Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/040406

New Analytical Solutions and Approximate Solution of the Space-Time Conformable Sharma-Tasso-Olver Equation

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Received: 14 Mar. 2017, Revised: 1 Jun. 2017, Accepted: 12 Jun. 2017
Published online: 1 Oct. 2018

Abstract: The main purpose of this article is to find the exact and approximate solutions of space-time conformable Sharma-Tasso-Olver equation using first integral method (FIM) and q-homotopy analysis method (q-HAM) respectively. The obtained exact and numerical solutions are compared with each other. Also, the numerical results obtained by q-HAM are compatible with the exact solutions obtained by FIM; hence, it is clearly seen that these techniques are powerful and efficient in finding approximate and exact solutions for nonlinear conformable partial differential equations.

Keywords: First integral method, Sharma-Tasso-Olver equation, q-homotopy analysis method, conformable derivative.

1 Introduction

The fractional differential equations and its solutions prevail in different branches of applied sciences such as engineering, applied mathematics, biology and physics [1,2,3,4,5,6]. So, mathematical viewpoints of fractional differential equations and the methods for their solutions have been discussed by many authors [7,8,9,10]. As a result of these discussions, it is deduced that fractional order differential equations require effective solution methods. Because of this prerequisite, scientists study on new effective methods or new definitions of fractional derivative which can be more applicable to real-life problems and to known methods. Up to now, many scientists have paid great attention to this new definition [12,13,14,15,16] which is presented as follows:

Definition 1. *Let $f : [0,\infty) \to \mathbb{R}$ be a function. The α^{th} order "conformable derivative" of f is defined by,*

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0, \alpha \in (0,1)$.

If f is α-differentiable in some $(0,a), a > 0$ and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists then define $f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t)$. The *"conformable integral"* of a function f starting from $a \geq 0$ is defined as:

$$I_\alpha^a(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$$

, where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$. The following properties of conformable derivative are given in [11].

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Theorem 1*Let* $\alpha \in (0,1]$ *and suppose* f,g *are* α*-differentiable at point* $t>0$. *Then*

1.$T_\alpha(cf+dg) = cT_\alpha(f)+cT_\alpha(g)$ *for all* $a,b\in\mathbb{R}$.
2.$T_\alpha(t^p) = pt^{p-\alpha}$ *for all* $p\in\mathbb{R}$.
3.$T_\alpha(\lambda) = 0$ *for all constant functions* $f(t)=\lambda$.
4.$T_\alpha(fg) = fT_\alpha(g)+gT_\alpha(f)$.
5.$T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.
6.*If , in addition* f *is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

Based on the practicability and easy applicability of this definition, considerable amount of studies have been made on this subject. For example; T. Abdeljawad [17] has presented fractional versions of the chain rule, exponential functions, Gronwalls inequality, integration by parts, Taylor power series expansions and Laplace transform. Conformable time-scale fractional calculus has been introduced by N. Benkhettoua *et al.* [18]. In addition to this, M.A. Hammad and R. Khalil [19] expressed the solution for the conformable heat equation and W.S. Chung [20] used the conformable derivative and integral to discuss fractional Newtonian mechanics. Also, the extended mean-value theorem and the Racetrack type principle are proven for the class of functions which are α-differentiable in the context of conformable derivatives and integral with application using D'Alambert approach by O.S. Iyiola [12]. Hence, it is clearly deduced that further studies and explanations can be made on this new subject area.

In recent years, scientists have applied many methods to solve nonlinear fractional PDEs such as quadratic spline collocation method [21], sub-equation method [22], homotopy analysis method [23], etc. Among these, a method which is based on the ring theory of commutative algebra called the first integral method is explained by Feng [24]. This method has been applied to different type of equations in various fields by many authors [25,26]. For this reason, in this article, the first integral method and the new fractional derivative are gathered to present analytic solution of space-time Sharma-Tasso-Olver (STO) equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + 3\beta\left(\frac{\partial^\sigma u}{\partial x^\sigma}\right)^2 + 3\beta u^2\frac{\partial^\sigma u}{\partial x^\sigma} + 3\beta u\frac{\partial^{2\sigma} u}{\partial x^{2\sigma}} + \beta\frac{\partial^{3\sigma} u}{\partial x^{3\sigma}} = 0, \tag{1}$$

where $t>0, 0<\alpha,\sigma<1$ and β is arbitrary constant.

Thereafter, the approximate analytical solution of the time conformable Sharma-Tasso-Olver equation is determined by using q-homotopy analysis method, a modified version of homotopy analysis method see [27,28,29,30]. This solution involves an auxiliary parameter h, which is also determined, and a fraction factor $\frac{1}{n}$. We show by comparison, that this method is powerful and efficient in finding the numerical solution of the time conformable Sharma-Tasso-Olver equation. The numerical solution of Sharma-Tasso-Olver equation with the analytical solution obtained by using the FIM are compared.

The rest of this article is organized as follows; in section (2), the analytical solution of the time conformable Sharma-Tasso-Olver equation using FIM is presented, in section (3), q-HAM is applied to obtain series solution of the equation involved. Numerical comparisons are made between the solutions obtained by FIM and q-HAM in section (4) and in section (5), we give some conclusion.

2 Analytical Solution of Space-time STO Equation by FIM

In this section, we give brief description of first integral method first and then apply it to solve the space-time Sharma-Tasso-Olver equation.

2.1 Fundamentals of FIM

We summarize the method of first integral applied to conformable partial differential equation in four steps.

Step1. Regard the nonlinear conformable partial differential equation in the form

$$P\left(\frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial^\sigma u}{\partial x^\sigma}, \frac{\partial u}{\partial y}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \dots\right) = 0, \tag{2}$$

where P is a polynomial and in its arguments and subscripts denote partial derivatives. **Step2.** Considering the transformation

$$u(x,t) = u(\xi), \xi = \frac{x^\sigma}{\sigma} - c\frac{t^\alpha}{\alpha} \tag{3}$$

in which c is a constant can be determined later. Based on this:

$$\frac{\partial^\alpha(.)}{\partial t^\alpha}=-c\frac{d(.)}{d\xi},\frac{\partial(.)}{\partial x}=\frac{d(.)}{d\xi},\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\alpha(.)}{\partial t^\alpha}\right)=c^2\frac{d(.)}{d\xi},\dots. \tag{4}$$

Using Eq.(3), Eq. (2) turns into a nonlinear ordinary differential equation

$$G(U,U',U'',U''',\dots)=0, \tag{5}$$

where the derivatives is with respect to ξ.
Step3. Now, we introduce new independent variables

$$X(\xi)=U(\xi),Y(\xi)=U_\xi(\xi) \tag{6}$$

which results in the system of nonlinear ordinary differential equations (ODE)

$$\frac{\partial X}{\partial\xi}=Y(\xi),\frac{\partial Y}{\partial\xi}=S(X(\xi),Y(\xi)). \tag{7}$$

Step4. If the integrals to (7) could be found under the same conditions by the qualitative theory of ordinary differential equations [32], then the general solutions to (7) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We apply the Division Theorem to obtain one first integral to (7), which reduces (4) to a first-order integrable ordinary differential equation. An exact solution to (2) is then obtained by solving this equation. Now, let us recall the Division Theorem:

Lemma 1._(Division Theorem)[24] Suppose that $P(x,y)$ and $Q(x,y)$ are polynomials in $\mathbb{C}[x,y]$ and $P(x,y)$ is irreducible in $\mathbb{C}[x,y]$. If $Q(x,y)$ vanishes at all zero points of $P(x,y)$, then there exists a polynomial $G(x,y)$ in $\mathbb{C}[x,y]$ such that_

$$Q(x,y)=P(x,y)G(x,y).$$

2.2 Implementation of the FIM on Space-time STO Equation

Consider the space-time fractional Sharma-Tasso-Olver equation (1) where $\alpha\in(0,1)$ and $\frac{\partial^\alpha u}{\partial t^\alpha}$ means conformable derivative of function $u(x,t)$. Using (3) and (4) and integrating obtained equation once, the Eq.(1) becomes

$$-cu+3\beta uu'+\beta u^3+\beta u''=0, \tag{8}$$

where the prime denotes the derivation with respect to ξ. Then choosing new variables using (6), we get

$$\begin{aligned}X_\xi(\xi)&=Y(\xi),\\ Y_\xi(\xi)&=\frac{c}{\beta}X(\xi)-3X(\xi)Y(\xi)-(X(\xi))^3.\end{aligned} \tag{9}$$

Now, we employ the division theorem to find the first integral of (9). We assume that $X(\xi)=X,Y(\xi)=Y$ are nontrivial solutions of (9) and $P(X,Y)=\sum_{i=0}^{m}a_i(X)Y^i$ is irreducible polynomial in the complex domain $\mathbb{C}[X,Y]$ such that

$$P(X(\xi),Y(\xi))=\sum_{i=0}^{m}a_i(X)Y^i=0, \tag{10}$$

where $a_i(X)(i=0,1,\dots,m)$ are polynomials of X and $a_m\neq0$. Eq.(10) is called the first integral of Eq. (9). Based on the division theorem, there exists a polynomial $g(X)+h(X)Y$ in $\mathbb{C}[X,Y]$ such that

$$\frac{\partial P}{\partial\xi}=\frac{\partial P}{\partial X}\frac{\partial X}{\partial\xi}+\frac{\partial P}{\partial Y}\frac{\partial Y}{\partial\xi}=(g(X)+h(X)Y)\left(\sum_{i=0}^{m}a_i(X)Y^i\right). \tag{11}$$

Case A: Assume that $m = 1$ in (10. When the coefficients of $Y^i (i = 0, 1, 2)$ in both sides of (11) are compared, we achieve

$$a_1'(X) = a_1(X)h(X), \tag{12}$$

$$a_0'(X) = g(X)a_1(X) + a_0(X)h(X) - 3Xa_1(X), \tag{13}$$

$$g(X)a_0(X) = \frac{c}{\beta}Xa_1(X) - X^3 a_1(X). \tag{14}$$

Due to $a_1(X)$ is a polynomial of X, we determine that $a_1(X)$ is a constant and $h(X) = 0$ from (12). For convenience, let's take $a_1(X) = 1$. Balancing the degrees of $a_0(X)$ and $g(X)$, it is comprehended that $degg(X) = 1$. Because, if the other choices are evaluated, it is clearly seen that degrees of the polynomials in two sides of Eq. (14) are not equal to each other. As a result of our evaluations, we can take $g(X) = AX + B$. By using $g(X)$ in Eq. (13) we handle,

$$a_0(X) = \frac{3+A}{2}X^2 + BX + \gamma,$$

where γ is integration constant. Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ in (14) and equaling all the coefficients of powers of X to zero, then a system of nonlinear algebraic equations is found out. Solving the algebraic equations with aid the Mathematica, we obtain the following set of solutions

$$A = -1; B = 0; c = -\gamma\beta, \tag{15}$$

$$A = -2; B = 0; c = -2\gamma\beta, \tag{16}$$

$$A = -1; B = \pm\sqrt{\frac{c}{\beta}}; \gamma = 0. \tag{17}$$

Setting Eq.(15) in Eq.(10), we obtain

$$Y(\xi) = -X(\xi)^2 - \gamma. \tag{18}$$

Combining (18) with (9), we obtain the exact solution to (1) as

$$u_1(x,t) = -\sqrt{\gamma}\tan\left(\left(\frac{x^\sigma}{\sigma} - \frac{ct^\alpha}{\alpha}\right)\sqrt{\gamma} - \sqrt{\gamma}\varepsilon\right).$$

Substituting Eq.(16) in Eq.(10),

$$Y(\xi) = \frac{1}{2}(-X(\xi)^2 - 2\gamma). \tag{19}$$

Similarly combining (19) with (9), we obtain the exact solution of Eq.(1) as following

$$u_2(x,t) = \sqrt{2\gamma}\tan\left(\frac{1}{2}\left(-\sqrt{2\gamma}\left(\frac{x^\sigma}{\sigma} - \frac{ct^\alpha}{\alpha}\right) + 2\varepsilon\sqrt{2\gamma}\right)\right).$$

The other solution (which can be obtained in a similar way) for Eq. (17) is following.
For $B = \sqrt{\frac{c}{\beta}}$ in (17)

$$u_3(x,t) = -\frac{\sqrt{c}\left(\cosh(\sqrt{c}\varepsilon) + \sinh(\sqrt{c}\varepsilon)\right)}{-\cosh\left(\frac{\sqrt{c}\left(\frac{x^\sigma}{\sigma} - \frac{ct^\alpha}{\alpha}\right)}{\sqrt{\beta}}\right) + \sqrt{\beta}\cosh(\sqrt{c}\varepsilon) - \sinh\left(\frac{\sqrt{c}\left(\frac{x^\sigma}{\sigma} - \frac{ct^\alpha}{\alpha}\right)}{\sqrt{\beta}}\right) + \sqrt{\beta}\sinh(\varepsilon\sqrt{c})}.$$

For $B = -\sqrt{\frac{c}{\beta}}$ in (17)

$$u_4(x,t) = \frac{\sqrt{c}\left(\cosh\left(\frac{\sqrt{c}\left(\frac{x^\sigma}{\sigma} - \frac{ct^\alpha}{\alpha}\right)}{\sqrt{\beta}}\right) - \sinh\left(\frac{\sqrt{c}\left(\frac{x^\sigma}{\sigma} - \frac{ct^\alpha}{\alpha}\right)}{\sqrt{\beta}}\right)\right)}{\sqrt{\beta}\cosh\left(\frac{\sqrt{c}\left(\frac{x^\sigma}{\sigma} - \frac{ct^\alpha}{\alpha}\right)}{\sqrt{\beta}}\right) - \cosh(\sqrt{c}\varepsilon) + \sqrt{\beta}\sinh\left(\frac{\sqrt{c}\left(\frac{x^\sigma}{\sigma} - \frac{ct^\alpha}{\alpha}\right)}{\sqrt{\beta}}\right) - \sinh(\sqrt{c}\varepsilon)},$$

where ε is integration constant.

Case B: Suppose that $m=2$. Comparing the coefficients of $Y^i (i=0,1,2,3)$ in both sides of (11), we handle

$$a_2'(X) = a_2(X)h(X), \tag{20}$$

$$a_1'(X) = 6Xa_2(X) + g(X)a_2(X) + h(X)a_1(X), \tag{21}$$

$$a_0'(X) = 3Xa_1(X) - 2\frac{c}{\beta}a_2(X) + 2X^3 a_2(X) + g(X)a_1(X) + h(X)a_0(X), \tag{22}$$

$$g(X)a_0(X) = \frac{c}{\beta}Xa_1(X) - X^3 a_1(X). \tag{23}$$

Since $a_2(X)$ is a polynomial of X, it is determined that $a_2(X)$ is a constant and $h(X)=0$ from (20). For convenience, lets take $a_2(X)=1$. Balancing the degrees of $a_0(X)$ and $g(X)$, it is realized that $degg(X)=1$. If the other choices evaluated for degree of $g(x)$, it is seen that degrees of the polynomials in two side of Eq. (23) are not equal to each other. As a result of our evaluations, we can take $g(X)=AX+B$. Regarding $g(X)$ in Eq. (21) we handle,

$$a_1(X) = \left(\frac{6+A}{2}\right)X^2 + BX + \gamma$$

and

$$a_0(X) = \left(\frac{11}{4} + \frac{9A}{8} + \frac{A^2}{8}\right)X^4 + \left(\frac{AB}{2} + 2B\right)X^3 + \left(\frac{B^2}{2} - \frac{c}{\beta} + \frac{3\gamma}{2} + \frac{A\gamma}{2}\right)X^2 + BX\gamma + r,$$

where γ, r are integration constants. Substituting $a_0(X)$, $a_1(X)$, $a_2(X)$ and $g(X)$ in (23) and setting all the coefficients of powers of X to zero, then a system of nonlinear algebraic equations is obtained. Solving the algebraic equations by the help of Mathematica, we get the following set of solutions

$$A=-2; B=0; \gamma=0; r=0, \tag{24}$$

$$A=-2; c=\frac{B^2\beta}{4}; \gamma=0; r=0, \tag{25}$$

$$A=-3; c=B^2\beta; \gamma=-\frac{B^2}{2}; r=0, \tag{26}$$

$$A=-2; c=B^2\beta; \gamma=-B^2; r=0, \tag{27}$$

$$A=-4; B=0; c=-\beta\gamma; r=\frac{\gamma^2}{4}, \tag{28}$$

$$A=-3; B=0; c=\frac{-2\beta\gamma}{3}; r=\frac{2\gamma^2}{9}, \tag{29}$$

$$A=-2; B=0; c=-\frac{\beta\gamma}{3}; r=\frac{\gamma^2}{6}. \tag{30}$$

Substituting Eq.(24) in Eq.(10), we obtain

$$Y_1(X) = \frac{-\sqrt{c}X - X^2\sqrt{\beta}}{\sqrt{\beta}} \tag{31}$$

and

$$Y_2(X) = \frac{\sqrt{c}X - X^2\sqrt{\beta}}{\sqrt{\beta}}. \tag{32}$$

Combining (31) and (32) with (9), we obtain the exact solutions to (1) successively as

$$u_5(x,t) = -\frac{\sqrt{c}(\cosh(\sqrt{c}\varepsilon)+\sinh(\sqrt{c}\varepsilon))}{-\cosh\left(\frac{\sqrt{c}\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)}{\sqrt{\beta}}\right)+\sqrt{\beta}\cosh(\sqrt{c}\varepsilon)-\sinh\left(\frac{\sqrt{c}\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)}{\sqrt{\beta}}\right)+\sqrt{\beta}\sinh(\sqrt{c}\varepsilon)}.$$

The other solution for (24) using (20) can be obtained in a similar way as

$$u_6(x,t) = \frac{\sqrt{c}\left(\cosh\left(\frac{\sqrt{c}\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)}{\sqrt{\beta}}\right)+\sinh\left(\frac{\sqrt{c}\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)}{\sqrt{\beta}}\right)\right)}{\sqrt{\beta}\cosh\left(\frac{\sqrt{c}\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)}{\sqrt{\beta}}\right)-\cosh(\sqrt{c}\varepsilon)+\sqrt{\beta}\sinh\left(\frac{\sqrt{c}\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)}{\sqrt{\beta}}\right)-\sinh(\sqrt{c}\varepsilon)}.$$

The other solutions (which can be obtained in the same manner) for Eq. (25)-(30) as following.
For (25), exact traveling wave solutions can be obtained as

$$u_7(x,t) = -\frac{B(\cosh(B\varepsilon)+\sinh(B\varepsilon))}{-\cosh\left(\frac{B\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)}{2}\right)+2\cosh(B\varepsilon)-\sinh\left(\frac{B\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)}{2}\right)+2\sinh(B\varepsilon)}.$$

For (26), exact solutions of Eq.(1) is evaluated as

$$u_8(x,t) = B\tanh\left(\frac{B}{2}\left(\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)+2\varepsilon\right)\right) \tag{33}$$

and

$$u_9(x,t) = -\frac{B(\cosh(B\varepsilon)+\sinh(B\varepsilon))}{-\cosh\left(B\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)\right)+\cosh(B\varepsilon)-\sinh\left(B\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)\right)+\sinh(B\varepsilon)}.$$

For (27) traveling wave solutions of Eq.(1) can be obtained as following

$$u_{10}(x,t) = B\tanh\left(B\left(\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)+\varepsilon\right)\right)$$

and

$$u_{11}(x,t) = -\frac{B(\cosh(B\varepsilon)+\sinh(B\varepsilon))}{-\cosh\left(B\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)\right)+\cosh(B\varepsilon)-\sinh\left(B\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)\right)+\sinh(B\varepsilon)}.$$

Now for (28) the exact solution to Eq.(1) as follows

$$u_{12}(x,t) = \sqrt{\gamma}\tan\left(\frac{1}{2}\left(-\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)\sqrt{\gamma}+2\sqrt{\gamma}\varepsilon\right)\right).$$

The exact solutions of Eq.(1) for (29) can be obtained

$$u_{13}(x,t) = \sqrt{\frac{2\gamma}{3}}\tan\left(\frac{1}{6}\left(-\sqrt{6\gamma}\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)+6\sqrt{6\gamma}\varepsilon\right)\right)$$

and

$$u_{14}(x,t) = \sqrt{\frac{2\gamma}{3}}\tan\left(\frac{1}{3}\left(-\sqrt{6\gamma}\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)+3\sqrt{6\gamma}\varepsilon\right)\right).$$

The last solutions of Eq.(1) for (30)

$$u_{15}(x,t) = -\frac{\sqrt{\gamma}\sin\left(\frac{\sqrt{\gamma}\left(\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)-6\varepsilon\right)}{\sqrt{(3)}}\right)}{-1+\sqrt{3}\cos\left(\frac{\sqrt{\gamma}\left(\left(\frac{x^\sigma}{\sigma}-\frac{ct^\alpha}{\alpha}\right)-6\varepsilon\right)}{\sqrt{3}}\right)}$$

and

$$u_{16}(x,t) = -\frac{\sqrt{\gamma}\sin\left(\frac{\sqrt{\gamma}\left(\left(\frac{x^{\sigma}}{\sigma}-\frac{ct^{\alpha}}{\alpha}\right)-6\varepsilon\right)}{\sqrt{(3)}}\right)}{1+\sqrt{3}\cos\left(\frac{\sqrt{\gamma}\left(\left(\frac{x^{\sigma}}{\sigma}-\frac{ct^{\alpha}}{\alpha}\right)-6\varepsilon\right)}{\sqrt{3}}\right)},$$

where ε is arbitrary integration constant in all of the solutions.

3 Approximate Analytical Solution of Time-fractional STO Equation by q-HAM

In this section, we give a brief description of the q-homotopy analysis method and then apply it to time-fractional Sharma-Tasso-Olver Equation. We make this particuler choice of parameters for comparison purposes.

3.1 Fundamentals of the q-HAM

Consider the following differential equation

$$\mathscr{N}\left[D_t^{\alpha}u(x,t)\right]-g(x,t)=0, \tag{34}$$

where $\mathscr{N}$ denotes non-linear operator, D_t^{α} is the the conformable derivative, g is a given function and $u(x,t)$ is an unknown function. To generalize the original homotopy method, the zeroth-order deformation equation is constructed as

$$(1-nq)\mathscr{L}\left(\varphi(x,t;q)-u_0(x,t)\right)=qhH(x,t)\left(\mathscr{N}\left[D_t^{\alpha}\varphi(x,t;q)\right]-g(x,t)\right), \tag{35}$$

where $n\geqslant 1$, $q\in\left[0,\frac{1}{n}\right]$ denotes the so-called embedded parameter, $h\neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $H(x,t)$ is a non-zero auxiliary function.

When $q=0$ and $q=\frac{1}{n}$, we have equation (35) to be

$$\varphi(x,t;0)=u_0(x,t) \quad and \quad \varphi\left(x,t;\frac{1}{n}\right)=u(x,t) \tag{36}$$

respectively. So, as q increases from 0 to $\frac{1}{n}$, the solution $\varphi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$.

If $u_0(x,t)$, $\mathscr{L}$, h, $H(x,t)$ are chosen appropriately, solution $\varphi(x,t;q)$ of equation (35) exists for $q\in\left[0,\frac{1}{n}\right]$.

The Taylor series expansion of $\varphi(x,t;q)$ gives

$$\varphi(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)q^m, \tag{37}$$

where

$$u_m(x,t)=\frac{1}{m!}\frac{\partial^m\varphi(x,t;q)}{\partial q^m}\bigg|_{q=0}. \tag{38}$$

Assume that the auxiliary linear operator L, the initial guess c_0, the auxiliary parameter h and $H(x,t)$ are properly chosen such that the series (37) converges at $q=\frac{1}{n}$, then we have

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{39}$$

Let the vector $\mathbf{u}_n$ be defined as follows:

$$\mathbf{u}_n=\left\{u_0(x,t),u_1(x,t),\cdots,u_n(x,t)\right\}. \tag{40}$$

Differentiating equation (35) m-times with respect to the (embedding) parameter q, then evaluating at $q=0$ and finally dividing them by $m!$, we have what is known as the m^{th}-order deformation equation

$$\mathscr{L}\left[u_m(x,t)-\chi_m^{*}u_{m-1}(x,t)\right]=hH(x,t)\mathscr{R}_m\left(\mathbf{u}_{m-1}\right). \tag{41}$$

with initial conditions

$$u_m^{(k)}(x,0)=0, \quad k=0,1,2,...,m-1, \tag{42}$$

where

$$\mathscr{R}_m(\mathbf{u}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}\left(\mathscr{N}[D_t^{\alpha}\varphi(x,t;q)]-g(x,t)\right)}{\partial q^{m-1}}\bigg|_{q=0} \tag{43}$$

and

$$\chi_m^{*}=\begin{cases}0 & m\leqslant 1\\ n & otherwise.\end{cases} \tag{44}$$

3.2 Application of the Method

Consider the time-fractional Sharma-Tasso-Olver equation (1) with $\sigma=1$ and initial condition

$$u(x,0)=\tanh\left(\frac{2+x}{2}\right), \tag{45}$$

where $\alpha\in(0,1)$ and the derivative means conformable derivative. For convenience and to shorten the article, the exact solution is taken as (33) and $B=1$, $\beta=1$, $\varepsilon=1$ are used for all calculations. One can easily make the calculations for all of the other solutions and other values of B, β, c.
To obtain the series solution of Eq. (1) with initial condition (45), the linear operator is chosen as

$$\mathscr{L}[\varphi(x,t;q)]=D_t^{\alpha}\varphi(x,t;q)$$

with the property

$$\mathscr{L}[s]=0,$$

where s is constant. From Eq. (1), the nonlinear operator can be defined as following,

$$\mathscr{N}[\varphi(x,t;q)]=\frac{\partial^{\alpha}\varphi(x,t;q)}{\partial t^{\alpha}}+3\left(\frac{\partial\varphi(x,t;q)}{\partial x}\right)^2+3\varphi(x,t;q)^2\frac{\partial\varphi(x,t;q)}{\partial x}+3\varphi(x,t;q)\frac{\partial^2\varphi(x,t;q)}{\partial x^2}+\frac{\partial^3\varphi(x,t;q)}{\partial x^3}.$$

From Theorem (1), the nonlinear operator can be written as follows,

$$\mathscr{N}[\varphi(x,t;q)]=t^{1-\alpha}\frac{\partial\varphi(x,t;q)}{\partial t}+3\left(\frac{\partial\varphi(x,t;q)}{\partial x}\right)^2+3\varphi(x,t;q)^2\frac{\partial\varphi(x,t;q)}{\partial x}+3\varphi(x,t;q)\frac{\partial^2\varphi(x,t;q)}{\partial x^2}+\frac{\partial^3\varphi(x,t;q)}{\partial x^3}.$$

Thus the zero-order deformation equation is set up as:

$$(1-nq)\mathscr{L}[\varphi(x,t;q)-u_0(x,t)]=qh\mathscr{N}[\varphi(x,t;q)].$$

Choosing $H(x,t)=1$, the mth-order deformation equation is

$$\mathscr{L}[u_m(x,t)-\chi_m^{*}u_{m-1}(x,t)]=hR_m(\mathbf{u}_{m-1}) \tag{46}$$

with initial condition for $m\geq 1$, $u_m(x,0)=0$, χ_m^{*} is as defined in (44) and

$$\begin{aligned}R_m(\mathbf{u}_{m-1})=&\,t^{1-\alpha}\frac{\partial u_{m-1}(x,t)}{\partial t}+3\sum_{n=0}^{m-1}\frac{\partial u_n(x,t)}{\partial x}\frac{\partial u_{m-1-n}(x,t)}{\partial x}+3\sum_{n=0}^{m-1}\left(\sum_{k=0}^{n}u_k(x,t)u_{n-k}(x,t)\right)\frac{\partial u_{m-1-n}(x,t)}{\partial x}\\&+3\sum_{n=0}^{m-1}u_n(x,t)\frac{\partial^2 u_{m-1-n}(x,t)}{\partial x^2}+\frac{\partial^3 u_{m-1}(x,t)}{\partial x^3}.\end{aligned}$$

The solutions of the mth-order deformation Eq. (46) for $m \geq 1$ result in

$$u_m(x,t) = \chi_m^* u_{m-1}(x,t) + h\mathscr{L}^{-1}\left[R_m\left(\mathbf{u}_{m-1}\right)\right]. \tag{47}$$

By using Eq.(47) with initial condition given by (45) we respectively obtain

$$\begin{aligned}
u_0(x,t) &= \tanh\left(\frac{2+x}{2}\right),\\
u_1(x,t) &= \frac{2ht^{\alpha}e^{2+x}}{\alpha(1+e^{2+x})^2},\\
u_2(x,t) &= \frac{2e^{2+x}hnt^{\alpha}}{\left(1+e^{2+x}\right)^2\alpha} - \frac{e^{2+x}h^2t^{\alpha}\left(\left(-1+e^{2+x}\right)t^{\alpha} - 2\left(1+e^{2+x}\right)\alpha\right)}{\left(1+e^{2+x}\right)^3\alpha^2},\\
u_3(x,t) &= \frac{e^{2+x}h^3t^{3\alpha}\left(\left(1+e^{2+x}\right)\left(1+e^{2+x}\left(-4+e^{2+x}\right)\right)\right)}{3\left(1+e^{2+x}\right)^5\alpha^3}\\
&\quad - \frac{e^{2+x}ht^{\alpha}\left(6\left(-1+e^{2+x}\right)h\left(h+2e^{2+x}(h-2n)+n+e^{4+2x}(h+n)\right)t^{\alpha}\alpha + 6\left(1+e^{2+x}\right)^3(h+n)^2\alpha^2\right)}{3\left(1+e^{2+x}\right)^5\alpha^3},\\
&\;\;\vdots
\end{aligned}$$

We can obtain $u_m(x,t)$ for $m = 4,5,6,\cdots$, following the same approach, using Mathematica, Maple or MATLAB.

Then the series solution expression by q-HAM can be written in the form

$$u(x,t,n,h) = \tanh\left(\frac{2+x}{2}\right) + \sum_{n=1}^{\infty} u_i(x,t;n;h)\left(\frac{1}{n}\right)^i. \tag{48}$$

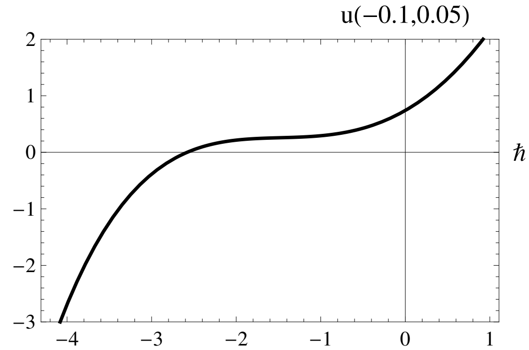
Equation (48) is an appropriate solution to the problem (1) in terms of convergence parameter h and n.

4 Numerical Results and Comparison

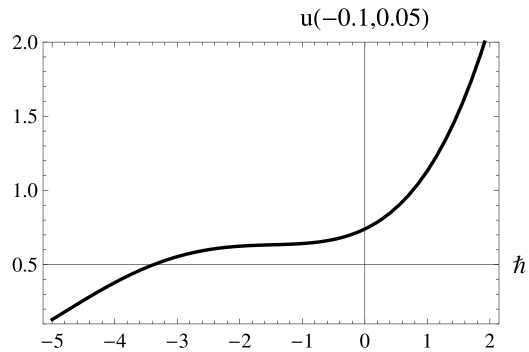
In this section, we discuss some numerical results. How the parameter h can be chosen to get to a good approximation is given. We make some comparisons between the solution obtain by q-HAM and the analytical solution by FIM. In addition, we give the error estimate for different fractional order α. Throughout this section, we use only the U_5 series solution given by q-HAM.

4.1 The $\hbar$-curve

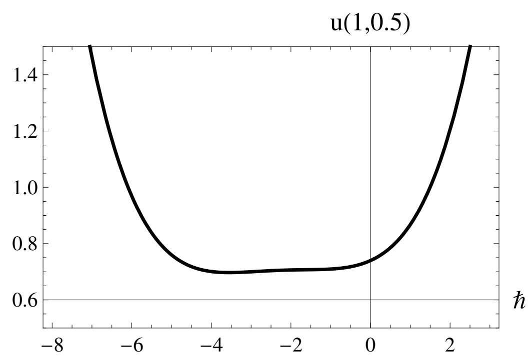
The auxiliary parameter h and n, which are involved in our q-HAM solution series, provide us with a simple way to adjust and control the convergence of the solution series. To obtain a suitable range for $\hbar$, we consider the so-called $\hbar$-curves which are shown in Figure (1). We choose an appropriate value of $\hbar$ which guarantees that the series solution is convergent, as pointed by Liao [31], by finding the valid region of $\hbar$ which corresponds to the line segments nearly parallel to the horizontal axis.



a) $\alpha = 0.25$

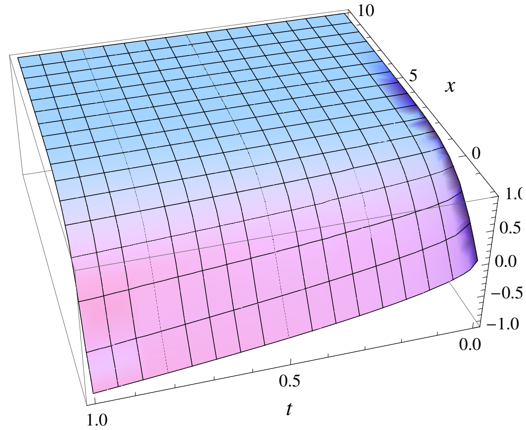


$\alpha = 0.5$

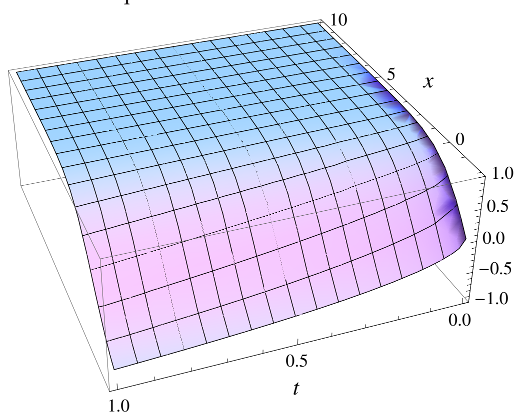


$\alpha = 0.75$

Fig. 1: The $\hbar$-curve of 4th-order q-HAM solution for different values of α and $n = 2$.



a) 4th-order qHAM solution with $\hbar = -2$ and $n = 2$.



b) Analytical solution.

Fig. 2: Comparison of the qHAM Solution and the analytical solution for $\alpha = 0.5$.

4.2 Exact Solution vs Approximate

Effectiveness and efficiency of q-HAM are displayed here by comparing the approximate solution, (U_5), obtained by q-HAM given in Equation (48) of the time conformable fractional Sharma-Tasso-Olver equation with its exact solution (33) for $x = 10$, $0 \le t \le 0.5$, $\alpha = 0.25$, $\hbar = -2$, $n = 2$, $x = 10$, $0 \le t \le 0.5$, $\alpha = 0.5$,i $\hbar = -2$, $n = 2$ and $x = 10$, $0 \le t \le 0.5$, $\alpha = 0.75$, $\hbar = -2$, $n = 2$ shown in Table (1)-(3) respectively for different values of $\hbar$. The graphs of both approximate solution and analytical solution are also shown in Figure (2) using $\alpha = 0.5$, $\hbar = -2$ and $n = 2$.

Table 1: Exact and approximate solutions for $\alpha = 0.25$, $n = 2$ and $\hbar = -2$.

x	t	q-HAM	Exact	Absolute error
10	0.1	0.999893	0.999883	9.09897×10^{-6}
	0.2	0.999846	0.999822	2.38116×10^{-5}
	0.3	0.999805	0.999763	4.22799×10^{-5}
	0.4	0.999768	0.999704	6.39294×10^{-5}
	0.5	0.999733	0.999645	8.84508×10^{-5}

Table 2: Exact and approximate solutions for $\alpha = 0.5$, $n = 2$ and $\hbar = -1.5$.

x	t	q-HAM	Exact	Absolute error
10	0.1	0.999977	0.999977	3.58582×10^{-7}
	0.2	0.999971	0.999970	9.69343×10^{-7}
	0.3	0.999965	0.999963	1.83308×10^{-6}
	0.4	0.999959	0.999956	2.95884×10^{-6}
	0.5	0.999954	0.999949	4.35933×10^{-6}

Table 3: Exact and Approximate solutions for $\alpha = 0.75$, $n = 2$ and $\hbar = -2$.

x	t	q-HAM	Exact	Absolute error
10	0.1	0.999985	0.999984	5.65812×10^{-7}
	0.2	0.999983	0.999982	1.21963×10^{-6}
	0.3	0.999981	0.999979	2.02684×10^{-6}
	0.4	0.999979	0.999976	3.00473×10^{-6}
	0.5	0.999977	0.999973	4.16941×10^{-6}

5 Conclusion

In this paper, we present new exact and approximate solutions of space-time conformable Sharma-Tasso-Olver equation, which are found by using FIM and q-HAM respectively. It has been also shown that the q-HAM solution of the problem converges very rapidly to the exact one which is obtained using FIM by choosing a convenient auxiliary parameter from given tables and figures. Thus, it is deduced that both methods give reliable and effective results for solving conformable nonlinear equations. In this way, we conclude that the applied methods can be used to solve many nonlinear time-fractional partial differential equations. So-called conformable derivative definition is a convenient definition in the exact solution procedure of fractional differential equations. Conformable fractional derivative provides convenience both in applicability of methods and solution procedures as its derivative definition does not include any integral term.

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