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Two Dimensional Legendre Wavelets: A Powerful Approach for a Variety of Systems of Integral Equations

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Abstract: In this paper, a numerical method is introduced to solve systems of integral equations. The method consists of reducing the system of integral equations to a system of algebraic one, by considering the solution as a series, in terms of Legendre wavelets with unknown coe?cients on the interval $[0,1]\times[0,1]$. The operational matrices of integration and product and the error bound are calculated for two dimensional Legendre wavelets and at the end some examples are presented to illustrate the efficiency and the simplicity of the method. It is expected the lesser computational costs compared to the other common methods. The results reveal that this method is very effective and can be applied for other problems in different fields of sciences.

Keywords: Mother wavelet, Legendre wavelets method, Operational matrix of integration, Operational matrix of product, Systems of integral equations

1 Introduction

Orthogonal functions and polynomials have been used by many authors for solving various functional equations. The main idea of using an orthogonal basis is that the problem under study reduces to a system of linear or nonlinear algebraic equations. This can be done by truncated series of orthogonal basis functions for the solution of a problem and using the operational matrices. Here we use two dimensional Legendre wavelets basis on interval $[0,1]\times[0,1]$ for solving systems of integral equations. These systems arise in mathematical modeling of many phenomena and some methods have been proposed in the literature for solving these systems on the interval $[0,1]\times[0,1]$ such as the Adomian decomposition method [9,10] and Homotopy perturbation method [11, 12]. It should be noted that global intervals $[a,b]\times[c,d]$ can be converted to the interval $[0,1]\times[0,1]$ by an appropriate change of variable.

This paper is organized as follows: Section 1 is devoted to introduction; in Section 2, the two dimensions Legendre wavelets are introduced and their operational matrices of integration and product are computed; Numerical examples are presented in Section 3; Conclusions are given in the final Section, 4.

2 Two Dimensional Legendre Wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet [1,2,3]. When the dilation parameter, a, and the translation parameter, b, vary continuously, the following family of continuous wavelets will appear,

$$\psi_{a,b}(x)=|a|^{-\frac{1}{2}}\ \psi(\frac{x-b}{a}),\qquad a,b\in\mathbb{R},\quad a\neq 0. \tag{1}$$

If we choose the dilation parameter as a^{-k}, and the translation parameter as nba^{-k}, where $a>1, b>0, n$, and k are positive integer numbers, then we will have the following family of discrete wavelets,

$$\psi_{k,n}(x)=|a|^{\frac{k}{2}}\ \psi(a^k x-nb). \tag{2}$$

These functions generate a wavelet basis for $L^2(\mathbb{R})$ and for special case $a=2$ and $b=1$, the functions $\psi_{k,n}(x)$ generate an orthonormal basis. The family of continuous two-dimensional wavelets is constructed from product of two one-dimensional wavelets as follows,

$$\psi_{k,n,k',n'}(x,y)=\psi_{k,n}(x)\ \psi_{k',n'}(y), \tag{3}$$

where k,n,k', and n' are positive integers. The family $\{\psi_{k,n,k',n'}(x,y)\}$ is a wavelet basis for $L^2(\mathbb{R}^2)$. Two

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dimensional Legendre wavelets are defined in $L^2(\mathbb{R}^2)$ on the interval $[0,1]\times[0,1]$ as follows [7,8],

$$\psi_{n,m,n',m'}(x,y)=\begin{cases}\sqrt{(m+\frac{1}{2})(m'+\frac{1}{2})}2^{\frac{k+k'}{2}}P_m(2^kx-2n+1)P_{m'}(2^{k'}y-2n'+1), & \frac{n-1}{2^{k-1}}\leqslant x<\frac{n}{2^{k-1}},\\ 0, & otherwise,\end{cases}\tag{4}$$

Where,

$$n=1,2,...,2^{k-1},\qquad n'=1,2,...,2^{k-1},$$
$$m=0,1,...,M-1,\qquad m'=0,1,...,M'-1.$$

k, k', M, and M' are any positive integers, m and m' are the degrees of Legendre polynomials. $P_m(x)$ and $P_{m'}(x)$ are the famous Legendre polynomials of orders m and m', which are orthogonal with respect to the weight function $w(x)=1$, on the interval $[-1,1]$. These polynomials satisfy the following differential equation, which named the Legendre differential equation,

$$(1-x^2)P''_m(x)-2xP'_m(x)+n(n+1)P_m(x)=0,\tag{5}$$

and can be obtained by recursive formula as follows,

$$\begin{cases}P_{m+1}(x)=(\frac{2m+1}{m+1})xP_m(x)-(\frac{m}{m+1})P_{m-1}(x),\ m=1,2,...\\ P_0(x)=1,\ \ P_1(x)=x.\end{cases}\tag{6}$$

2.1 Function approximation

The continuous function $f(x,y)$, defined on the interval $[0,1]\times[0,1]$, can be expanded as,

$$f(x,y)\simeq\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n'=0}^{\infty}\sum_{m'=0}^{\infty}c_{n,m,n',m'}\psi_{n,m,n',m'}(x,y).\tag{7}$$

The series representation of $f(x,y)$ in (7) is called a wavelet series and the wavelet coefficients $c_{n,m,n',m'}$ are given by,

$$c_{n,m,n',m'}=\int_{\frac{n'-1}{2^{k'-1}}}^{\frac{n'}{2^{k'-1}}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}f(x,y)\psi_{n,m,n',m'}(x,y)dxdy.\tag{8}$$

The convergence of the series (7) in $L^2([0,1]\times[0,1])$ means that,

$$\lim_{s_1,s_2,s_3,s_4\to\infty}||f(x,y)-\sum_{n=1}^{s_1}\sum_{m=0}^{s_2}\sum_{n'=0}^{s_3}\sum_{m'=0}^{s_4}c_{n,m,n',m'}\psi_{n,m,n',m'}(x,y)||=0.\tag{9}$$

Therefore, one can consider the following truncated series for series (7),

$$f(x,y)\simeq\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{n'=0}^{2^{k'-1}}\sum_{m'=0}^{M'-1}c_{n,m,n',m'}\psi_{n,m,n',m'}(x,y)=C^T\ \Psi(x,y),\tag{10}$$

where C and $\Psi(x,y)$ are $2^{k-1}2^{k'-1}MM'\times 1$ matrices and given by,

$$\begin{aligned}C=[&c_{1,0,1,0},c_{1,0,1,1},...,c_{1,0,1,M'-1},c_{1,0,2,0},c_{1,0,2,1},...,\\ &c_{1,0,2,M'-1},...,c_{1,0,2^{k'-1},0},...,c_{1,0,2^{k'-1},M'-1},\cdots\\ &c_{2^{k-1},M-1,2^{k'-1},M'-1}]^T\\ =&[c_1,c_2,c_3,...,c_{2^{k-1}2^{k'-1}MM'}]^T\end{aligned}\tag{11}$$

and

$$\begin{aligned}\Psi(x,y)=[&\psi_{1,0,1,0},\psi_{1,0,1,1},...,\psi_{1,0,1,M'-1},\psi_{1,0,2,0},\psi_{1,0,2,1},...,\\ &\psi_{1,0,2,M'-1},...,\psi_{1,0,2^{k'-1},0},...,\psi_{1,0,2^{k'-1},M'-1},\cdots\\ &\psi_{2^{k-1},M-1,2^{k'-1},M'-1}]^T\\ &=[\psi_1(x,y),\psi_2(x,y),\psi_3(x,y),...,\psi_{2^{k-1}2^{k'-1}MM'}(x,y)]^T\end{aligned}\tag{12}$$

The integration of the product of two Legendre wavelets vector functions leads to the following matrix,

$$\int_0^1\int_0^1\Psi(x,y)\ \Psi^T(x,y)\ dx\ dy=I,\tag{13}$$

where I is an identity matrix.

2.2 The operational matrix of integration for x

In [6] the operational matrix of integration for one dimensional Legendre wavelets has been computed and in [7] the operational matrices of integration for two dimensional Legendre wavelets have been computed that are wrong. Now, we want to achieve the operational matrix of integration for two dimensional Legendre wavelets correctly. The integration of the vector $\Psi(x,y)$ with respect to variable x, defined by (12), can be achieved as,

$$\int_0^x\Psi(x',y)\ dx'\simeq P_x\ \Psi(x,y),\tag{14}$$

where P_x is the $2^{k-1}2^{k'-1}MM'\times 2^{k-1}2^{k'-1}MM'$ operational matrix for the variable x. Frist, for example, we find matrix P_x for $k=k'=M=M'=2$. There are sixteen basis functions as follows,

$$\begin{cases}\psi_{1010}=2\\ \psi_{1011}=2\sqrt{3}(4y-1)\\ \psi_{1110}=2\sqrt{3}(4x-1)\\ \psi_{1111}=6(4x-3)(4y-1)\end{cases}\qquad 0\leqslant x,y<\frac{1}{2},$$

$$\begin{cases}\psi_{1020}=2\\ \psi_{1021}=2\sqrt{3}(4y-3)\\ \psi_{1120}=2\sqrt{3}(4x-1)\\ \psi_{1121}=6(4x-1)(4y-3)\end{cases}\qquad 0<x<\frac{1}{2},\ \frac{1}{2}\leqslant y<1,$$

$$\begin{cases} \psi_{2020} = 2 \\ \psi_{2021} = 2\sqrt{3}(4y-3) \\ \psi_{2120} = 2\sqrt{3}(4x-3) \\ \psi_{2121} = 6(4x-3)(4y-3) \end{cases} \qquad \frac{1}{2} \leqslant x, y < 1,$$

$$\begin{cases} \psi_{2010} = 2 \\ \psi_{2011} = 2\sqrt{3}(4y-1) \\ \psi_{2110} = 2\sqrt{3}(4x-3) \\ \psi_{2111} = 6(4x-3)(4y-1) \end{cases} \qquad \frac{1}{2} \leqslant x < 1, 0 \leqslant y < \frac{1}{2}.$$

By integrating the above equations from 0 to x we have,

$$\int_0^x \Psi_{16}(x', y)\, dx' \simeq P_{x_{16\times 16}}\, \Psi_{16}(x, y), \tag{15}$$

where

$$\begin{aligned} \Psi_{16}(x,y) = [&\psi_{1010}, \psi_{1011}, \psi_{1020}, \psi_{1021}, \psi_{1110}, \psi_{1111}, \\ &\psi_{1120}, \psi_{1121}, \psi_{2010}, \psi_{2011}, \psi_{2020}, \psi_{2021}, \\ &\psi_{2110}, \psi_{2111}, \psi_{2120}, \psi_{2121}]^T, \end{aligned}$$

and

$$P_{x_{16\times 16}} = \frac{1}{4} \begin{bmatrix}
1 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
-\frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0
\end{bmatrix}.$$

The matrix $P_{x_{16\times 16}}$ can be rewritten as follows,

$$P_{x_{16\times 16}} = \frac{1}{4} \begin{bmatrix} L_{8\times 8} & F_{8\times 8} \\ O_{8\times 8} & L_{8\times 8} \end{bmatrix}, \tag{16}$$

where,

$$L_{8\times 8} = \begin{bmatrix} I_{4\times 4} & \frac{\sqrt{3}}{3} I_{4\times 4} \\ -\frac{\sqrt{3}}{3} I_{4\times 4} & O'_{4\times 4} \end{bmatrix}, \tag{17}$$

$$F_{8\times 8} = \begin{bmatrix} 2I_{4\times 4} & O'_{4\times 4} \\ O'_{4\times 4} & O'_{4\times 4} \end{bmatrix}, \tag{18}$$

I is the identity matrix, O and O' are the null matrices. Therefore, the matrix P_x can be presented, in general, as follows,

$$P_x = \frac{1}{2^k} \begin{bmatrix} L & F & F & \cdots & F \\ O & L & F & \cdots & F \\ O & O & L & \cdots & F \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & L \end{bmatrix}, \tag{19}$$

where O is the $2^{k'-1}MM' \times 2^{k'-1}MM'$ null matrix, L and F are the following $2^{k'-1}MM' \times 2^{k'-1}MM'$ matrices,

$$L = \begin{bmatrix} I & \frac{\sqrt{3}}{3} I & O' & \cdots & O' \\ -\frac{\sqrt{3}}{3} I & O' & \frac{\sqrt{3}}{3\sqrt{5}} I & \cdots & O' \\ O' & \frac{\sqrt{5}}{5\sqrt{3}} I & O' & \cdots & O' \\ \vdots & \vdots & \vdots & \ddots & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} I \\ O' & O' & \cdots & \frac{\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} I & O' \end{bmatrix}, \tag{20}$$

$$F = \begin{bmatrix} 2I & O' & O' & \cdots & O' \\ O' & O' & O' & \cdots & O' \\ O' & O' & O' & \cdots & O' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O' & O' & O' & \cdots & O' \end{bmatrix}. \tag{21}$$

I is the $2^{k'-1}M' \times 2^{k'-1}M'$ identity matrix, and O' is the $2^{k'-1}M' \times 2^{k'-1}M'$ null matrix.

2.3 The operational Matrix of integration for y

The integration of the vector $\Psi(x,y)$ with respect to variable y, is computed as follows,

$$\int_0^y \Psi(x, y')\, dy' \simeq P_y\, \Psi(x, y), \tag{22}$$

where P_y is the $2^{k-1}2^{k'-1}MM' \times 2^{k-1}2^{k'-1}MM'$ operational matrix for the variable y. Similar to part (2.2), the matrix P_y for $k = k' = M = M' = 2$ will be found. This matrix is obtained as follows,

$$P_{y_{16\times 16}} = \frac{1}{4} \begin{bmatrix}
1 & \frac{\sqrt{3}}{3} & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & \frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \frac{\sqrt{3}}{3} & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{\sqrt{3}}{3} & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{\sqrt{3}}{3} & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{3} & 0
\end{bmatrix}. \tag{23}$$

Therefore, in general, the matrix P_y can be presented as follows,

$$P_y = \frac{1}{2^{k'}} \begin{bmatrix} P & O' & O' & \cdots & O' \\ O' & P & O' & \cdots & O' \\ O' & O' & P & \cdots & O' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O' & O' & O' & \cdots & P \end{bmatrix}, \tag{24}$$

where O' is the $2^{k'-1}M' \times 2^{k'-1}M'$ null matrix and P is $2^{k'-1}M' \times^{k'-1} M'$ matrix and given by,

$$P = \begin{bmatrix} L & F & F & \cdots & F \\ O & L & F & \cdots & F \\ O' & O' & L & \cdots & F \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O' & O' & O' & \cdots & L \end{bmatrix}, \tag{25}$$

Here L, F, and O are $M' \times M'$ matrices as follows,

$$L = \begin{bmatrix} 1 & \frac{\sqrt{3}}{3} & 0 & \cdots & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & \cdots & 0 \\ 0 & \frac{\sqrt{5}}{5\sqrt{3}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & 0 & \cdots & \frac{\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0 \end{bmatrix}, \tag{26}$$

$$F = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{27}$$

$$O = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{28}$$

2.4 The operational Matrix of product

The property of the product of two Legendre wavelets vector functions will be as follows,

$$\Psi(x,y)\,\Psi^t(x,y) \simeq \tilde{C}\,\Psi(x,y), \tag{29}$$

where C is the vector defined in (11) and $\tilde{C}$ is a $2^{k-1}2^{k'-1}MM' \times 2^{k-1}2^{k'-1}MM'$ matrix. This matrix is called the operational matrix of product, which in solving the integral equations is applied, and defined as follows,

$$\tilde{C} = \begin{bmatrix} \tilde{C}_1 & O & O & \cdots & O \\ O & \tilde{C}_2 & O & \cdots & O \\ O & O & \tilde{C}_3 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & O & \tilde{C}_{2^{k-1}} \end{bmatrix}, \tag{30}$$

where O is the $2^{k'-1}MM' \times 2^{k'-1}MM'$ null matrix and $\tilde{C}_i, i = 1,2,...,2^{k'-1}$ are the $2^{k'-1}MM' \times 2^{k'-1}MM'$ symmetric matrices as,

$$\tilde{C}_i = \begin{bmatrix} S^i_{11} & S^i_{12} & \cdots & S^i_{1M} \\ S^i_{21} & S^i_{22} & \cdots & S^i_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ S^i_{1M} & S^i_{2M} & \cdots & S^i_{MM} \end{bmatrix}, \quad i = 1,2,...,2^{k-1}. \tag{31}$$

In (31), S^i_{tj}, $i = 1,2,...,2^{k-1}$, $t, j = 1,2,...,M$, are the $2^{k'-1}M' \times 2^{k'-1}M'$ diagonal matrices which given by,

$$S^i_{tj} = \begin{bmatrix} \tilde{S}_1^{\,i\,tj} & O' & \cdots & O' \\ O' & \tilde{S}_2^{\,i\,tj} & \cdots & O' \\ \vdots & \vdots & \ddots & \vdots \\ O' & O' & \cdots & \tilde{S}_{2^{k'-1}}^{\,i\,tj} \end{bmatrix}, \tag{32}$$
$$i = 1,2,...,2^{k-1}, \quad t, j = 1,2,...,M,$$

where O' is the $M' \times M'$ null matrix, and $\tilde{S}_r^{\,i\,tj}$, $r = 1,2,...,2^{k'-1}$, $i = 1,2,...,2^{k^{-}1}$, $t, j = 1,2,...,M$, are $M' \times M'$ symmetric matrices with the following entries

$$(\tilde{S}_r^{\,i\,tj})_{s,w} = \Big((\Psi\,\Psi^T C)_n, \Psi_m \Big),$$
$$r = 1,2,...,2^{k'-1}, \quad i = 1,2,...,2^{k-1}, \tag{33}$$
$$t, j = 1,2,...,M, \quad s, w = 1,2,...,M',$$

where $n = (i-1)2^{k'-1}MM' + (t-1)2^{k'-1}M' + (r-1)M' + s$ and $m = (i-1)2^{k'-1}MM' + (j-1)2^{k'-1}M' + (r-1)M' + w$. For example, the operational matrix of product for $k = k' = M = M' = 2$ can be presented as follows,

$$\tilde{C}_{16\times16} = \begin{bmatrix} \tilde{C}_1 & O \\ O & \tilde{C}_2 \end{bmatrix}, \tag{34}$$

where,

$$\tilde{C}_1 = \begin{bmatrix} 2c_1 & 2c_2 & 0 & 0 & 2c_5 & 2c_6 & 0 & 0 \\ 2c_2 & 2c_1 & 0 & 0 & 2c_6 & 2c_5 & 0 & 0 \\ 0 & 0 & 2c_3 & 2c_4 & 0 & 0 & 2c_7 & 2c_8 \\ 0 & 0 & 2c_4 & 2c_3 & 0 & 0 & 2c_8 & 2c_7 \\ 2c_5 & 2c_6 & 0 & 0 & 2c_1 & 2c_2 & 0 & 0 \\ 2c_6 & 2c_5 & 0 & 0 & 2c_2 & 2c_1 & 0 & 0 \\ 0 & 0 & 2c_7 & 2c_8 & 0 & 0 & 2c_3 & 2c_4 \\ 0 & 0 & 2c_8 & 2c_7 & 0 & 0 & 2c_4 & 2c_3 \end{bmatrix}, \tag{35}$$

and

$$\tilde{C}_1 = \begin{bmatrix} 2c_9 & 2c_{10} & 0 & 0 & 2c_{13} & 2c_{14} & 0 & 0 \\ 2c_{10} & 2c_9 & 0 & 0 & 2c_{14} & 2c_{13} & 0 & 0 \\ 0 & 0 & 2c_{11} & 2c_{12} & 0 & 0 & 2c_{15} & 2c_{16} \\ 0 & 0 & 2c_{12} & 2c_{11} & 0 & 0 & 2c_{16} & 2c_{15} \\ 2c_{13} & 2c_{14} & 0 & 0 & 2c_9 & 2c_{10} & 0 & 0 \\ 2c_{14} & 2c_{13} & 0 & 0 & 2c_{10} & 2c_9 & 0 & 0 \\ 0 & 0 & 2c_{15} & 2c_{16} & 0 & 0 & 2c_{11} & 2c_{12} \\ 0 & 0 & 2c_{16} & 2c_{15} & 0 & 0 & 2c_{12} & 2c_{11} \end{bmatrix}. \tag{36}$$

3 Error analysis

In this section, the error bound of the approximate solution via Legendre wavelets series is illustrated via the following theorem.

Theorem 3.1. Suppose that $f(x,y) \in C^M[0,1] \times C^M[0,1]$ and $C^T\ \Psi(x,y)$ is the approximate solution via two dimensional Legendre wavelets method. Then, the error bound can be presented as follows,

$$||f(x,y)-C^T\ \Psi(x,y)|| \leqslant \frac{D}{P!2^{P(K-2)}}, \tag{37}$$

where $P=M+M'$,

$$D=\max\left\{ \max_{\eta_{x_i},\eta_{y_i}\in[0,1]\times[0,1]}\left|\left|\frac{\partial^P f(\eta_{x_i},\eta_{y_i})}{\partial x^{P-i}\partial y^i}\right|\right|\ \middle|\ i=0,1,...,P\right\},$$

and $K=\min\{k_1,k_2\}$.

Proof. By taking norm, we have,

$$||f(x,y)-C^T\ \Psi(x,y)||^2=\int_0^1\int_0^1 (f(x,y)-C^T\ \Psi(x,y))^2\ dx\ dy. \tag{38}$$

The interval $[0,1]\times[0,1]$ can be divided into $2^{k+k'-2}$ subintervals as $I_{nn'}=\left[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}}\right]\times\left[\frac{n'-1}{2^{k'-1}},\frac{n'}{2^{k'-1}}\right]$ that the function $f(x,y)$ can be approximated on these subintervals by Legendre wavelets method, as a polynomial at most of $(M-1)$th degree with respect to x and at most of M'th degree with respect to y with the least-square property. So, one has,

$$\begin{aligned}
&\int_0^1\int_0^1 (f(x,y)-C^T\ \Psi(x,y))^2\ dx\ dy\\
&=\sum_{n'=1}^{2^{k'-1}}\sum_{n=1}^{2^{k-1}}\int_{\frac{n'-1}{2^{k'-1}}}^{\frac{n'}{2^{k'-1}}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}(f(x,y)-C^T\ \Psi(x,y))^2\ dx\ dy\\
&\leqslant\sum_{n'=1}^{2^{k'-1}}\sum_{n=1}^{2^{k-1}}\int_{\frac{n'-1}{2^{k'-1}}}^{\frac{n'}{2^{k'-1}}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}(f(x,y)-S_P(x,y))^2\ dx\ dy,
\end{aligned} \tag{39}$$

$$\begin{aligned}
&||f(x,y)-S_P(x,y)||\\
&\leqslant\sum_{i=0}^{P}\frac{|x-\frac{2n-1}{2^k}|^{P-i}|y-\frac{2n'-1}{2^{k'}}|^i}{(P-i)!\ i!}\max_{(\xi_{x_i},\xi_{y_i})\in I_{nn'}}\left|\left|\frac{\partial^P f(\xi_{x_i},\xi_{y_i})}{\partial x^{P-i}\partial y^i}\right|\right|.
\end{aligned} \tag{40}$$

On the subintervals $I_{nn'}$ one has,

$$\begin{gathered}
|x-\frac{2n-1}{2^k}|^{P-i}\leqslant\frac{1}{2^{k-1}},\\
|y-\frac{2n'-1}{2^{k'}}|^{i}\leqslant\frac{1}{2^{k'-1}},\\
\max_{(\xi_{x_i},\xi_{y_i})\in I_{nn'}}\left|\left|\frac{\partial^P f(\xi_{x_i},\xi_{y_i})}{\partial x^{P-i}\partial y^i}\right|\right|\leqslant\max_{(\eta_{x_i},\eta_{y_i})\in[0,1]\times[0,1]}\left|\left|\frac{\partial^P f(\eta_{x_i},\eta_{y_i})}{\partial x^{P-i}\partial y^i}\right|\right|=M_i,\\
i=0,1,...,P.
\end{gathered}$$

Using the above inequalities for (40) we would have,

$$\begin{aligned}
&||f(x,y)-S_P(x,y)||\\
&\qquad\leqslant\sum_{i=0}^{P}\frac{M_i}{(P-i)!\ i!\ 2^{(P-i)(k-1)}2^{i(k'-1)}}.
\end{aligned} \tag{41}$$

By considering $M=\max\{M_1,M_2,...,M_P\}$, one has,

$$||f(x,y)-S_P(x,y)||\leqslant\frac{M}{P!}\left(\frac{1}{2^{k-1}}+\frac{1}{2^{k'-1}}\right)^2. \tag{42}$$

Whereas $K=\min\{k_1,k_2\}$, the following result is obtained,

$$||f(x,y)-S_P(x,y)||\leqslant\frac{D}{P!\ 2^{P(K-2)}}. \tag{43}$$

And finally, by substituting (43) into (39) the reliable result will be acquired as follows,

$$\begin{aligned}
&||f(x,y)-C^T\ \Psi(x,y)||\\
&\leqslant\sum_{n'=1}^{2^{k'-1}}\sum_{n=1}^{2^{k-1}}\int_{\frac{n'-1}{2^{k'-1}}}^{\frac{n'}{2^{k'-1}}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\left(\frac{D}{P!\ 2^{P(K-2)}}\right)^2\ dx\ dy\\
&=\int_0^1\int_0^1\left(\frac{D}{P!\ 2^{P(K-2)}}\right)^2\ dx\ dy\\
&=\left(\frac{D}{P!\ 2^{P(K-2)}}\right)^2. \qquad\Box
\end{aligned}$$

The above theorem implies when P increases the error decreases. In other words, when $P\to\infty$ one has $||f(x,y)-C^T\ \Psi(x,y)||\to\infty$.

4 Numerical Examples

In this section, some examples are considered for systems of integral equations on the interval $[0,1]\times[0,1]$ and will be solved by using the two dimensional Legendre wavelets method. The method consists of expanding the solution as a series in terms of two dimensional Legendre wavelets with unknown coefficients and reducing the system of integral equations to a system of algebraic equations. These examples are solved for $k=k'=1$ and $M=M'=4$. **Example 4.1.** Consider the following system of Volterra integral equations,

$$\begin{cases} u(x,y)=\frac{3}{2}x^2\ y+\frac{1}{3}x^3\ y-\int_0^x(u(t,y)+v(t,y))\ dt,\\ v(x,y)=x\ y-\frac{1}{2}x^2\ y+\frac{1}{3}x^3\ y+\int_0^x(v(t,y)-u(t,y))\ dt,\\ \qquad 0\leqslant x,y\leqslant 1.\end{cases} \tag{44}$$

The exact solutions are $u(x,y)=x^2\ y$ and $v(x,y)=x\ y$. According to the wavelets method, let's consider the following approximations,

$$\begin{gathered}
u(x,y)\simeq C_1^T\ \Psi(x,y),\quad v(x,y)\simeq C_2^T\ \Psi(x,y),\\
\frac{3}{2}x^2\ y+\frac{1}{3}x^3\ y\simeq F_1^T\ \Psi(x,y),\quad x\ y-\frac{1}{2}x^2\ y+\frac{1}{3}x^3\ y\simeq F_2^T\ \Psi(x,y),
\end{gathered}$$

where C_1, C_2, F_1, F_2, and $\Psi(x,y)$ are 16×1 matrices. Substituting the above approximations into the system (44) leads to the following system,

$$\begin{cases} C_1^T-F_1^T+(C_1^T+C_2^T)P_x\approx 0,\\ C_2^T-F_2^T-(C_2^T-C_1^T)P_x\approx 0.\end{cases} \tag{45}$$

By solving the system (45), the unknown coefficients can be determined as follows,

$$
\begin{aligned}
& c_{1,1}=\frac{1}{6},\ c_{1,2}=\frac{\sqrt{3}}{18},\ c_{1,3}=0,\ c_{1,4}=0,\ c_{1,5}=\frac{\sqrt{3}}{12},\\
& c_{1,6}=\frac{1}{12},\ c_{1,7}=0,\ c_{1,8}=0,\ c_{1,9}=\frac{\sqrt{5}}{60},\ c_{1,10}=\frac{\sqrt{15}}{180},\\
& c_{1,11}=0,\ c_{1,12}=0,\ c_{1,13}=0,\ c_{1,14}=0,\ c_{1,15}=0,\\
& c_{1,16}=0,\\
& c_{2,1}=\frac{1}{4},\ c_{2,2}=\frac{\sqrt{3}}{12},\ c_{2,3}=0,\ c_{2,4}=0,\ c_{2,5}=\frac{\sqrt{3}}{12},\\
& c_{2,6}=\frac{1}{12},\ c_{2,7}=0,\ c_{2,8}=0,\ c_{2,9}=0,\ c_{2,10}=\frac{\sqrt{15}}{180},\\
& c_{2,11}=0,\ c_{2,12}=0,\ c_{2,13}=0,\ c_{2,14}=0,\ c_{2,15}=0,\\
& c_{2,16}=0.
\end{aligned}
$$

Therefore, the solutions will be achieved as follows,

$$
\begin{aligned}
u(x,y) &= \sum_{i=1}^{16} c_{1,i}\ \psi_i(x,y) = x^2\, y,\\
v(x,y) &= \sum_{i=1}^{16} c_{2,i}\ \psi_i(x,y) = x\, y,
\end{aligned}
$$

which are the exact solutions.

Example 4.2. Consider the following linear system of Volterra-Fredholm integral equations, with the exact solutions $u(x,y)=x^2+x\,y$ and $v(x,y)=x^2\,y-y^2\,e^x$

$$
\begin{cases}
u(x,y)=x^2+x\,y-y+\frac{1}{3}y^3(e-1)-\frac{11}{12}y^2\\
\qquad\qquad +\int_0^y\int_0^1 (3u(t,s)+v(t,s))\,dt\,ds,\\
v(x,y)=x^2\,y-y^2\,e^x+\frac{1}{3}y+\frac{2}{3}y^3(1-e)+\frac{7}{12}y^2\\
\qquad\qquad -\int_0^y\int_0^1 (u(t,s)+2v(t,s))\,dt\,ds, \qquad 0\leqslant x,y\leqslant 1.
\end{cases}
\tag{46}
$$

By using the proposed method for system (46), the following algebraic system will be obtained,

$$
\begin{cases}
C_1^T-F_1^T-(3C_1^T+C_2^T)K\,P_y\approx 0,\\
C_2^T-F_2^T+(C_2^T+2C_1^T)K\,P_y\approx 0,
\end{cases}
\tag{47}
$$

Where matrix K can be computed as follows,

$$
\int_0^1 \Psi(t,y)\,dt \simeq K\,\Psi(x,y). \tag{48}
$$

The following values for the entries of the vectors C_1 and C_2 are resulted by solving the system (47), which consists of 32 equations for the same number of unknowns,

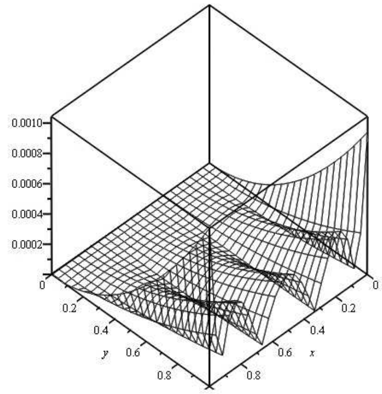

Fig. 1: The absolute error function of $v(x,y)$ in Example 4.2

$$
\begin{aligned}
& c_{1,1}=\frac{7}{12},\ c_{1,2}=\frac{\sqrt{3}}{12},\ c_{1,3}=0,\ c_{1,4}=0,\ c_{1,5}=\frac{\sqrt{3}}{4},\\
& c_{1,6}=\frac{1}{12},\ c_{1,7}=0,\ c_{1,8}=0,\ c_{1,9}=\frac{\sqrt{5}}{30},\ c_{1,10}=\frac{\sqrt{15}}{180},\\
& c_{1,11}=0,\ c_{1,12}=0,\ c_{1,13}=0,\ c_{1,14}=0,\ c_{1,15}=0,\\
& c_{1,16}=0,\\
& c_{2,1}=-\frac{e}{3}+\frac{1}{2},\ c_{2,2}=-\frac{\sqrt{3}}{18}(3e-4),\ c_{2,3}=-\frac{\sqrt{5}}{30}(e-1),\\
& c_{2,4}=0,\ c_{2,5}=\frac{\sqrt{3}}{12}(4e-11),\ c_{2,6}=\frac{e}{2}-\frac{17}{12},\\
& c_{2,7}=\frac{\sqrt{5}}{60}(140e-381),\ c_{2,8}=0,\ c_{2,9}=-\frac{\sqrt{5}}{60}(140e-381),\\
& c_{2,10}=-\frac{\sqrt{15}}{180}(210e-571),\ c_{2,11}=\frac{7}{6}e+\frac{19}{6},\ c_{2,12}=0,\\
& c_{2,13}=\frac{\sqrt{7}}{3}(71e-193),\ c_{2,14}=\frac{\sqrt{21}}{6}(71e-193),\\
& c_{2,15}=\frac{\sqrt{35}}{30}(71e-193),\ c_{2,16}=0.
\end{aligned}
$$

So, the following approximate solutions will be obtained,

$$
\begin{aligned}
u(x,y) &= \sum_{i=1}^{16} c_{1,i}\ \psi_i(x,y) = x^2+x\,y,\\
v(x,y) &= \sum_{i=1}^{16} c_{2,i}\ \psi_i(x,y) = y(x^2+1456y-16800x\,y\\
&\quad -27020x^3\,y+41100y\,x^2-536y\,e-15120x^2\,y\,e\\
&\quad +6180x\,y\,e+9940x^3\,y\,e),
\end{aligned}
$$

The absolute error function of $v(x,y)$ is plotted in Fig. 1.

Example 4.3. Consider the following nonlinear system of Fredholm integral equations,

$$\begin{cases} u(x,y) = x^2\cos(y) - y\sin(x) - y\cos(y)\sin(1) \\ \qquad +2y\cos(y)\cos(1) + v(x,y) \\ \qquad -\int_0^1 u(t,y)\frac{\partial v(t,y)}{\partial t}\,dt, \\ v(x,y) = y\sin(x) + 3x^2\cos(y) - 2y\cos(y)\sin(1) \\ \qquad +2y\cos(y)\cos(1) - 3u(x,y) \\ \qquad +\int_0^1 v(t,y)\frac{\partial u(t,y)}{\partial t}\,dt, \qquad 0 \leqslant x,y \leqslant 1. \end{cases} \tag{49}$$

With the exact solutions $u(x,y) = x^2\cos(y)$ and $v(x,y) = y\sin(x)$ on the interval $[0,1]\times[0,1]$. First, let's consider the following approximations,

$$\frac{\partial u(x,y)}{\partial x} \simeq C_1^T\ \Psi(x,y), \quad u(x,y) \simeq C_1^T\ P_x\ \Psi(x,y),$$
$$\frac{\partial v(x,y)}{\partial x} \simeq C_2^T\ \Psi(x,y), \quad v(x,y) \simeq C_2^T\ P_x\ \Psi(x,y),$$
$$u(x,y)\frac{\partial v(x,y)}{\partial x} \simeq Y_1^T\ \Psi(x,y),$$
$$\frac{\partial u(x,y)}{\partial x}v(x,y) \simeq Y_2^T\ P_x\ \Psi(x,y),$$
$$x^2\cos(y) - y\sin(x) - y\cos(y)\sin(1) + 2y\cos(y)\cos(1) \simeq F_1^T\ \Psi(x,y),$$
$$y\sin(x) + 3x^2\cos(y) - 2y\cos(y)\sin(1) + 2y\cos(y)\cos(1) \simeq F_2^T\ \Psi(x,y)$$

Substituting into the system (49) results the following system,

$$\begin{cases} (C_1^T - C_2^T)P_x - F_1^T + Y_1^T\ K \approx 0, \\ (3C_1^T + C_2^T)P_x - F_2^T - Y_2^T\ K \approx 0. \end{cases} \tag{50}$$

Where matrix K can be obtained by (48). Therefore, one gets the following approximate solutions,

$$\begin{aligned} u(x,y) = {} & (3.055050463\times10^{-39} - 6.110100928\times10^{-39}y)x^3 \\ & + (0.07880040004y^3 - 0.5492104648y^2 \\ & + 0.9994691925 + 0.0107436901y)x^2 \\ & + (7.016\times10^{-8}y - 7.52\times10^{-8}y - 8.104\times10^{-9} \\ & - 4\times10^{-11}y^3)x - 0.0003740641413y^3 \\ & + 0.0005751663880y^2 - 0.0002360041636y \\ & + 0.00002049300335 \end{aligned}$$

$$\begin{aligned} v(x,y) = {} & (5\times10^{-11} - 0.1442434000)x^3 + (1.7\times10^{-11}y^3 \\ & - 4.5107\times10^{-8}y^2 - 7.1873\times10^{9} - 0.0190966498y)x^2 \\ & + 0.9994691925 + 0.0107436901y)x^2 \\ & + (4.5103\times10^{-8}y^2 + 5.8875\times10^{-9} + 1.004753694y \\ & - 2.4\times10^{-11}y^3)x - 0.0005792916244y^3 \\ & + 0.0008741147659y^2 - 0.0006246130433y \\ & + 0.0000277346412. \end{aligned}$$

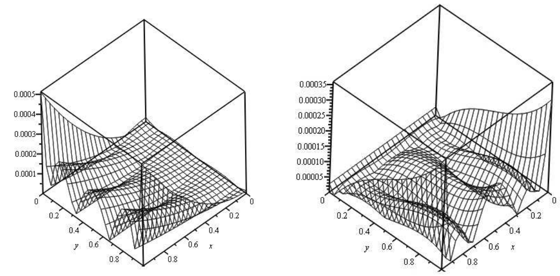

Fig. 2: The absolute error function of $u(x,y)$ (left), The absolute error function of $v(x,y)$ (right) in Example 4.3

The absolute error functions of approximate solutions are plotted in Figs. 2 and 3.

Example 4.4. Consider the following nonlinear system of Volterra-Fredholm integral equations with the exact solutions $u(x,y) = -\frac{x}{2}\sin(y)$ and $v(x,y) = y\tan(x)$ [9, 11],

$$\begin{cases} u(x,y) = -\frac{1}{6}(x^2+y^2)(y\cos(y)+\sin(y)) - \frac{1}{2}x\sin(y) \\ \qquad +(x^2+y^2)\int_0^y\int_0^1 t\ su(t,s)\ dt\ ds, \\ v(x,y) = 0.14726y^3(y-x) + y\tan(x) \\ \qquad +(x-y)\int_0^y\int_0^1 t\ v(t,s)\ dt\ ds, \qquad 0 \leqslant x,y \leqslant 1. \end{cases} \tag{51}$$

By applying the two dimensional Legendre wavelets approach to this system, the following results would be obtained,

$$\begin{aligned} u(x,y) = {} & (6.399548864\times10^{-7} - 0.00001226982627y \\ & + 0.00003752203787y^2 - 0.00002798867783y^3)x^2 \\ & + (0.0001263034 - 0.5023759227y + 0.0095459319y^2 \\ & + 0.07212333341y^3)x + 1.41768\times10^{-7} \\ & - 0.00000348540y + 0.00001300718946y^2 \\ & - 0.00001099643y^3, \end{aligned}$$

$$\begin{aligned} v(x,y) = {} & (0.001088549648y^2 + 0.00003559050798 + \\ & + .1055964073y - 0.0007233333332y^3)x^3 \\ & + (0.001085y^3 - 0.0000534712 - 0.001633424472y^2 \\ & - 0.6942187016y)x^2 + (0.00000214275 \\ & - 0.00004524975958y^3 + 0.00006519464040y^2 \\ & + 0.1187152247y)x + 0.00007457242022y^3 \\ & - 0.00007823847367y^2 - 0.0105290851y \\ & - 0.00000205100. \end{aligned}$$

The absolute error functions of approximate solutions are plotted in Figs. 4 and 5.

5 Conclusion

In this paper, the operational matrices of integration and product for two dimensional Legendre wavelets were

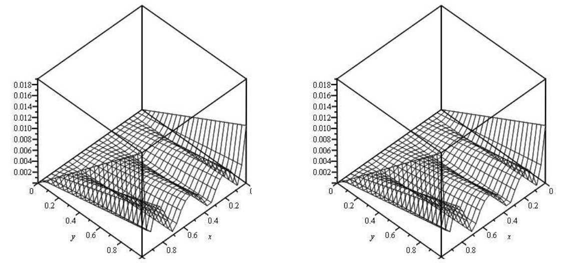

Fig. 3: The absolute error function of $u(x,y)$ (left), The absolute error function of $v(x,y)$ (right) in Example 4.4

obtained and the two dimensional Legendre wavelets method was applied to find approximate solutions for four systems of integral equations successfully. It can be concluded that the method is a powerful tool for solving two dimensional problems. The obtained solutions and plots of the absolute error functions via examples confirm this claim. One of the advantages of Legendre wavelet method compared to other common methods, such as Adomian decomposition, homotopy perturbation, and variational iteration methods, is that does not need an initial approximate for exact solutions. Finding more applications of this method and other orthogonal basis functions is one of the research fields in our research group. The computations associated with these examples are performed using the package Maple16.

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