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Fractional Quantization of Podolsky Electrodynamics Using Fractional Hamilton-Jacobi Formulation

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Abstract: For fractional derivative order constrained systems, the Hamilton-Jacobi formulation in terms Riemann-Liouville fractional derivative was developed. The equations of motion are written as total differential fractional equations fractional in many variables using this formalism. We use the Hamilton-Jacobi formulation in terms of Riemann-Liouville fractional derivative to study Podolsky electrodynamics, comparing our results to those obtained using the Euler-Lagrange Riemann-Liouville fractional derivative method. A fractional difference will be presented as a minor adjustment to a Hamilton-Jacobi derivation formula that is more compatible with the traditional similarity. After generalizing Podolsky electrodynamics for constrained systems with fractional second-order Lagrangians, a new formulation is used to help the reader understand the conclusions.

Keywords: Fractional Hamilton-Jacobi formulation, second order Lagrangian, fractional electromagnetic Lagrangian, Podolsky electrodynamics.

1 Introduction

Several methods have been developed to generalize the concept of derivative and integral to a non-integer order, and several alternative definitions of fractional derivatives have appeared [1−3]. Fractional calculus has been successfully applied in various fields in recent years, including chemistry, biology, modeling and recognition, electronics, and wave propagation. Fractional calculus has been successfully applied in various fields in recent years, including chemistry, biology, modeling and recognition, electronics, and wave propagation. Technology, physics, pure and applied mathematics, have benefited from fractional calculus [4, 5]. Latest studies of scaling dynamics have used fractional derivatives [6, 7]. Fractional calculus has a wide range of applications in classical mechanics [8−11]. Ostrogradskii [12] was the first to establish a treatment for higher-order derivatives theories, which leads to the Euler Lagrange equations and the Hamilton equations of motion. The Hamilton-Jacobi formalism of systems with higher order derivatives was studied in ref.[12], where the action function was investigated for both constrained and unconstrained systems to find a solution to the appropriate set of Hamilton-Jacobi partial differential equations. They used it to obtain the solution of the equations of motion by the WKB approximation. By a different approach some authors studied the Hamiltonian formulation of higher order dynamical systems using Dirac's approach to constrained dynamics [13, 14]. The path integral quantization of systems with higher order derivatives is investigated [15], where these systems are transformed into first order systems with constraints using the Hamilton–Jacobi [HJ] method [16−21]. Important applications of the [HJ] formalism, including systems of higher order derivatives, were also made in [22−24]. The action function is obtained for both constrained and unconstrained systems by solving the required series of Hamilton–Jacobi partial differential equations [HJPDEs] and used to determine the solution of the equations of motion by using the WK formalism [25]. The following are the main characteristics of the novel ideas presented in this manuscript.

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The Lagrange characteristics approach is used to solve a class of nonlinear fractional partial differential equations. All of this knowledge is required to solve the Hamilton-Jacobi formulation for first order, specifically to uncover the properties of its fractional Hamilton-Jacobi equation, and thus to construct a unique formulation of the Hamilton-Jacobi formulation of the Podolsky electrodynamics. According to the findings of this study, because of the order of the fractional derivative and the fractional operator, fractional calculus allows for more flexible models than classical calculus. The fractional equation model It may be of interest to get further insight into the modeling of fractional equations using integrability requirements, which would allow us to generalize the formalism using equation motion of order n. First or second classes will be crucial in the current formalism for Lagrangians who order more than two. They now have a fractional calculus that is exactly identical to conventional calculus and is driven by a fractional Podolsky electrodynamics. To get this result, we had to return to the definition of fractional derivative, and we primarily suggested a concept based on fractional order difference, slightly modified the so-called Riemann-Liouville definition in terms $(A_{i}, \overline{A}_{i}, A_{i}, \overline{A}_{0})$ .The Riemann-Liouville fractional derivative is a well-known extension of ordinary calculus, and practically all other fractional derivative formulations are subsets of the Riemann-Liouville fractional derivative. The Riemann-Liouville definition, on the other hand, satisfied all the mathematical principles within the scope of fractional calculus, and more importantly, when using Podolsky's generalized electrodynamics, we obtain initial conditions with Riemann-Liouville derivatives, which is both practical and mathematically realistic because we are within the scope of fractional calculus. The method is used to solve the Podolsky generalized electrodynamics problem.

We extend the recently formed Hamilton-Jacobi formalism [17, 18] to include continuous second order Lagrangians. The following is the paper's main contribution: We now have a fractional calculus that is fully equivalent to normal calculus and is based on fractional Podolsky electrodynamics. To achieve this result, we had to return to the Hamilton-Jacobi formulation concept for first-order constrained systems, and we primarily proposed equations of motion of the system as total differential equations for the characteristics, and we rewrote the so-called Riemann-Liouville definition to generalize the problem caused by the nature of the constraints first or second class. It is structured as follows:

- First and foremost, because we have introduced the derivative of Riemann Definitions and Properties, we will provide some background information on the Riemann Definitions (Section 2) so that the reader can relate our method to it.

- This new fractional derivative concept will allow us to provide a solid foundation for the recent fractional Podolsky proposal while also allowing us to add some new results about (Section 3), specifically the Hamilton-Jacobi formulation for a general second order system.

- We will then have all we need to generalize Lagrange's method of characteristics to solve a wide range of Lagrangian and Hamiltonian equation problems. This approach is applied to a continuous second-order system in Section 4. The approach entails solving a problem using both the Euler and Hamilton-Jacobi formalisms. As a result, we will have a considerably broader technique for solving Podolsky generalized electrodynamics than we do presently, which is limited to a few extreme examples [28, 29]. The conclusions are discussed in Section 6.

2 Basic definitions

Some basic concepts used in this work are briefly presented in this part of the study. The fractional derivatives of the left and right Riemann-Liouville are defined as follows:
The fractional derivative left of Riemann-Liouville:

$${}_{a}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^{n} \int_{a}^{x} (x - \tau)^{n-\alpha+1} f(\tau) d\tau. \tag{1}$$

The right Riemann- Liouville fractional derivative

$${}_{x}D_{b}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^{n} \int_{a}^{x} (\tau - x)^{n-\alpha+1} f(\tau) d\tau. \tag{2}$$

where Γ denotes the Gamma function, and α is the order of the derivative such that $n-1 < \alpha < n$. If α is an integer, these derivatives are defined in the usual sense, i.e.

$$ {}_aD_x^{\alpha} f(x) = \left(\frac{d}{dx}\right)^n f(x) \tag{3} $$

$$ {}_aD_t^{\alpha} f(x) = \left(\frac{d}{dx}\right)^n f(t)\alpha $$
$$ = 1,2,. \tag{4} $$

3 The Lagrangian formulation of second order systems with fractional derivative

The fractional Euler-Lagrange and Hamilton equations for variational problems were discovered in recent years [28]. He also offered some original thoughts and highlighted some of their potential applications. The method described in [29] is used in the present paper to obtain the Podolsky electrodynamic equations. In the Lagrangian, the dynamics of a physical system are encoded as a function of the positions and velocity of all the degrees of freedom comprising the system[28] . One considers paths in the configuration space to extract the dynamics. The position and velocity at each time and also the value of the Lagrangian is determined for a given direction. The continuous structure of Lagrangian density denoted by the dynamic field variables, generalized coordinate $\boldsymbol{\psi_\rho}$ and its second-order derivatives, generalized velocities${}_{x_\mu}D_b^{\beta}\,{}_{x_\sigma}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t)$defined as

$$ \mathcal{L} = \mathcal{L}\begin{bmatrix} \boldsymbol{\psi_\rho},\ {}_aD_{x_\mu}^{\alpha}\boldsymbol{\psi_\rho}(x,t),\ {}_{x_\mu}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t), \\ {}_aD_{x_\mu}^{\alpha}\,{}_aD_{x_\sigma}^{\alpha}\boldsymbol{\psi_\rho}(x,t),\ {}_{x_\mu}D_b^{\beta}\,{}_{x_\sigma}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t) \end{bmatrix} \tag{5} $$

For this Lagrangian density in fractional form, the Euler-Lagrange equation can be given as

$$ \begin{bmatrix} \dfrac{\partial\mathcal{L}}{\partial\boldsymbol{\psi_\rho}} + \dfrac{\partial\mathcal{L}}{\partial\,{}_aD_{x_\mu}^{\alpha}\boldsymbol{\psi_\rho}(x,t)} + \dfrac{\partial\mathcal{L}}{\partial\,{}_{x_\mu}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t)} \\ + \dfrac{\partial\mathcal{L}}{\partial\,{}_aD_{x_\mu}^{\alpha}\,{}_aD_{x_\sigma}^{\alpha}\boldsymbol{\psi_\rho}(x,t)} + \dfrac{\partial\mathcal{L}}{\partial\,{}_{x_\mu}D_b^{\beta}\,{}_{x_\sigma}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t)} \end{bmatrix} = 0 \tag{6} $$

Using the variational principle, we can write:

$$ \delta S = \int \delta\mathcal{L}\ d^4x = 0 \tag{7} $$

Using Eq. (5), the variation of $\mathcal{L}$ is:

$$ \delta\mathcal{L} = \begin{bmatrix} \dfrac{\partial\mathcal{L}}{\partial\boldsymbol{\psi_\rho}}\delta\boldsymbol{\psi_\rho} \\ + \dfrac{\partial\mathcal{L}}{\partial\,{}_aD_{x_\mu}^{\alpha}\boldsymbol{\psi_\rho}(x,t)}\delta\,{}_aD_{x_\mu}^{\alpha}\boldsymbol{\psi_\rho}(x,t) + \dfrac{\partial\mathcal{L}}{\partial\,{}_{x_\mu}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t)}\delta\,{}_{x_\mu}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t) \\ + \dfrac{\partial\mathcal{L}}{\partial\,{}_aD_{x_\mu}^{\alpha}\,{}_aD_{x_\sigma}^{\alpha}\boldsymbol{\psi_\rho}(x,t)}\delta\,{}_aD_{x_\mu}^{\alpha}\,{}_aD_{x_\sigma}^{\alpha}\boldsymbol{\psi_\rho}(x,t) + \\ \dfrac{\partial\mathcal{L}}{\partial\,{}_{x_\mu}D_b^{\beta}\,{}_{x_\sigma}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t)}\delta\,{}_{x_\mu}D_b^{\beta}\,{}_{x_\sigma}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t) \end{bmatrix} \tag{8} $$

Substituting Eq. (6) into Eq. (7), and using the following commutation relation

$$ \begin{bmatrix} \delta\,{}_aD_{x_\mu}^{\alpha}\boldsymbol{\psi_\rho}(x,t) = {}_aD_{x_\mu}^{\alpha}\delta\boldsymbol{\psi}_\rho(x,t) \\ \delta\,{}_{x_\mu}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t) = {}_{x_\mu}D_b^{\beta}\delta\boldsymbol{\psi}_\rho(x,t) \end{bmatrix} \tag{9} $$

$$ \begin{bmatrix} \delta\,{}_aD_{x_\mu}^{\alpha}\,{}_aD_{x_\sigma}^{\alpha}\boldsymbol{\psi_\rho}(x,t) = {}_aD_{x_\mu}^{\alpha}\,{}_aD_{x_\sigma}^{\alpha}\delta\boldsymbol{\psi}_\rho(x,t) \\ \delta\,{}_{x_\mu}D_b^{\beta}\,{}_{x_\sigma}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t) = {}_{x_\mu}D_b^{\beta}\,{}_{x_\sigma}D_b^{\beta}\delta\boldsymbol{\psi}_\rho(x,t) \end{bmatrix} \tag{10} $$

we get,

$$\int \left[\begin{array}{c} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\psi_\rho}} \delta \boldsymbol{\psi_\rho} + \frac{\partial \mathcal{L}}{\partial \ {}_aD^\alpha_{x_\mu} \boldsymbol{\psi_\rho}(x,t)} \delta \ {}_aD^\alpha_{x_\mu} \boldsymbol{\psi_\rho}(x,t) \\ + \frac{\partial \mathcal{L}}{\partial_{x_\mu} D^\beta_b \boldsymbol{\psi_\rho}(x,t)} \delta_{x_\mu} D^\beta_b \boldsymbol{\psi_\rho}(x,t) \\ + \frac{\partial \mathcal{L}}{\partial \ {}_aD^\alpha_{x_\mu} \ {}_aD^\alpha_{x_\sigma} \boldsymbol{\psi_\rho}(x,t)} \delta \ {}_aD^\alpha_{x_\mu} \ {}_aD^\alpha_{x_\sigma} \boldsymbol{\psi_\rho}(x,t) + \\ \frac{\partial \mathcal{L}}{\partial_{x_\mu} D^\beta_b \ {}_{x_\sigma}D^\beta_b \ \partial_\mu \boldsymbol{\psi}_\rho(x,t)} \delta_{x_\mu} D^\beta_b \ {}_{x_\sigma} D^\beta_b \boldsymbol{\psi_\rho}(x,t) \end{array} \right] d^4x = 0 \tag{11}$$

Integrating by parts the second and the third and the fourth and the fifth terms in Eq. (11) , This lead to Euler – Lagrange equations.

$$\left[\begin{array}{c} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\psi_\rho}} - \ {}_aD^\alpha_{x_\mu} \frac{\partial \mathcal{L}}{\partial \ {}_aD^\alpha_{x_\mu} \boldsymbol{\psi_\rho}(x,t)} - \ {}_{x_\mu}D^\beta_b \frac{\partial \mathcal{L}}{\partial_{x_\mu} D^\beta_b \boldsymbol{\psi_\rho}(x,t)} \\ + \ {}_aD^\alpha_{x_\mu} \ {}_aD^\alpha_{x_\sigma} \frac{\partial \mathcal{L}}{\partial \ {}_aD^\alpha_{x_\mu} \ {}_aD^\alpha_{x_\sigma} \boldsymbol{\psi_\rho}(x,t)} + \ {}_{x_\mu}D^\beta_b \ {}_{x_\sigma}D^\beta_b \frac{\partial \mathcal{L}}{\partial_{x_\mu} D^\beta_b \ {}_{x_\sigma} D^\beta_b \boldsymbol{\psi_\rho}(x,t)} \end{array} \right] = 0 \tag{12}$$

As a special case, taking σ,μ= 0,i. the Euler – Lagrange reduce to the original relations like:

$$\left[\begin{array}{c} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\psi_\rho}} - \ {}_aD^\alpha_t \frac{\partial \mathcal{L}}{\partial \ {}_aD^\alpha_t \boldsymbol{\psi_\rho}(x,t)} - \ {}_tD^\beta_b \frac{\partial \mathcal{L}}{\partial \ {}_tD^\beta_b \boldsymbol{\psi_\rho}(x,t)} \\ - \ {}_aD^\alpha_{x_i} \frac{\partial \mathcal{L}}{\partial \ {}_aD^\alpha_{x_i} \boldsymbol{\psi_\rho}(x,t)} - \ {}_{x_i}D^\beta_b \frac{\partial \mathcal{L}}{\partial_{x_i} D^\beta_b \boldsymbol{\psi_\rho}(x,t)} \\ + \ {}_aD^\alpha_t \ {}_aD^\alpha_t \frac{\partial \mathcal{L}}{\partial \ {}_aD^\alpha_t \ {}_aD^\alpha_t \boldsymbol{\psi_\rho}(x,t)} + \ {}_tD^\beta_b \ {}_tD^\beta_b \frac{\partial \mathcal{L}}{\partial \ {}_tD^\beta_b \ {}_tD^\beta_b \boldsymbol{\psi_\rho}(x,t)} \\ + \ {}_aD^\alpha_{x_i} \ {}_aD^\alpha_{x_i} \frac{\partial \mathcal{L}}{\partial \ {}_aD^\alpha_{x_i} \ {}_aD^\alpha_{x_i} \boldsymbol{\psi_\rho}(x,t)} + \ {}_{x_i}D^\beta_b \ {}_{x_i}D^\beta_b \frac{\partial \mathcal{L}}{\partial_{x_i} D^\beta_b \ {}_{x_i} D^\beta_b \boldsymbol{\psi_\rho}(x,t)} \end{array} \right] = 0 \tag{13}$$

Using the equation Euler-Lagrange, other motion equations are derived from the variables of the other fields ($\boldsymbol{\phi}$ and $\boldsymbol{A_i}$), as we get:

$$\left[\begin{array}{c} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\phi}} - \ {}_aD^\alpha_t \frac{\partial \mathcal{L}}{\partial \ {}_aD^\alpha_t \boldsymbol{\phi}} - \ {}_aD^\alpha_{x_i} \frac{\partial \mathcal{L}}{\partial_{x_\mu} D^\beta_b \boldsymbol{\phi}} \\ + \ {}_aD^\alpha_t \ {}_aD^\alpha_t \frac{\partial \mathcal{L}}{\partial \ {}_aD^\alpha_t \ {}_aD^\alpha_t \boldsymbol{\phi}} + \ {}_{x_i}D^\beta_b \ {}_{x_i}D^\beta_b \frac{\partial \mathcal{L}}{\partial_{x_i} D^\beta_b \ {}_{x_i} D^\beta_b \boldsymbol{\phi}} \end{array} \right] = 0 \tag{14}$$

$$\left[\begin{array}{c} \frac{\partial \mathcal{L}}{\partial \boldsymbol{A_i}} - \ {}_aD^\alpha_t \frac{\partial \mathcal{L}}{\partial \ {}_aD^\alpha_t \boldsymbol{A_i}} - \ {}_aD^\alpha_{x_i} \frac{\partial \mathcal{L}}{\partial \ {}_aD^\alpha_{x_i} \boldsymbol{A_i}} \\ + \ {}_aD^\alpha_t \ {}_aD^\alpha_t \frac{\partial \mathcal{L}}{\partial \ {}_aD^\alpha_t \ {}_aD^\alpha_t \boldsymbol{A_i}} + \ {}_{x_i}D^\beta_b \ {}_{x_i}D^\beta_b \frac{\partial \mathcal{L}}{\partial_{x_i} D^\beta_b \ {}_{x_i} D^\beta_b \boldsymbol{A_i}} \end{array} \right] = 0 \tag{15}$$

If $\alpha \rightarrow 1$ is taken, we get the classical case equations for integer order[27].

The canonical variables ($\boldsymbol{\psi_\rho}, \boldsymbol{\pi^1_\alpha}$) and ($\boldsymbol{\bar{\psi}_\rho}, \boldsymbol{\pi^2_\alpha}$), where ($\boldsymbol{\bar{\psi}_\rho} = \ {}_aD^\alpha_t \boldsymbol{\psi_\rho}$), will then span the phase-space, and the Hamiltonian will be given by

$$H = \boldsymbol{\pi^1_\alpha} \boldsymbol{\bar{\psi}_\rho} + \boldsymbol{\pi^2_\alpha} \ {}_aD^{2\alpha}_t \boldsymbol{\psi_\rho} - \boldsymbol{L} \tag{16}$$

This means that, the Hamiltonian is a function of the form

$$H = \boldsymbol{H}\left(\boldsymbol{\psi_\rho}, \boldsymbol{\pi^1_\alpha}, \boldsymbol{\bar{\psi}_\rho}, \boldsymbol{\pi^2_\alpha}, \boldsymbol{t}\right) \tag{17}$$

4 Fractional Hamiltonian and Hamilton–Jacobi formulations

Gfiler developed a new formalism for Connected first-order systems based on Caratheodory's equivalent Lagrangians approach for obtaining a series of Hamilton-Jacobi partial differential equations and writing the equations of motion as total differential equations.

The Hamilton-Jacobi formalism will be developed to a general second-order Lagrangian using Caratheodory's method in this section. Any second-order Lagrangian will derive from this formalism. Caratheodory's second-order equivalent Lagrangians approach According to Lagrangians, we can get acompletely equivalent Lagrangian L($\boldsymbol{\psi_\rho}$, ${}_aD_t^{\alpha}\boldsymbol{\psi_\rho}$, ${}_aD_t^{2\alpha}\boldsymbol{\psi_\rho}$, t) by:

$$L' = L\left(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},,\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho},\mathrm{t}\right) - \frac{ds\left(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},\mathrm{t}\right)}{dt} \tag{18}$$

Since their action integrals have simultaneous extremes, these Lagrangians are equivalent. As a result, we can choose S($\boldsymbol{\psi_\rho}$, ${}_aD_t^{\alpha}\boldsymbol{\psi_\rho}$, t) in such a way that L′ becomes an extreme, and we can then reduce the variational problem of finding an extreme for the Lagrangian L to a differential calculus problem. To do so, we must find a set of functions $\phi_p(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},\mathrm{t})$ $\beta_p(\boldsymbol{\psi_\rho},\mathrm{t})$,and S($\boldsymbol{\psi_\rho}$, ${}_aD_t^{\alpha}\boldsymbol{\psi_\rho}$, t) that satisfy the following conditions.

$$L'\left(\boldsymbol{\psi_\rho},\phi_p,\beta_p,t\right) = 0 \tag{19}$$

and for all neighborhood of ${}_aD_t^{\alpha}\boldsymbol{\psi_\rho} = \beta_p(\boldsymbol{\psi_\rho},\mathrm{t})$ and ${}_aD_t^{2\alpha}\boldsymbol{\psi_\rho} = \phi_p(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},\mathrm{t})$

$$L'\left(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho},t\right) > 0 \tag{20}$$

With these conditions satisfied, the Lagrangian L′ would have a minimum in ${}_aD_t^{\alpha}\boldsymbol{\psi_\rho} = \beta_p(\boldsymbol{\psi_\rho},\ \mathrm{t})$ and ${}_aD_t^{2\alpha}\boldsymbol{\psi_\rho} = \phi_p(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},\mathrm{t})$, resulting in a minimum in the action integral. As a result, the solutions of the differential equations will correspond to the action integral's extreme.

We can deduce the following from the definition of L′:

$$L' = \left|\begin{array}{c} L\left(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},,\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho},\mathrm{t}\right) - \dfrac{dS\left(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},\mathrm{t}\right)}{dt} \\ -\dfrac{dS(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},\mathrm{t})}{d\boldsymbol{\psi_\rho}}\dfrac{d\boldsymbol{\psi_\rho}}{dt} - \dfrac{dS(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},\mathrm{t})}{d\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho}}\dfrac{d\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho}}{dt} \end{array}\right| \tag{21}$$

Using condition (19) we have

$$\left.\left[\begin{array}{c} L\left(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},,\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho},\mathrm{t}\right) - \dfrac{dS\left(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},\mathrm{t}\right)}{dt} \\ -\dfrac{dS(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},\mathrm{t})}{d\boldsymbol{\psi_\rho}}\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho} - \dfrac{dS(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},\mathrm{t})}{d\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho}}\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho} \end{array}\right]\right|_{\substack{{}_aD_t^{\alpha}\boldsymbol{\psi_\rho} = \beta_p \\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho} = \phi_p}} = 0 \tag{22}$$

$$\left.\frac{dS}{d\boldsymbol{\psi_\rho}}\right|_{\substack{{}_aD_t^{\alpha}\boldsymbol{\psi_\rho} = \beta_p \\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho} = \phi_p}}$$

$$= \left.\left[\begin{array}{c} L\left(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},,\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho},\mathrm{t}\right) - \dfrac{dS\left(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},\mathrm{t}\right)}{dt} \\ -\dfrac{dS(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},\mathrm{t})}{d\boldsymbol{\psi_\rho}}\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho} - \dfrac{dS(\boldsymbol{\psi_\rho},\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho},\mathrm{t})}{d\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho}}\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho} \end{array}\right]\right|_{\substack{{}_aD_t^{\alpha}\boldsymbol{\psi_\rho} = \beta_p \\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho} = \phi_p}} \tag{23}$$

Since ${}_aD_t^{\alpha}\boldsymbol{\psi_\rho} = \beta_p$and ${}_aD_t^{2\alpha}\boldsymbol{\psi_\rho} = \phi_p$are minimum points of L' we must have

$$\left.\frac{dL\prime}{d\boldsymbol{\psi_\rho}}\right|_{\substack{{}_aD_t^{\alpha}\boldsymbol{\psi_\rho} = \beta_p \\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho} = \phi_p}} = 0 \tag{24}$$

As a result, $L' = L(\boldsymbol{\psi_\rho}, {}_aD_t^{\alpha}\boldsymbol{\psi_\rho})$ and the equation L′ are represented as

$$\left.\left[\frac{dL}{d\,{}_aD_t^{2\alpha}\boldsymbol{\psi_\rho}} - \frac{dS(\boldsymbol{\psi_\rho}, {}_aD_t^{\alpha}\boldsymbol{\psi_\rho}, \mathrm{t})}{d\,{}_aD_t^{\alpha}\boldsymbol{\psi_\rho}}\right]\right|_{\substack{{}_aD_t^{\alpha}\boldsymbol{\psi_\rho} = \beta_p \\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho} = \phi_p}} = 0 \tag{25}$$

Or

$$\left.\frac{dS}{d\,{}_aD_t^{\alpha}\boldsymbol{\psi_\rho}}\right|_{\substack{{}_aD_t^{\alpha}\boldsymbol{\psi_\rho} = \beta_p \\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho} = \phi_p}} = \left.\frac{dL}{d\,{}_aD_t^{2\alpha}\boldsymbol{\psi_\rho}}\right|_{\substack{{}_aD_t^{\alpha}\boldsymbol{\psi_\rho} = \beta_p \\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho} = \phi_p}} \tag{26}$$

Similarly, we would have

$$\left.\frac{d\mathrm{L}'}{d\boldsymbol{\psi_\rho}}\right|_{\substack{{}_aD_t^{\alpha}\boldsymbol{\psi_\rho} = \beta_p \\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho} = \phi_p}} = 0$$

Using eq. (21), takes the form

$$\left.\left[\frac{dL}{d\,{}_aD_t^{\alpha}\boldsymbol{\psi_\rho}} - \frac{dS}{d\,{}_aD_t^{\alpha}\boldsymbol{\psi_\rho}}\left(\frac{dS}{dt}\right)\right]\right|_{\substack{{}_aD_t^{\alpha}\boldsymbol{\psi_\rho} = \beta_p \\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho} = \phi_p}} = 0 \tag{27}$$

In terms of Lagrangian density and Hamiltonian principal function, we can express Eq. (27) as follows:

$$\left.\left[\frac{\partial L}{\partial\,{}_aD_t^{\alpha}\boldsymbol{\psi_\rho}} - \frac{\partial}{\partial t}\frac{\partial S}{\partial\,{}_aD_t^{\alpha}\boldsymbol{\psi_\rho}} - \frac{\partial S}{\partial\boldsymbol{\psi_\rho}} - \frac{\partial^2 S}{\partial\,{}_aD_t^{\alpha}\boldsymbol{\psi_\rho}\partial\boldsymbol{\psi_\rho}}\,{}_aD_t^{\alpha}\boldsymbol{\psi_\rho} - \frac{\partial^2 S}{\partial\,{}_aD_t^{\alpha}\boldsymbol{\psi_\rho}\partial\,{}_aD_t^{\alpha}\boldsymbol{\psi_\rho}\boldsymbol{\psi_\rho}}\,{}_aD_t^{2\alpha}\boldsymbol{\psi_\rho}\right]\right|_{\substack{{}_aD_t^{\alpha}\boldsymbol{\psi_\rho} = \beta_p \\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho} = \phi_p}} = 0 \tag{28}$$

$$\left.\left[\frac{\partial L}{\partial\,{}_aD_t^{\alpha}\boldsymbol{\psi_\rho}} - \frac{\partial S}{\partial\boldsymbol{\psi_\rho}} - \frac{d}{dt}\frac{\partial S}{\partial\,{}_aD_t^{\alpha}\boldsymbol{\psi_\rho}}\right]\right|_{\substack{{}_aD_t^{\alpha}\boldsymbol{\psi_\rho} = \beta_p \\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho} = \phi_p}} = 0 \tag{29}$$

Or

$$\left.\frac{dS}{d\boldsymbol{\psi_\rho}}\right|_{{}_aD_t^{\alpha}\boldsymbol{\psi_\rho} = \beta_p} = \left.\left[\frac{\partial L}{\partial\,{}_aD_t^{\alpha}\boldsymbol{\psi_\rho}} - \frac{d}{dt}\frac{\partial S}{\partial\,{}_aD_t^{\alpha}\boldsymbol{\psi_\rho}}\right]\right|_{\substack{{}_aD_t^{\alpha}\boldsymbol{\psi_\rho} = \beta_p \\ {}_aD_t^{2\alpha}\boldsymbol{\psi_\rho} = \phi_p}} \tag{30}$$

Using the definitions for conjugated momenta given in equations (13) and (16) and writing ${}_aD_t^{\alpha}\boldsymbol{\psi_\rho}=\overline{\boldsymbol{\psi_\rho}}$, we can deduce from equation (24) that in action to obtain an extreme of the operation, we must obtain a function S($\boldsymbol{\psi_\rho}$, ${}_aD_t^{\alpha}\boldsymbol{\psi_\rho}$, t) such that**:**

$$\frac{\partial S}{\partial t} = -H_0 \tag{31}$$

$$H_0 = \boldsymbol{p_i}\overline{\boldsymbol{\psi}}_{\boldsymbol{\rho}} + \boldsymbol{\pi_i}\ {}_aD_t^{\alpha}\overline{\boldsymbol{\psi}}_{\boldsymbol{\rho}} - L \tag{32}$$

$$\boldsymbol{p_i} = \frac{\partial S}{\partial \boldsymbol{\psi_\rho}}\ ;\boldsymbol{\pi_i} = \frac{\partial S}{\partial \overline{\boldsymbol{\psi}}_{\boldsymbol{\rho}}} \tag{33}$$

The Hamilton-Jacobi partial differential equation, or simply the HJPDE, has become one of the fundamental equations of the equivalent Lagrangian method.

5 Fractional electromagnetic lagrangian density

In this section, we'll focus at a continuous system with Lagrangian density that is dependent on the dynamical field variables, as well as its second-order derivatives: $\mathcal{L} = \mathcal{L}(\boldsymbol{\psi_\rho}, \partial\boldsymbol{\psi_\rho}, \partial^2\boldsymbol{\psi_\rho})$. With Greek indices ranging from 0 to 3 and Latin indices ranging from 1 to 3, we use the metric $\eta_{\mu\nu}$= diag(+1, -1, -1, -1). As previously mentioned, generalizing the formalism provided in Sections 5 is simple; all that is required is to consider that the Euler-Lagrange equations of motion are now given by

$$\begin{bmatrix} \dfrac{\partial\mathcal{L}}{\partial\boldsymbol{\psi_\rho}} - {}_aD_{x_\mu}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\ {}_aD_{x_\mu}^{\alpha}\boldsymbol{\psi_\rho}(x,t)} - {}_{x_\mu}D_b^{\beta}\dfrac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t)} + \\ {}_aD_{x_\mu}^{\alpha}\ {}_aD_{x_\sigma}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\ {}_aD_{x_\mu}^{\alpha}\ {}_aD_{x_\sigma}^{\alpha}\boldsymbol{\psi_\rho}(x,t)} + {}_{x_\mu}D_b^{\beta}\ {}_{x_\sigma}D_b^{\beta}\dfrac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D_b^{\beta}\ {}_{x_\sigma}D_b^{\beta}\boldsymbol{\psi_\rho}(x,t)} \end{bmatrix} = 0 \tag{34}$$

and that the momenta, conjugated to ${}_aD_t^{\alpha}\boldsymbol{\psi_\rho}$ and ${}_aD_t^{2\alpha}\boldsymbol{\psi_\rho}$, are as follows:

$$\boldsymbol{p_a} = \frac{\partial\mathcal{L}}{\partial\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho}} - 2\ {}_aD_{x_k}^{\alpha}\left[\frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_k}^{\alpha}\ {}_aD_t^{\alpha}\boldsymbol{\psi_\rho}\right)}\right] - {}_aD_t^{\alpha}\left[\frac{\partial\mathcal{L}}{\partial\left({}_aD_t^{2\alpha}\boldsymbol{\psi_\rho}\right)}\right] \tag{35}$$

$$\boldsymbol{\pi_a} = \frac{\partial\mathcal{L}}{\partial\left({}_aD_t^{2\alpha}\boldsymbol{\psi_\rho}\right)} \tag{36}$$

The most general form of Lagrangian density for a four-vector field is given by the so-called Podolsky Lagrangian density[28].

$$\mathcal{L}_{LW} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - a^2\partial_\lambda F^{\alpha\lambda}\partial^\alpha F_{\rho\alpha} \tag{37}$$

Where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

So, we can write the Lagrangian (24) as:

$$\mathcal{L} = -\frac{1}{2}F_{0i}F^{0i} - a^2[(\partial_i F^{0i})^2 - (\partial_0 F^{0i})^2] \tag{38}$$

To rewrite the Podolsky Lagrangian density in Riemann – Liouville fractional form we use these relations :

$$\begin{bmatrix} F_{\mu\nu} = {}_aD_{x_\mu}^{\alpha}\boldsymbol{A_\nu} - {}_aD_{x_\nu}^{\alpha}\boldsymbol{A_\mu} \\ F^{\mu\nu} = {}_aD_{x^\mu}^{\alpha}\boldsymbol{A^\nu} - {}_aD_{x^\nu}^{\alpha}\boldsymbol{A^\mu} \end{bmatrix} \tag{39}$$

$$\begin{bmatrix} \partial_\alpha = {}_aD_{x_\mu}^{\alpha} = \left({}_aD_t^{\alpha},\ {}_aD_{x_i}^{\alpha}\right) \\ \partial^\alpha = {}_aD_{x^\mu}^{\alpha} = \left({}_aD_t^{\alpha}, -{}_aD_{x^i}^{\alpha}\right) \end{bmatrix} \tag{40}$$

$$F_{\mu\nu}F^{\mu\nu} = 2\left[{}_aD_{x_\mu}^{\alpha}A_\nu\ {}_aD_{x^\mu}^{\alpha}A^\nu - {}_aD_{x_\mu}^{\alpha}A_\nu\ {}_aD_{x^\nu}^{\alpha}A^\mu\right] \tag{41}$$

$$\begin{bmatrix} \mathbf{A^\alpha} = \left(\phi, \vec{A}\right) \\ \mathbf{A_\alpha} = \left(\phi, -\vec{A}\right) \end{bmatrix} \tag{42}$$

using definition of left Riemann – Liouville fractional derivative, the fractional electromagnetic Lagrangian density formulation takes the form :

$$\mathcal{L} = -\frac{1}{2}\begin{bmatrix} \left({}_aD_t^\alpha \boldsymbol{A_i}\right)^2 - 2\,{}_aD_t^\alpha \boldsymbol{A_i}\,{}_aD_{x_i}^\alpha \phi \\ +\left({}_aD_{x_i}^\alpha \boldsymbol{\phi}\right)^2 \end{bmatrix} - a^2\begin{bmatrix} \left[{}_aD_{x_i}^\alpha\,{}_aD_t^\alpha \boldsymbol{A_i} - {}_aD_t^\alpha\,{}_aD_{x_i}^\alpha \boldsymbol{\phi}\right]^2 \\ -\left[{}_aD_t^\alpha\,{}_aD_{x_i}^\alpha \boldsymbol{A_i} - {}_aD_{x_i}^\alpha\,{}_aD_t^\alpha \boldsymbol{\phi}\right]^2 \end{bmatrix} \quad (43)$$

Equations of motion, using Euler- Lagrange equation Eq.(14), by taking the derivative with respect to ϕ , we get:

$$\left[1 + 2a^2\left({}_aD_{x_i}^\alpha\right)^2\right]\,{}_aD_{x_i}^\alpha\left[{}_aD_{x_i}^\alpha\,{}_aD_t^\alpha \boldsymbol{A_i} - {}_aD_t^\alpha\,{}_aD_{x_i}^\alpha \boldsymbol{\phi}\right] = 0 \quad (44)$$

Now use the general formula (15) to obtain other equations of motion from the other field variable $\boldsymbol{A^i}$.

$$\left[1 + 2a^2\left({}_aD_t^\alpha\right)^2\right]\,{}_aD_t^\alpha\left[{}_aD_{x_i}^\alpha\,{}_aD_t^\alpha \boldsymbol{A_i} - {}_aD_t^\alpha\,{}_aD_{x_i}^\alpha \boldsymbol{\phi}\right] = 0 \quad (45)$$

Adding equations (35) and (36) to get

$$\left[1 + 2a^2\left(\left({}_aD_t^\alpha\right)^2 - \left({}_aD_{x_i}^\alpha\right)^2\right)\right]\left[{}_aD_t^\alpha + {}_aD_{x_i}^\alpha\right]\left[{}_aD_{x_i}^\alpha \boldsymbol{\phi} - {}_aD_t^\alpha \boldsymbol{A_i}\right] = 0 \quad (46)$$

This represents the second non- homogeneous equation in fractional form.

If α goes to 1,then Eq. (35) and (36) become:

$$[1 + 2a^2(\partial_0^2 - \partial_i^2)][\partial_0 + \partial_i]F_{i0} = 0 \quad (47)$$

The result obtained in (44,45) is the same as obtained in[29,30].

Using (34) ,the Euler-Lagrange equations of motion are now given by

$$\begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\psi_\rho}} - {}_aD_{x_\mu}^\alpha \frac{\partial \mathcal{L}}{\partial\,{}_aD_{x_\mu}^\alpha \boldsymbol{\psi_\rho}(x,t)} - {}_{x_\mu}D_b^\beta \frac{\partial \mathcal{L}}{\partial\,{}_{x_\mu}D_b^\beta \boldsymbol{\psi_\rho}(x,t)} \\ + {}_aD_{x_\mu}^\alpha\,{}_aD_{x_\sigma}^\alpha \frac{\partial \mathcal{L}}{\partial\,{}_aD_{x_\mu}^\alpha\,{}_aD_{x_\sigma}^\alpha \boldsymbol{\psi_\rho}(x,t)} + {}_{x_\mu}D_b^\beta\,{}_{x_\sigma}D_b^\beta \frac{\partial \mathcal{L}}{\partial\,{}_{x_\mu}D_b^\beta\,{}_{x_\sigma}D_b^\beta \boldsymbol{\psi_\rho}(x,t)} \end{bmatrix} = 0 \quad (48)$$

and that the momenta, conjugated respectively to ${}_aD_{x_\mu}^\alpha \boldsymbol{\psi_\rho}(x,t)$ and ${}_aD_{x_\mu}^\alpha\,{}_aD_{x_\sigma}^\alpha \boldsymbol{\psi_\rho}(x,t)$ are:

$$\boldsymbol{p_\mu} = \frac{\partial \mathcal{L}}{\partial\,{}_aD_t^\alpha \boldsymbol{\psi_\rho}(x,t)} - 2\partial_k\left[\frac{\partial \mathcal{L}}{\partial\,{}_aD_k^\alpha\,{}_aD_t^\alpha \boldsymbol{\psi_\rho}(x,t)}\right] - 2\partial_0\left[\frac{\partial \mathcal{L}}{\partial\,{}_aD_t^{2\alpha} \boldsymbol{\psi_\rho}(x,t)}\right] \quad (49)$$

$$\boldsymbol{\pi_\mu} = \left[\frac{\partial \mathcal{L}}{\partial\,{}_aD_t^{2\alpha} \boldsymbol{\psi_\rho}(x,t)}\right] \quad (50)$$

The Hessian matrix is now:

$$H_{ab} = \frac{\partial^2 \mathcal{L}}{{}_aD_t^{2\alpha} \boldsymbol{\psi_\rho^a}\,{}_aD_t^{2\alpha} \boldsymbol{\psi_\rho^b}} \quad (51)$$

and $\boldsymbol{\bar{A}^\mu} = {}_aD_t^\alpha \boldsymbol{A}$ are the dynamical variables we've chosen. Definitions (49) and (50) give the conjugated momenta as $\mathbf{A^\mu}$ follows:

$$\mathbf{p_\mu} = -\left({}_aD_t^\alpha \mathbf{A_\mu} - {}_aD_{x_\mu}^\alpha \mathbf{A_0}\right) - 2a^2\left({}_aD_{x_k}^\alpha\,{}_aD_\lambda^\alpha\left({}_aD_t^\alpha \mathbf{A_\lambda} - {}_aD_{x_\lambda}^\alpha \mathbf{A_0}\right)\delta_\mu^k\right) - {}_aD_t^\alpha\,{}_aD_{x_\lambda}^\alpha\left({}_aD_{x_\mu}^\alpha \mathbf{A_\lambda} - {}_aD_{x_\lambda}^\alpha \mathbf{A_\mu}\right)$$

$$\boldsymbol{\pi_\mu} = 2a^2\left({}_aD_{x_\lambda}^\alpha\left({}_aD_t^\alpha \mathbf{A_\lambda} - {}_aD_{x_\lambda}^\alpha \mathbf{A_0}\right)\delta_\mu^0 - {}_aD_{x_\lambda}^\alpha\left({}_aD_{x_\mu}^\alpha \mathbf{A_\lambda} - {}_aD_{x_\lambda}^\alpha \mathbf{A_\mu}\right)\right) \quad (53)$$

The below are the primary constraints:

$$\lambda_1 = \pi_0 \approx 0 \quad (54)$$

$$\lambda_2 = p_0 - {}_aD_{x^k}^\alpha \boldsymbol{\pi_k} \approx 0 \quad (55)$$

We can write the accelerations ${}_aD_t^\alpha \boldsymbol{\bar{A}}$ as: using the definition of π

$${}_aD_t^\alpha \boldsymbol{\bar{A}} = \frac{1}{2a^2}(\pi_i) + {}_aD_{x_k}^\alpha\left({}_aD_{x_i}^\alpha \mathbf{A_k} - {}_aD_{x_k}^\alpha \mathbf{A_i}\right) + {}_aD_{x_i}^\alpha \mathbf{\bar{A}_0} \quad (56)$$

The canonical Hamiltonian is defined as follows:

$$H_c = \int d^3x\left(\boldsymbol{P_\alpha^1 \bar{A}^\mu} + \boldsymbol{\pi_\mu}\,{}_aD_t^\alpha \boldsymbol{\bar{A}^\mu} - \boldsymbol{\mathcal{L}}\right) \quad (57)$$

Using equation (56) as a basis, we can arrive at the following:

$$H_c = \int \begin{bmatrix} \bar{\boldsymbol{A}}^0 \ {}_aD_{x_i}^{\alpha} \boldsymbol{\pi}_i \quad + \boldsymbol{P}_i \bar{\boldsymbol{A}}^i + \\ \frac{1}{2a^2}(\pi_i)(\pi^i) + \pi_i \ {}_aD_{x_k}^{\alpha}\left({}_aD_{x^i}^{\alpha}\mathbf{A}^{\boldsymbol{k}} - {}_aD_{x^k}^{\alpha}\mathbf{A}^{\boldsymbol{i}}\right) + \pi_i \ {}_aD_{x^i}^{\alpha}\bar{\boldsymbol{A}}_0 \\ +\frac{1}{4}\left({}_aD_{x_\mu}^{\alpha}\boldsymbol{A}_{\boldsymbol{v}} - {}_aD_{x_v}^{\alpha}\boldsymbol{A}_{\boldsymbol{\mu}}\right)\left({}_aD_{x^\mu}^{\alpha}\boldsymbol{A}^{\boldsymbol{v}} - {}_aD_{x^v}^{\alpha}\boldsymbol{A}^{\boldsymbol{\mu}}\right) \\ +\frac{1}{2a^2}\left(\bar{\boldsymbol{A}}_{\boldsymbol{i}} - {}_aD_{x_i}^{\alpha}\mathbf{A_0}\right)\left(\bar{\boldsymbol{A}}^i - {}_aD_{x^i}^{\alpha}\mathbf{A_0}\right) \\ -a^2\left({}_aD_{x_k}^{\alpha}\bar{\boldsymbol{A}}^k - {}_aD_{x_k}^{\alpha} \ {}_aD_{x^k}^{\alpha}\bar{\boldsymbol{A}}_0\right)\left({}_aD_{x_i}^{\alpha}\bar{\boldsymbol{A}}^i - {}_aD_{x_i}^{\alpha} \ {}_aD_{x^i}^{\alpha}\mathbf{A_0}\right) \end{bmatrix} d^3x \tag{58}$$

The total Hamiltonian, according to Dirac's formalism, is:

$$H_T = H_c + \int (C_1(x)\lambda_1 + C_2(x)\lambda_2)d^3x \tag{59}$$

As a consequence of the consistency conditions, we get:

$${}_aD_t^{\alpha}\lambda_1 = \{\lambda_1, H_T\} \approx 0 \tag{60}$$

$${}_aD_t^{\alpha}\lambda_2 = \{\lambda_2, H_T\} = {}_aD_{x^k}^{\alpha}P_k \approx 0 \tag{61}$$

As a result, we have a second constraint given by

$$\lambda_3 = {}_aD_{x^k}^{\alpha}P_k \approx 0 \tag{62}$$

As a consequence of the consistency condition, ${}_aD_t^{\alpha}\lambda_2 = \{\lambda_2, H_T\} \approx 0$ is obtained. Since all constraints are of the first order, the expanded Hamiltonian is:

$$H_T = H_c + \int (C_1(x)\lambda_1 + C_2(x)\lambda_2 + C_3(x)\lambda_3)d^3x \tag{63}$$

The dynamical variables' equations of motion, given by ${}_aD_t^{\alpha}A^{\alpha} = \{A^{\alpha}, H_E\}$, are as follows:

$${}_aD_t^{\alpha}A^0 = \bar{A}^0 + C_2\,; \ {}_aD_t^{\alpha}A^i = \bar{A}^i - {}_aD_{x_i}^{\alpha}C_3 \tag{64}$$

Simply put, $\bar{A}^{\alpha}$ equals ${}_aD_t^{\alpha}A^{\alpha}$ plus additive arbitrary functions. In addition, ${}_aD_t^{\alpha}A^0 = \{A^{\alpha}, H_E\}$ gives

$${}_aD_t^{\alpha}\bar{A}^0; \ {}_aD_t^{\alpha}\bar{A}^i = \frac{1}{2a^2}(\pi_i) + {}_aD_{x_k}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_k - {}_aD_{x_k}^{\alpha}A_i\right) + {}_aD_{x_i}^{\alpha}\bar{A}_0 \tag{65}$$

As a result, both $\bar{A}^0$ and A^0 are arbitrary, while we have again obtained equation (56).

In the case of the momenta variables ${}_aD_t^{\alpha}\pi_i = \{\pi_i, H_E\}$ and ${}_aD_t^{\alpha}P_{\alpha} = \{P_{\alpha}, H_E\}$, the following results are obtained :

$${}_aD_t^{\alpha}\pi_i = -\left({}_aD_t^{\alpha}A_i - {}_aD_{x_i}^{\alpha}A_0\right) - 2a^2 \ {}_aD_{x_i}^{\alpha} \ {}_aD_{x_k}^{\alpha}\left({}_aD_t^{\alpha}A_k - {}_aD_{x_k}^{\alpha}A_0\right) - P_i \tag{66}$$

$${}_aD_t^{\alpha}p_0 = -\left({}_aD_t^{\alpha}A_i - {}_aD_{x_i}^{\alpha}A_0\right) - 2a^2 \ {}_aD_{x^i}^{\alpha} \ {}_aD_{x_i}^{\alpha} \ {}_aD_{x_k}^{\alpha}\left({}_aD_t^{\alpha}A_k - {}_aD_{x_k}^{\alpha}A_0\right) \tag{67}$$

$${}_aD_t^{\alpha}p_i = - {}_aD_{x_i}^{\alpha} \ {}_aD_{x^k}^{\alpha}\pi_k + {}_aD_{x_k}^{\alpha} \ {}_aD_{x^k}^{\alpha}\pi_i - {}_aD_{x_k}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_k - {}_aD_{x_k}^{\alpha}A_i\right) \tag{68}$$

Equation (66) is the concept of p_i given by equation (67), which when combined with (68) yields constraint λ_3.

Using Hamilton-Jacobi formalism, we can derive the following:

$$H_0' = H_C + P_0\,; \ P_0 = \frac{\partial S}{\partial t} \tag{69}$$

$$H_1' = \pi_0; H_2' = P_0 - {}_aD_{x^k}^{\alpha}\pi_k \tag{70}$$

In the case of A^i, the total differential equation is:

$$dA^i = \frac{\partial H_0'}{\partial p_i}dt + \frac{\partial H_1'}{\partial p_i}d\bar{A}_0 + \frac{\partial H_2'}{\partial p_i}dA_0 \tag{71}$$

$$dA^i = \frac{\partial H_0'}{\partial p_i}dt = \frac{\partial H_c'}{\partial p_i}dt \qquad \text{Then the Eq. (71) will take the form} \qquad dA^i = \bar{A}^i dt \tag{72}$$

Since C3 is arbitrary, it is completely equivalent to equation (64). With respect to $\bar{A}^i$, the total differential equation is:

$$d\bar{A}^i = \frac{\partial H_0'}{\partial p_i}dt + \frac{\partial H_1'}{\partial p_i}d\bar{A}_0 + \frac{\partial H_2'}{\partial p_i}dA_0 \tag{73}$$

$$d\bar{A}^i = \left(\frac{1}{2a^2}(\pi^i) + {}_aD_{x_k}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_k - {}_aD_{x^k}^{\alpha}A_i\right) + {}_aD_{x_i}^{\alpha}\bar{A}_0\right)dt \tag{74}$$

The fractional form of the above equation is the same as the fractional form of the equation obtained from Euler method (Eq. (55)). p_i and p_0 can be determined as follows:

$$dp^i = -\frac{\partial H_0'}{\partial A_i}dt - \frac{\partial H_1'}{\partial A_i}d\bar{A}_0 - \frac{\partial H_2'}{\partial A_i}dA_0 = -\frac{\partial H_0'}{\partial A_i}dt = -\frac{\partial H_c}{\partial A_i}dt \tag{75}$$

$$\mathrm{dp}^{\mathrm{i}} = -\int \mathrm{d}^3\mathrm{x}\begin{bmatrix} \pi_{\mathrm{j}}\ {}_{\mathrm{a}}\mathrm{D}^{\alpha}_{\mathrm{x_k}}\left(\dfrac{\partial\left({}_{\mathrm{a}}\mathrm{D}^{\alpha}_{\mathrm{x_i}}\mathrm{A_k} - {}_{\mathrm{a}}\mathrm{D}^{\alpha}_{\mathrm{x^k}}\mathrm{A_i}\right)}{\partial \mathrm{A_i}}\right) \\ -\dfrac{1}{2}\left({}_{\mathrm{a}}\mathrm{D}^{\alpha}_{\mathrm{x_j}}\mathrm{A_n} - {}_{\mathrm{a}}\mathrm{D}^{\alpha}_{\mathrm{x^n}}\mathrm{A_j}\right)\left(\dfrac{\partial\left({}_{\mathrm{a}}\mathrm{D}^{\alpha}_{\mathrm{x^j}}\mathrm{A_n} - {}_{\mathrm{a}}\mathrm{D}^{\alpha}_{\mathrm{x^n}}\mathrm{A_j}\right)}{\partial \mathrm{A_i}}\right)\end{bmatrix}\mathrm{dt} \tag{76}$$

$$\mathrm{dp}^{\mathrm{i}} = \left[-\ {}_{\mathrm{a}}\mathrm{D}^{\alpha}_{\mathrm{x^i}}\ {}_{\mathrm{a}}\mathrm{D}^{\alpha}_{\mathrm{x^k}}\pi_{\mathrm{k}} + {}_{\mathrm{a}}\mathrm{D}^{\alpha}_{\mathrm{x_k}}\ {}_{\mathrm{a}}\mathrm{D}^{\alpha}_{\mathrm{x^k}}\pi^{\mathrm{i}} - {}_{\mathrm{a}}\mathrm{D}^{\alpha}_{\mathrm{x_k}}\left({}_{\mathrm{a}}\mathrm{D}^{\alpha}_{\mathrm{x_i}}\mathrm{A_k} - {}_{\mathrm{a}}\mathrm{D}^{\alpha}_{\mathrm{x^k}}\mathrm{A_i}\right)\right] \tag{77}$$

And

$$\mathrm{dp}^{0} = -\frac{\partial \mathrm{H}'_0}{\partial \mathrm{A}_0}\mathrm{dt} - \frac{\partial \mathrm{H}'_1}{\partial \mathrm{A}_0}\mathrm{d}\bar{\mathrm{A}}_0 - \frac{\partial \mathrm{H}'_2}{\partial \mathrm{A}_0}\mathrm{dA}_0 = -\frac{\partial \mathrm{H}'_0}{\partial \mathrm{A}_0}\mathrm{dt} = -\frac{\partial \mathrm{H_c}}{\partial \mathrm{A}_0}\mathrm{dt} \tag{78}$$

$$dp^0 = -\int d^3x\begin{bmatrix} \left(\bar{\mathbf{A}}^{i} - {}_{a}D^{\alpha}_{x^i}\mathbf{A_0}\right)\left(\dfrac{\partial\left(\bar{\mathbf{A}}_{\iota} - {}_{a}D^{\alpha}_{x_i}\mathbf{A_0}\right)}{\partial \mathbf{A_0}}\right) \\ -2a^2\left({}_{a}D^{\alpha}_{x_i}\bar{\mathbf{A}}^{i} - {}_{a}D^{\alpha}_{x_i}\ {}_{a}D^{\alpha}_{x^i}\mathbf{A_0}\right)\left(\dfrac{\partial\left({}_{a}D^{\alpha}_{x_k}\bar{\mathbf{A}}^{k} - {}_{a}D^{\alpha}_{x_i}\ {}_{a}D^{\alpha}_{x^i}\mathbf{A_0}\right)}{\partial \mathbf{A_0}}\right)\end{bmatrix}dt \tag{79}$$

$$dp^0 = \left[-\ {}_{a}D^{\alpha}_{x_i}\left(\bar{\mathbf{A}}^{i} - {}_{a}D^{\alpha}_{x^i}\mathbf{A_0}\right) - 2a^2\ {}_{a}D^{\alpha}_{x^k}\ {}_{a}D^{\alpha}_{x^k}\left({}_{a}D^{\alpha}_{x_i}\bar{\mathbf{A}}^{i} - {}_{a}D^{\alpha}_{x_i}\ {}_{a}D^{\alpha}_{x^i}\mathbf{A_0}\right)\right]dt \tag{80}$$

Finally for π^i we have:

$$d\pi^{i} = -\frac{\boldsymbol{\partial} H'_0}{\boldsymbol{\partial}\bar{\mathbf{A}}_{\mathbf{i}}}dt - \frac{\boldsymbol{\partial} H'_1}{\boldsymbol{\partial}\bar{\mathbf{A}}_{\mathbf{i}}}d\bar{\mathbf{A}}_{\mathbf{0}} - \frac{\boldsymbol{\partial} H'_2}{\boldsymbol{\partial}\bar{\mathbf{A}}_{\mathbf{i}}}d\mathbf{A_0} = -\frac{\boldsymbol{\partial} H'_0}{\boldsymbol{\partial}\bar{\mathbf{A}}_{\mathbf{i}}}dt = -\frac{\boldsymbol{\partial} H_c}{\boldsymbol{\partial}\bar{\mathbf{A}}_{\mathbf{i}}}dt \tag{81}$$

$$d\pi^{i} = -\int d^3x\begin{bmatrix} p^j\dfrac{\partial \bar{\mathbf{A}}_{\boldsymbol{j}}}{\partial \mathbf{A}_{\boldsymbol{i}}} + \left(\bar{\mathbf{A}}^{\boldsymbol{j}} - {}_{a}D^{\alpha}_{x^j}\mathbf{A_0}\right)\left(\dfrac{\partial\left(\bar{\mathbf{A}}_{\boldsymbol{j}} - {}_{a}D^{\alpha}_{x_j}\mathbf{A_0}\right)}{\partial \mathbf{A_0}}\right) \\ -2a^2\left({}_{a}D^{\alpha}_{x_j}\bar{\mathbf{A}}^{\boldsymbol{j}} - {}_{a}D^{\alpha}_{x_j}\ {}_{a}D^{\alpha}_{x^j}\mathbf{A_0}\right)\left(\dfrac{\partial\left({}_{a}D^{\alpha}_{x_k}\bar{\mathbf{A}}^{\boldsymbol{k}} - {}_{a}D^{\alpha}_{x_k}\ {}_{a}D^{\alpha}_{x^k}\mathbf{A_0}\right)}{\partial \mathbf{A_0}}\right)\end{bmatrix}dt \tag{82}$$

$$\begin{aligned} &d\pi^{i} \\ &= \left[-p^j - \left({}_{a}D^{\alpha}_{t}\mathbf{A}^{\boldsymbol{i}} - {}_{a}D^{\alpha}_{x^i}\mathbf{A}^{\mathbf{0}}\right)\right. \\ &- 2a^2\ {}_{a}D^{\alpha}_{x^i}\ {}_{a}D^{\alpha}_{x_k}\left({}_{a}D^{\alpha}_{t}\mathbf{A}^{\boldsymbol{k}}\right. \\ &\left.\left.-\ {}_{a}D^{\alpha}_{x^k}\mathbf{A}^{\mathbf{0}}\right)\right]dt \end{aligned} \tag{83}$$

Equations (77), (80), and (83) are completely equivalent to equations (66), (67), and (68); as a result, equations (80) and (83) have the secondary limit that is absent from the total differential equations.

7 Conclusion

We obtained a set of Hamilton-Jacobi partial differential equations in terms of these variables, and we used this set to obtain the system's equations of motion as total differential equations for the characteristics. The presence of constraints (first class or second class) will play a crucial role in ensuring that these conditions are satisfied, since these total differential equations must satisfy integrability conditions. These integrability conditions are also being investigated, as is the generalization of the existing formalism to Lagrangians of order greater than two. We found that a generalization of Hamilton-Jacobi fractional formalism yields findings that are consistent with Euler fractional formalism. Our findings would match those obtained in [29, 30] for the case n ⟶ 1.

References

[1] K. B. Oldham and J. Spanier, *The fractional calculus*, Academic, Press, New-York-London, 1974.

[2] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives*, Gordon and Breach , Yverdon Yverdon-les-Bains, Switzerland, 1993.

[3] Y. M. Alawaideh and B. M. Al-khamiseh, The new formulation of Hamiltonian second order continuous systems of Riemann-Liouville fractional derivatives, *Phys. Scripta*, 125210 (2022)

[4] O. P. Agrawal, Analytical solution for stochastic response of a fractionally damped beam, *J. Vibr. Acoust.* **126**, 561-566 (2004).

[5] D. Baleanu and A. Fernandez, On fractional operators and their classifications, *Math.* **7**, 830 (2019).

[6] T. F. Nonnenmacher, Fractional integral and differential equations for a class of Levy-type probability densities, *J. Phys. A: Math. Gen.* **23**, L697S (1990).

[7] F. Riewe, Nonconservative lagrangian and hamiltonian mechanics, *Phys. Rev. E.* **53**, (1996).

[8] F. Riewe, Mechanics with fractional derivatives, *Phys. Rev. E.* **55**, 3581 (1997).
[9] D. Baleanu, A. Kashuri, P. O. Mohammed and B. Meftah, General Raina fractional integral inequalities on coordinates of convex functions, *Adv. Differ. Equ.*, 1-23 (2021).
[10] Y. Alawaideh and B. Al-Khamiseh, Hamilton formulation for generalized Proca electrodynamics using fractional derivative, *J. Interdis. Math.*, 1-13 (2022).
[11] A. A. Diab, R. Hijjawi, J. Asad and J. M. Khalifeh, Hamiltonian formulation of classical fields with fractional derivatives: revisited, *Meccanica* **48**, 323-330 (2013).
[12] B. Podolsky and P. Schwed, Review of a generalized electrodynamics, *Rev. Mod. Phy.* **20**, 40 (1948).
[13] B. Podolsky and C. Kikuchi, A generalized electrodynamics part II—quantum, *Phys. Rev.* **65**, 228 (1944).
[14] Y. Alawaideh, B. Al-khamiseh and W. Al-Awaida, A new approach for the generalized Dirac Lagrangian density with Atangana--Baleanu fractional derivative, *J. MESA*. **13**, 497-509 (2022).
[15] S. I. Muslih and H. A. El-Zalan, Hamiltonian formulation of systems with higher order derivatives, *Int. J. Theor. Phy.* **46**, 3150-3158 (2007).
[16] N. McClamroch, Singular systems of differential equations as dynamic models for constrained robot systems, *Proc. 1986 IEEE Int. Conf. Robot. Autom., IEEE.*, 21-28 (1986).
[17] Y. Güler, Integration of singular systems, Nuovo Cim. B , **107**, 1143-1149 (1992).
[18] S. Muslih and Y. Güler, On the integrability conditions of constrained systems, Nuovo Cim. B, **110**, 307-315 (1995).
[19] E. M. Rabei and Y. Güler, Hamilton-Jacobi treatment of second-class constraints, *Phys. Rev. A.* **46**, 3513 (1992).
[20] S.I. Muslih, Canonical path integral quantization of Einstein's gravitational field, *Gen. Rel. Grav.* **34**, 1059-1065 (2002).
[21] S. I. Muslih, On the path integral of constrained systems, AIP Conference Proceedings, *American Ins. Phy.*, 240-247 (2004).
[22] D. Baleanu, Cosmological perturbations in FRW model with scalar field within Hamilton-Jacobi formalism and symplectic projector method, *Centr. Eur. J. Phy.* **4**, 503-510 (2006).
[23] Y. M Alawaideh, Hamiltonian formulation for continuous systems using functional fractional derivatives, M. S. Thesis, The University of Jordan, Jordan, (2019).
[24] M. Bertin, B. Pimentel and P. Pompeia, Formalismo de Hamilton-Jacobi à la Carathéodory, *Revista Bras. Ensino Fís.* **29**, 393-403 (2007).
[25] E. H. Hasan, E. M. Rabei and H. B. Ghassib, Quantization of higher-order constrained Lagrangian systems using the WKB approximation, *Int. J. Theor. Phy.* **43,** 2285-2298 (2004).
[26] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, Princeton university press, 1992.
[27] D. M. Gitman and I. V. Tyutin, *Quantization of singular theories with higher derivatives and theories with time-dependent constraints*, Quantization of Fields with Constraints, Springer, 217-253, 1990.
[28] D. Baleanu, Fractional variational principles in action, *Phys. Scripta* **2009**, 014006 (2009).
[29] H. A. El-Zalan, S. I. Muslih, E. M. Rabei and D. Baleanu, Hamilton formulation for continuous systems with second order derivatives, *Int. J. Theor. Phys.* **47**, 2195-2202 (2008).
[30] C. A. Galvao and B. Pimentel, The canonical structure of Podolsky generalized electrodynamics, *Canad. J. Phy.* **66**, 460-466 (1988).