Spectrum and Fine Spectrum of the Lower Triangular Matrix $B(r,0,s)$ over the Sequence Space $c_{s}$

Binod Chandra Tripathy$^{1,*}$ and Rituparna Das$^{2,*}$

1 Mathematical Sciences Division; Institute of Advanced Study in Science and Technology, Guwahati, Assam, India.
2 Department of Mathematics, Sikkim Manipal Institute of Technology, Sikkim, India.

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Abstract: Fine spectra of various matrix operators on different sequence spaces have been examined by several authors. Recently, some authors have determined the approximate point spectrum, the defect spectrum and the compression spectrum of various matrix operators on different sequence spaces. Here in this article we have determined the spectrum and fine spectrum of the lower triangular matrix $B(r,0,s)$ on the sequence space $c_{s}$. In a further development, we have also determined the approximate point spectrum, the defect spectrum and the compression spectrum of the operator $B(r,0,s)$ on the sequence space $c_{s}$.

Keywords: Spectrum of an operator; matrix mapping; sequence space

1 Introduction

By $w$, we denote the space of all real or complex valued sequences. Throughout the paper $c$, $c_{0}$, $bv$, $cs$, $bs$, $\ell_{1}$, $\ell_{\infty}$ represent the spaces of all convergent, null, bounded variation, convergent series, bounded series, absolutely summable and bounded sequences respectively. Also $bv_{0}$ denotes the sequence space $bv \cap c_{0}$.

The fine spectra of the Cesàro operator $C_{1}$ over the sequence space $bv_{p}$, $(1 \leq p < \infty)$ was determined by Akhmedov and Başar [2]. Okutoyi [23] determined the spectrum of the Cesàro operator $C_{1}$ on the sequence space $bv_{0}$. The spectrum and fine spectrum of the Zweier Matrix on the sequence spaces $\ell_{1}$ and $bv$ were studied by Altay and Karakuş [5]. Altay and Başar [3,4] determined the fine spectrum of the difference operator $\Delta$ and the generalized difference operator $B(r,s)$ on the sequence spaces $c_{0}$ and $c$. Furkan, Bilgiç and Kayaduman [14] have determined the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces $\ell_{1}$ and $bv$. Akhmedov and El-Shabrawy [1] determined the fine spectrum of the operator $\Delta_{a,b}$ on the sequence space $c$. Fine spectra of operator $B(r,s,t)$ over the sequence spaces $\ell_{1}$ and $bv$ and generalized difference operator $B(r,s)$ over the sequence spaces $\ell_{p}$ and $bv_{p}$, $(1 \leq p < \infty)$ were studied by Bilgiç and Furkan [11,12]. Altun [6,7] determined the fine spectrum of triangular Toeplitz operators and tridiagonal symmetric matrices over some sequence spaces. Fine spectrum of the generalized difference operator $\Delta_{v}$ on the sequence space $\ell_{1}$ was investigated by Srivastava and Kumar [28]. Panigrahi and Srivastava [24,25] studied the spectrum and fine spectrum of the second order difference operator $\Delta_{p,v}$ on the sequence space $c_{0}$ and generalized second order forward difference operator $\Delta_{s,v}$ on the sequence space $\ell_{1}$. Fine spectra of upper triangular double-band matrix $U(r,s)$ over the sequence spaces $c_{0}$ and $c$ were studied by Karakaya and Altun [20]. Karaisa and Başar [19] have determined the spectrum and fine spectrum of the upper triangular matrix $A(r,s,t)$ over the sequence space $\ell_{p}$, $(0 < p < \infty)$. In a further development, they have also determined the approximate point spectrum, defect spectrum and compression spectrum of the operator $A(r,s,t)$ on the sequence space $\ell_{p}$, $(0 < p < \infty)$. The approximate point spectrum, defect spectrum and compression spectrum of the operator $B(r,s)$ on the sequence spaces $c_{0}$, $c$, $\ell_{p}$ and $bv_{p}$, $(1 < p < \infty)$ were studied by Başar, Durna and Yıldırım [9].

The notion of matrix transformations over sequence space has been studied from various aspects. Banach algebra of matrix maps have been investigated by Rath and Tripathy [26]. Besides the above listed workers, the spectrum and fine spectrum for various matrix operators

* Corresponding author e-mail: tripathybc@yahoo.com, rituparnadas_ghy@rediffmail.com , ri2p.das@gmail.com

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has been investigated by Tripathy and Pal \[29,30\], Tripathy and Saikia \[31\] and many others in the recent years.

In this paper, we shall determine the spectrum and fine spectrum of the lower triangular matrix \(B(r,0,s)\) on the sequence space \(cs\). Also, we will determine the approximate point spectrum, the defect spectrum and the compression spectrum of the operator \(B(r,0,s)\) on the sequence space \(cs\). Clearly, \(cs = \{x = (x_n) \in w : \lim_{n \to \infty} \sum_{i=0}^{n} x_i \text{ exists}\}\) is a Banach space with respect to the norm \(\|x\|_{cs} = \sup_{n} \left| \sum_{i=0}^{n} x_i \right|\).

### 2 Preliminaries and Background

Let \(X\) and \(Y\) be Banach spaces and \(T : X \to Y\) be a bounded linear operator. By \(R(T)\), we denote the range of \(T\), i.e.

\[
R(T) = \{y \in Y : y = Tx, x \in X\}.
\]

By \(B(X)\), we denote the set of all bounded linear operators on \(X\) into itself. If \(T \in B(X)\), then the adjoint \(T^*\) of \(T\) is a bounded linear operator on the dual \(X^*\) of \(X\) defined by \((T^* f)(x) = f(Tx)\), for all \(f \in X^*\) and \(x \in X\). Let \(X \neq \{\theta\}\) be a complex normed linear space, where \(\theta\) is the zero element and \(T : D(T) \to X\) be a linear operator with domain \(D(T) \subseteq X\). With \(T\), we associate the operator

\[
T_\lambda = T - \lambda I,
\]

where \(\lambda\) is a complex number and \(I\) is the identity operator on \(D(T)\). If \(T_\lambda\) has an inverse which is linear, we denote it by \(T_\lambda^{-1}\), that is

\[
T_\lambda^{-1} = (T - \lambda I)^{-1},
\]

and call it the resolvent operator of \(T\).

A regular value \(\lambda\) of \(T\) is a complex number such that

(R1) \(T_\lambda^{-1}\) exists,

(R2) \(T_\lambda\) is bounded,

(R3) \(T_\lambda^{-1}\) is defined on a set which is dense in \(X\) i.e. \(R(T_\lambda) = X\).

The resolvent set of \(T\), denoted by \(\rho(T,X)\), is the set of all regular values \(\lambda\) of \(T\). Its complement \(\sigma(T,X) = C - \rho(T,X)\) in the complex plane \(C\) is called the spectrum of \(T\). Furthermore, the spectrum \(\sigma(T,X)\) is partitioned into three disjoint sets as follows:

The point (discrete) spectrum \(\sigma_p(T,X)\) is the set such that \(T_\lambda^{-1}\) does not exist. Any such \(\lambda \in \sigma_p(T,X)\) is called an eigenvalue of \(T\).

The continuous spectrum \(\sigma_c(T,X)\) is the set such that \(T_\lambda^{-1}\) exists and satisfies (R3), but not (R2), that is, \(T_\lambda^{-1}\) is unbounded.

The residual spectrum \(\sigma_r(T,X)\) is the set such that \(T_\lambda^{-1}\) exists (and may be bounded or not), but does not satisfy (R3), that is, the domain of \(T_\lambda^{-1}\) is not dense in \(X\).

From Goldberg \[17\], if \(X\) is a Banach space and \(T \in B(X)\), then there are three possibilities for \(R(T)\) and \(T^{-1};\)

(I) \(R(T) = X\),

(II) \(R(T) \neq R(T) = X\),

(III) \(R(T) \neq X\).

and

(1) \(T^{-1}\) exists and is continuous,

(2) \(T^{-1}\) exists but is discontinuous,

(3) \(T^{-1}\) does not exist.

Applying Goldberg \[17\] classification to \(T_\lambda\), we have three possibilities for \(T_\lambda^{-1}\):

(I) \(T_\lambda\) is surjective,

(II) \(R(T_\lambda^{-1}) \neq R(T_\lambda) = X\),

(III) \(R(T_\lambda^{-1}) \neq X\),

and

(1) \(T_\lambda\) is injective and \(T_\lambda^{-1}\) is continuous,

(2) \(T_\lambda\) is injective but \(T_\lambda^{-1}\) is discontinuous,

(3) \(T_\lambda^{-1}\) is not injective.

If these possibilities are combined in all possible ways, nine different states are created which may be shown as in the Table 1.

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\rho(T,X))</td>
<td>(\sigma_c(T,X))</td>
<td>(\sigma_p(T,X))</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

These are labeled by: \(I_1, I_2, I_3, H_1, H_2, H_3, H_1, H_2\) and \(H_3\). If \(\lambda\) is a complex number such that \(T_\lambda \in I_1\) or \(T_\lambda \in I_2\), then \(\lambda\) is in the resolvent set \(\rho(T,X)\) of \(T\). The further classification gives rise to the fine spectrum of \(T\).

If an operator is in state \(H_2\) for example, then \(R(T) \neq R(T) = X\) and \(T^{-1}\) exists but is discontinuous and we write \(\lambda \in H_2\sigma(T,X)\). The state \(H_1\) is impossible as if \(T_\lambda\) is injective, then from Kreyszig \[22\], Problem 6, p.290 \(T_\lambda^{-1}\) is bounded and hence continuous if and only if \(R(T_\lambda)\) is closed.

Again, following Appell et al. \[8\], we define the three more subdivisions of the spectrum called as the approximate point spectrum, defect spectrum and compression spectrum.

Given a bounded linear operator \(T\) in a Banach space \(X\), we call a sequence \((x_k)\) in \(X\) as a Weyl sequence for \(T\) if \(\|x_k\| = 1\) and \(\|Tx_k\| \to 0\) as \(k \to \infty\).

The approximate point spectrum of \(T\), denoted by \(\sigma_{ap}(T,X)\), is defined as the set

\[
\sigma_{ap}(T,X) = \{\lambda \in C : \text{there is a Weyl sequence for } T_\lambda\}\tag{1}
\]

The defect spectrum of \(T\), denoted by \(\sigma_{\delta}(T,X)\), is defined as the set

\[
\sigma_{\delta}(T,X) = \{\lambda \in C : T_\lambda \text{ is not surjective}\}\tag{2}
\]
The two subspectra given by equations (1) and (2) form a (not necessarily disjoint) subdivisions
\[ \sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_{p}(T,X) \] (3)
of the spectrum. There is another subspectrum
\[ \sigma_{co}(T,X) = \{ \lambda \in \mathbb{C} : R(T_{\lambda}) \neq X \} \]
which is often called the compression spectrum of T. The compression spectrum gives rise to another (not necessarily disjoint) decomposition
\[ \sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_{co}(T,X) \] (4)
Clearly, \( \sigma_{p}(T,X) \subseteq \sigma_{ap}(T,X) \) and \( \sigma_{co}(T,X) \subseteq \sigma_{p}(T,X) \). Moreover, it is easy to verify that
\[ \sigma_r(T,X) = \sigma_{co}(T,X) \setminus \sigma_{p}(T,X) \]
and
\[ \sigma_c(T,X) = \sigma(T,X) \setminus [\sigma_{p}(T,X) \cup \sigma_{co}(T,X)] \].

By the definitions given above, we can illustrate the subdivisions spectrum in the Table 2.

### Table 2: Subdivisions of spectrum of a linear operator

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( R(T_{\lambda}) = X )</td>
<td>( \lambda \in \sigma_{p}(T,X) )</td>
<td>( \lambda \in \sigma_{ap}(T,X) )</td>
</tr>
<tr>
<td>II</td>
<td>( R(T_{\lambda}) = X )</td>
<td>( \lambda \in \sigma_{p}(T,X) )</td>
<td>( \lambda \in \sigma_{co}(T,X) )</td>
</tr>
<tr>
<td>III</td>
<td>( R(T_{\lambda}) \neq X )</td>
<td>( \lambda \in \sigma_{p}(T,X) )</td>
<td>( \lambda \in \sigma_{ap}(T,X) )</td>
</tr>
</tbody>
</table>

Proposition 2.1 [Appell et al. [8], Proposition 1.3, p. 28] Spectra and subspectra of an operator \( T \in B(X) \) and its adjoint \( T^* \in B(X^*) \) are related by the following relations:

(a) \( \sigma(T^*, X^*) = \sigma(T,X) \)
(b) \( \sigma_{p}(T^*, X^*) \subseteq \sigma_{ap}(T,X) \).
(c) \( \sigma_{ap}(T^*, X^*) = \sigma(T,X) \).
(d) \( \sigma_{p}(T^*, X^*) = \sigma_{ap}(T,X) \).
(e) \( \sigma_{co}(T^*, X^*) \subseteq \sigma_{p}(T,X) \).
(f) \( \sigma_{co}(T^*, X^*) \subseteq \sigma_{ap}(T,X) \).
(g) \( \sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_{p}(T,X) \cup \sigma_{co}(T,X) \).

The relations (c)-(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum. The equality (g) implies, in particular, that \( \sigma(T,X) = \sigma_{ap}(T,X) \) if \( X \) is a Hilbert space and \( T \) is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (Appell et al. [8]).

Let \( \lambda \) and \( \mu \) be two sequence spaces and \( A = (a_{nk}) \) be an infinite matrix of real or complex numbers \( a_{nk}, \) where \( n,k \in \mathbb{N}_0 = \{0,1,2,\ldots\} \). Then, we say that \( A \) defines a matrix mapping from \( \lambda \) into \( \mu \), and we denote it by \( A : \lambda \rightarrow \mu \), if for every sequence \( x = (x_k) \in \lambda \), the sequence \( Ax = \{ (Ax)_n \} \), the A-transform of \( x \), is in \( \mu \), where
\[ (Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, n \in \mathbb{N}_0. \] (5)

By \( (\lambda : \mu) \), we denote the class of all matrices such that \( A : \lambda \rightarrow \mu \). Thus, \( \lambda \in (\lambda : \mu) \) if and only if the series on the right hand side of equation (5) converges for each \( n \in \mathbb{N}_0 \) and every \( x \in \lambda \) and we have \( Ax = \{ (Ax)_n \}_{n \in \mathbb{N}_0} \in \mu \) for all \( x \in \lambda \). The lower triangular matrix \( B(r,0,s) \) is an infinite matrix of the form

\[
B(r,0,s) = \begin{pmatrix}
r & 0 & 0 & 0 & \cdots \\
0 & r & 0 & 0 & \cdots \\
0 & 0 & s & 0 & \cdots \\
0 & 0 & 0 & s & \cdots \\
& \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

where \( s \neq 0 \).

The following results will be used in order to establish the results of this article.

**Lemma 2.2** [Wilansky [32], Example 6B, Page 130] The matrix \( A = (a_{nk}) \) gives rise to a bounded linear operator \( T \in B(cs) \) from cs to itself if and only if

(i) \( \sup_{n} \sum_{k=1}^{m} |a_{nk} - a_{n,k-1}| < \infty \).

(ii) \( \sum_{n} a_{nk} \) is convergent for each \( k \).

**Lemma 2.3** [Goldberg [17], Page 59] \( T \) has a dense range if and only if \( T^* \) is one to one.

**Lemma 2.4** [Goldberg [17], Page 60] \( T \) has a bounded inverse if and only if \( T^* \) is onto.

### 3 Spectrum and Fine Spectrum of the operator \( B(r,0,s) \) on the sequence space cs

**Theorem 3.1** \( B(r,0,s) : cs \rightarrow cs \) is a bounded linear operator and \( \| B(r,0,s) \|_{cs,cs} \leq |r| + |s| \).

**Proof:** From Lemma 2.2, it is easy to show that \( B(r,0,s) : cs \rightarrow cs \) is a bounded linear operator. Now,

\[
|B(r,0,s)(x)| = \left| \sum_{i=0}^{n} r x_i + \sum_{i=0}^{n-2} s x_i \right| \\
\leq |r| \sum_{i=0}^{n} |x_i| + |s| \sum_{i=0}^{n-2} |x_i| \\
\leq (|r| + |s|) \| x \|_{cs}
\]
and hence, \( \| B(r,0,s) \|_{(cs : cs)} \leq |r| + |s| \). \( \square \)

**Theorem 3.2** The spectrum of the operator \( B(r,0,s) \) over \( cs \) is given by

\[
\sigma(B(r,0,s), cs) = \{ \alpha \in \mathbb{C} : |\alpha - r| \leq |s| \}.
\]

**Proof:** We prove this theorem by showing that 
\( (B(r,0,s) - \alpha I) \) exists and is in \( (cs : cs) \) for 
\( |\alpha - r| > |s| \), and then show that the operator \( B(r,0,s) - \alpha I \) is not invertible for 
\( |\alpha - r| \leq |s| \).

Let \( \alpha \) be such that \( |\alpha - r| > |s| \). Since \( s \neq 0 \) we have \( \alpha \neq r \) and so \( B(r,0,s) - \alpha I \) is a triangle, therefore \( (B(r,0,s) - \alpha I)^{-1} \) exists. Let \( y = (y_n) \in cs \).

Solving \( (B(r,0,s) - \alpha I)x = y \) for \( x \) in terms of \( y \) we get

\[
(B(r,0,s) - \alpha I)^{-1} = \begin{pmatrix}
\frac{1}{r-\alpha} & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{1}{r-\alpha} & 0 & 0 & 0 & \cdots \\
\frac{s}{(r-\alpha)^2} & 0 & \frac{1}{r-\alpha} & 0 & 0 & \cdots \\
0 & \frac{s}{(r-\alpha)^2} & 0 & \frac{1}{r-\alpha} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots 
\end{pmatrix}
\]

It is easy to show that for all \( m \),

\[
\sum_k \left| \sum_{n=1}^{m} (a_{nk} - a_{n,k-1}) \right| \leq \frac{1}{|r-\alpha|} \left( 1 + \frac{|s|}{|r-\alpha|^2} + \cdots + \frac{|s|^m}{|r-\alpha|^{m+1}} \right)
\]

and hence, \( \sum \sum_{k} \left( \sum_{n=1}^{m} (a_{nk} - a_{n,k-1}) \right) \) is not convergent and hence, \( x = (x_n) \notin cs \). Therefore, 
\( (B(r,0,s) - \alpha I)^{-1} \) is not in \( (cs : cs) \) and so \( \alpha \in \sigma(B(r,0,s), cs) \).

If \( \alpha = r \), then the operator \( B(r,0,s) - \alpha I = B(0,0,s) \) is represented by the matrix

\[
B(0,0,s) = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
s & 0 & 0 & 0 & \cdots \\
0 & s & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}
\]

Since, the range of \( B(r,0,s) - \alpha I = B(0,0,s) \) is not dense, so \( \alpha \in \sigma(B(r,0,s), cs) \).

Hence,

\[
\{ \alpha \in \mathbb{C} : |\alpha - r| \leq |s| \} \subseteq \sigma(B(r,0,s), cs).
\]

This completes the proof. \( \square \)

**Theorem 3.3** The point spectrum of the operator \( B(r,0,s) \) over \( cs \) is given by

\[
\sigma_p(B(r,0,s), cs) = \phi.
\]

**Proof:** Let \( \alpha \) be an eigenvalue of the operator \( B(r,0,s) \).

Then there exists \( x \neq 0 \) in \( cs \) such that 
\( B(r,0,s)x = \alpha x \). Then, we have

\[
x_0 = \alpha x_0,
\]

\[
x_1 = \alpha x_1
\]

\[
x_{n+1} + x_n = \alpha x_n,
\]

\[
x_{n-2} + x_{n-1} = \alpha x_n, \quad n \geq 2
\]

If \( x_{n_{0}} \) is the first non-zero entry of the sequence \( (x_n) \), then \( \alpha = r \). Then from the relation \( x_{n_{0}} + x_{n_{0}+2} = \alpha x_{n_{0}+2} \), we have \( x_{n_{0}} = 0 \). But \( s \neq 0 \) and hence, \( x_{n_{0}} = 0 \), a contradiction.

Hence, \( \sigma_p(B(r,0,s), cs) = \phi \). \( \square \)

If \( T : cs \rightarrow cs \) is a bounded linear operator represented by a matrix \( A \), then it is known that the adjoint operator 
\( T^* : cs^* \rightarrow cs^* \) is defined by the transpose \( A^t \) of the matrix \( A \). It should be noted that the dual space \( cs^* \) of \( cs \) is isometrically isomorphic to the Banach space \( bv \) of all bounded variation sequences normed by 
\( \| x \|_{bv} = \sum_{n=0}^{\infty} |x_{n+1} - x_n| + \lim_{n \rightarrow \infty} |x_n| \).

**Theorem 3.4** The point spectrum of the operator \( B(r,0,s)^* \) over \( cs^* \) is given by

\[
\sigma_p(B(r,0,s)^*, cs^* \equiv bv) = \{ \alpha \in \mathbb{C} : |\alpha - r| < |s| \}.
\]

**Proof:** Let \( \alpha \) be an eigenvalue of the operator \( B(r,0,s)^* \).

Then there exists \( x \neq 0 \) in \( bv \) such that
\[ B(r,0,s)^{\ast}x = \alpha x. \] Then, we have
\[
B(r,0,s)x = \alpha x
\]
\[
\Rightarrow r\alpha + x\alpha = \alpha x_0
\]
\[
x_2 + x_3 = \alpha x_1
\]
\[
x_2 + x_4 = \alpha x_2
\]
\[
\cdots
\]
\[
r_n + x_{n+2} = \alpha x_n, \quad n \geq 0
\]
Then, we have
\[
x_{2n} = \left( \frac{\alpha - r}{s} \right)^n x_0
\]
\[
x_{2n+1} = \left( \frac{\alpha - r}{s} \right)^n x_1
\]
Since \( x = (x_n) \in bv \), so \( x = (x_n) \in c \) and hence the subsequences \( (x_{2n}) \) and \( (x_{2n+1}) \) are also convergent. Now, the subsequences \( (x_{2n}) \) and \( (x_{2n+1}) \) are convergent if and only if \( |\alpha - r| < |s| \). Hence,
\[
\sigma_{p}(B(0,0,s)^{\ast}, cs) = \{ \alpha \in \mathbb{C} : |\alpha - r| < |s| \}.
\]
\[ \text{Theorem 3.5} \quad \text{The residual spectrum of the operator } B(r,0,s) \text{ over } cs \text{ is given by}
\]
\[ \sigma_r(B(r,0,s), cs) = \{ \alpha \in \mathbb{C} : |\alpha - r| = |s| \}.
\]
\[ \text{Proof:} \quad \text{Since,}
\]
\[ \sigma_r(B(r,0,s), cs) = \sigma_p(B(r,0,s)^{\ast}, cs) \setminus \sigma_p(B(0,0,s), cs),
\]
so we get the required result by using Theorem 3.4 and Proposition 2.1. \( \Box \)
\[ \text{Theorem 3.6} \quad \text{The continuous spectrum of the operator } B(r,0,s) \text{ over } cs \text{ is given by}
\]
\[ \sigma_c(B(r,0,s), cs) = \{ \alpha \in \mathbb{C} : |\alpha - r| = |s| \}.
\]
\[ \text{Proof:} \quad \text{Since,} \sigma(B(r,0,s), cs) \text{ is the disjoint union of}
\]
\[ \sigma_p(B(r,0,s), cs), \sigma_c(B(r,0,s), cs) \text{ and } \sigma_c(B(r,0,s), cs),
\]
\[ \text{therefore, by Theorem 3.2, Theorem 3.3 and Theorem 3.5, we get}
\]
\[ \sigma_c(B(r,0,s), cs) = \{ \alpha \in \mathbb{C} : |\alpha - r| = |s| \}.
\]
\[ \text{Theorem 3.7} \quad \text{If } \alpha = r, \text{ then } \alpha \in III_1 \sigma(B(r,0,s), cs).
\]
\[ \text{Proof:} \quad \text{If } \alpha = r, \text{ the range of } B(r,0,s) \text{ is not dense. So, from Table 2 and Theorem 3.3, we have}
\]
\[ \alpha \in \sigma_c(B(r,0,s), cs).
\]
\[ \text{From Table 2,}
\]
\[ \sigma_c(B(r,0,s), cs) = III_1 \sigma(B(r,0,s), cs) \cup III_2 \sigma(B(r,0,s), cs).
\]
\[ \text{Therefore,}
\]
\[ \alpha \in III_1 \sigma(B(r,0,s), cs)
\]
or
\[ \alpha \in III_2 \sigma(B(r,0,s), cs).
\]
Also for \( \alpha = r, B(r,0,s) - \alpha I = B(0,0,s), \) the proof is straightforward. To prove the result, it is enough to show that the operator \( B(0,0,s) \) is bounded below. It is easy to verify that for all \( x \in cs, \) we have
\[
\| B(0,0,s) x \| \geq \frac{|s|}{2} \| x \|
\]
which shows that the operator \( B(0,0,s) \) is bounded below and so \( B(0,0,s) \) has a bounded inverse. This completes the theorem. \( \square \)
\[ \text{Theorem 3.8} \quad \text{If } \alpha \neq r \text{ and } \alpha \in \sigma_c(B(r,0,s), cs), \text{ then}
\]
\[ \alpha \in III_2 \sigma(B(r,0,s), cs).
\]
\[ \text{Proof:} \quad \text{Since,} \alpha \in \sigma_c(B(r,0,s), cs), \text{ therefore, from Table 2,}
\]
\[ \alpha \in III_1 \sigma(B(r,0,s), cs)
\]
or
\[ \alpha \in III_2 \sigma(B(r,0,s), cs).
\]
Now, \( \alpha \in \sigma_c(B(r,0,s), cs) \) implies that \( |\alpha - r| < |s| \). Therefore, the series (6) in Theorem 3.2 is not convergent and hence, the operator \( B(r,0,s) \) has no bounded inverse. Therefore, \( \alpha \in III_2 \sigma(B(r,0,s), cs). \) \( \square \)
\[ \text{Theorem 3.9} \quad \text{If } \alpha \in \sigma_c(B(r,0,s), cs), \text{ then}
\]
\[ \alpha \in III_1 \sigma(B(r,0,s), cs).
\]
\[ \text{Proof:} \quad \text{If } \alpha \in \sigma_c(B(r,0,s), cs) \text{ then } |\alpha - r| = |s|. \text{ Therefore, the series (6) in Theorem 3.2 is not convergent and hence, the operator}
\]
\[ B(r,0,s) \text{ has no bounded inverse. So, } \alpha \in II.
\]
\[ \text{Now we shall show that the operator } B(r,0,s) - \alpha I \text{ is not onto.}
\]
\[ \text{Let } y = (y_n) = (1,0,0,0,...). \text{ Clearly, } (y_n) \in cs.
\]
\[ \text{Let } x = (x_n) \text{ be a sequence such that } B(r,0,s)x = y.
\]
\[ \text{Solving, we get}
\]
\[ x_{2n} = \frac{(-s)^n}{(r - \alpha)^{n+1}}
\]
\[ \text{and}
\]
\[ x_{2n+1} = 0.
\]
\[ \text{Now, the series}
\]
\[ \sum_{n=0}^{\infty} x_n = \sum_{n=0}^{\infty} \frac{(-s)^n}{(r - \alpha)^{n+1}} = \frac{1}{r - \alpha} \sum_{n=0}^{\infty} \left( \frac{s}{r - \alpha} \right)^n
\]
\[ \text{is not convergent as } |\alpha - r| = |s| \text{ and hence the operator}
\]
\[ B(r,0,s) - \alpha I \text{ is not onto. So, } \alpha \in II.
\]
\[ \text{This completes the proof.} \]
Theorem 3.11 The compression spectrum of the operator $B(r,0,s)$ over $cs$ is given by
\[\sigma_{co}(B(r,0,s),cs) = \{\alpha \in \mathbb{C} : |\alpha - r| < |s|\}.\]

Proof: By Proposition 2.1 (e), we get
\[\sigma_p(B(r,0,s)^*,cs^*) = \sigma_{co}(B(r,0,s),cs).\]
Using Theorem 3.4, we get the required result. □

Theorem 3.12 The defect spectrum of the operator $B(r,0,s)$ over $cs$ is given by
\[\sigma_d(B(r,0,s),cs) = \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}.\]

Proof: From Table 2, we have
\[\sigma_d(B(r,0,s),cs) = \sigma(B(r,0,s),cs) \setminus I_3\sigma(B(r,0,s),cs).\]
Also,
\[\sigma_p(B(r,0,s),cs) = I_3\sigma(B(r,0,s),cs) \cup I_3\sigma(B(r,0,s),cs) \cup I_{11}\sigma(B(r,0,s),cs)\]
By Theorem 3.3, we have $\sigma_p(B(r,0,s),cs) = \emptyset$ and so $I_3\sigma(B(r,0,s),cs) = \emptyset$. Hence $\sigma_d(B(r,0,s),cs) = \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}$. □

Corollary 3.13 The following statements hold:
(i) $\sigma_{ap}(B(r,0,s)^*,cs^*) \equiv bv = \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}$.
(ii) $\sigma_d(B(r,0,s)^*,cs^*) \equiv bv = \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\} \setminus \{r\}$.

Proof: Using Proposition 2.1 (c) and (d), we get
\[\sigma_{ap}(B(r,0,s)^*,cs^*) \equiv bv = \sigma_d(B(r,0,s),cs)\]
and
\[\sigma_d(B(r,0,s)^*,cs^*) \equiv bv = \sigma_{ap}(B(r,0,s),cs).\]
Using Theorem 3.10 and Theorem 3.12, we get the required results. □

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References
[12] H. Bilgic and H. Furkan, On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces $\ell_1$ and $bv$, Nonlinear Anal., 68(2008), 499-506.
[18] M. Gonzalez, The fine spectrum of the Cesáro operator $B(r,s,t)$ over the sequence spaces $\ell_1$ and $bv$, Arch. Math., 44(1985), 353-358.
operator $\Delta^2_{uvw}$ on the sequence space $\ell_1$, Demonstratio Mathematica, Vol. XLV, No. 3, 2012.


[28] P. D. Srivastava and S. Kumar, Fine spectrum of the generalized difference operator $\Delta_v$ on the sequence space $\ell_1$, Thai J. Math., 8(2) (2010), 221-233.


[31] B. C. Tripathy and P. Saikia, On the spectrum of the Cesàro operator $C_1$ on $bv \cap \ell_\infty$, Math. Slovaca, 63(3)(2013), 563-572.


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Binod Chandra Tripathy is a Professor in Mathematical Sciences Division, Institute of Advanced Study in Science and Technology, Guwahati, India. He received his M.Sc and PhD degree in Mathematics from Berhampur University, India. He is referee and Editor of several national and international journals. His main research interests are: sequence space, series, summability theory, analysis, functional analysis, fuzzy set theory, operator theory and topology.

Rituparna Das is an Assistant Professor of Mathematics in Sikkim Manipal Institute of Technology, India. He received the M.Sc degree in Mathematics from Guwhati University, India and currently doing his PhD work in the same university. His main research interests are: functional analysis, operator theory and topology.