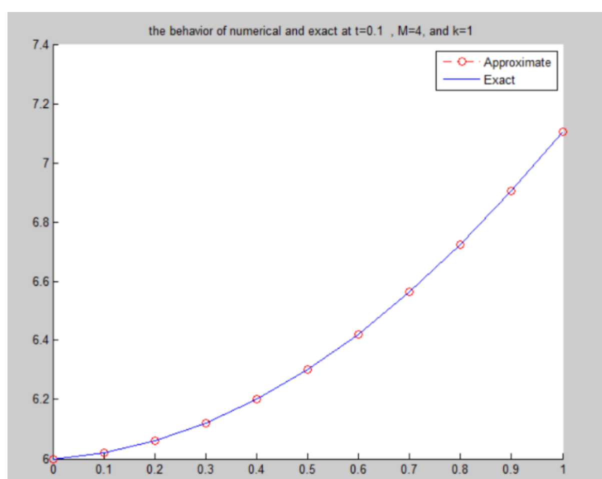

Figure 5.1: Exact and numerical solution

Example 2:

Consider the following partial differential equation

$$u_t(x,t) - u_{xx}(x,t) + \gamma u_x(x,t) = x^2 - 2t + 2xt, \quad (5.4)$$

with the boundary conditions

$$u(0,t) = 3, \qquad u(1,t) = t + 3 \tag{5.5}$$

the exact solution of equation (5.4) is as follows

$$u(x,t) = x^2 t + 3 \tag{5.6}$$

Table 5.2. Comparison between exact and numerical solution.

x	$t = 0.1, \gamma = 1, M = 4, \text{and} k = 1$		
	Exact solution	numerical solution	Error
0.0	3.000000	3.000000	0.000000E+00
0.1	3.010000	3.010000	5.048582E-09
0.2	3.040000	3.040000	2.355715E-09
0.3	3.090000	3.090000	3.213047E-08
0.4	3.160000	3.160000	7.419323E-08
0.5	3.250000	3.250000	1.184615E-07
0.6	3.360000	3.360000	1.548529E-07
0.7	3.490000	3.490000	1.732849E-07
0.8	3.640000	3.640000	1.636750E-07
0.9	3.810000	3.810000	1.159409E-07
1.0	4.000000	4.000000	0.000000E+00

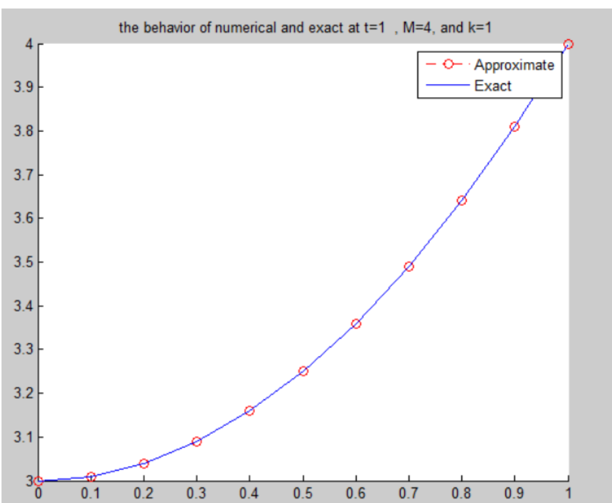

Figure 5.2: Exact and numerical solution

From the numerical results in Tables and figures 5.1 and 5.2, it is easy to conclude that the obtained results by the proposed method are in perfect concord with the exact solution.

6 Conclusions

The present work aims to improve an efficient and accurate method for solving the partial differential equation. The problem has been reduced to solving a system of linear algebraic equations. Two test problems were presented to demonstrate the validity of the technique.

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