Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100511

Some Identities with Generalized Hypergeometric Functions

*Vasily E. Tarasov**

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow 119991, Russia

Received: 17 Aug. 2015, Revised: 28 Feb. 2016, Accepted: 29 Feb. 2016
Published online: 1 Sep. 2016

Abstract: In this paper, we consider some identities of a special form of the generalized hypergeometric functions ${}_1F_2(a;b,c;z)$ of real argument z. This special form is important for application in fractional calculus and fractional dynamics. The suggested functions stand out among other generalized hypergeometric functions by the power-law form of its Fourier transforms. Identities for infinite series and integrals, which include these generalized hypergeometric functions, are proved.

Keywords: Special functions, generalized hypergeometric function, fractional calculus

1 Introduction

Generalized hypergeometric function is one of the most important special functions. Moreover the generalized hypergeometric function includes many other functions as special cases, such as the Bessel functions, the classical orthogonal polynomials and elementary functions. This function has been proposed as a generalization of the Gauss and Kummer hypergeometric functions [1,2]. The generalized hypergeometric functions have a lot of application in physics [3]. Special functions including generalized hypergeometric functions play an important role in the fractional calculus [4,5,6,7] that has a long history [8]. The fractional calculus as a theory of differentiation and integrations of non-integer orders has a wide application in mechanics and physics (for example, see [9] and references therein) since it allows us to describe processes in nonlocal and hereditary media. Recently the generalized hypergeometric functions have been suggested to describe lattice models with power-law long-range interactions [10,11,12].

In this paper, we consider some properties and identities of a special form of the function ${}_1F_2(a;b,c;z)$ that are important for application in fractional dynamics of media with power-law nonlocality and corresponding discrete models. Mathematically the considered functions stand out among other the generalized hypergeometric functions by the power-law form of its Fourier transforms.

2 Definition of generalized hypergeometric function

In this section we give a definition and some properties of the generalized hypergeometric function to fix notation for further consideration.

Definition. *The generalized hypergeometric function is define by the equation*

$$ {}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) := \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!}, \quad (1)$$

where $a_l \in \mathbb{C}$ $(l = 1,\dots,p)$, $b_j \in \mathbb{C}$ and $b_j \neq 0,-1,-2...$ $(j = 1,\dots,q)$, and $(a)_k$ is the Pochhammer symbol (rising factorial) that is defined by

$$ (a)_0 = 1, \quad (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)(a+2)...(a+k-1) \quad (k \in \mathbb{N}). \quad (2)$$

The series (1) is absolutely convergent for all values of $z \in \mathbb{C}$ if $p \leq q$.

* Corresponding author e-mail: tarasov@theory.sinp.msu.ru

The function (1) has an integral representation in terms of the Mellin-Barnes contour integral

$$
{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) = \frac{1}{2\pi i}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{l=1}^{p}\Gamma(a_l)}
\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\prod_{l=1}^{p}\Gamma(a_l-s)}{\prod_{j=1}^{q}\Gamma(b_j-s)}(-z)^{-s}\Gamma(s)\,ds, \quad (3)
$$

where $|\arg(-z)|<\pi$ and the path of integration separates all the poles $s=-k$ ($k\in\mathbb{N}_0$) to the left and all the poles $s=a_j+n$ ($n\in\mathbb{N}_0$, $j=1,\dots,p$) to the right.

In this paper, we are interested in a special form of the ${}_1F_2$ function

$$
{}_1F_2(a;b,c;z) := \sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k\,(c)_k}\frac{z^k}{k!}. \quad (4)
$$

Note the symmetry of this function

$$
{}_1F_2(a;b,c;z) = {}_1F_2(a;c,b;z). \quad (5)
$$

For generalized hypergeometric function (4) the following differentiation relation holds

$$
\left(\frac{d^n}{dx^n}\right){}_1F_2(a;b,c;x) = \frac{(a)_n}{(b)_n\,(x)_n}\,{}_1F_2(a+n;b+n,c+n;x) \quad (n\in\mathbb{N}). \quad (6)
$$

It is possible to write the product formula for this function

$$
{}_1F_2(a_1;b_1,c_1;z)\,{}_1F_2(a_2;b_2,c_2;z) = \sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{(a_1)_m\,(a_2)_{k-m}}{(b_1)_m\,(b_2)_{k-m}\,(c_1)_m\,(c_2)_{k-m}}\,\frac{z^k}{m!\,(k-m)!}. \quad (7)
$$

In this paper, we consider some identities of the following generalized hypergeometric functions

$$
T_{\pm}[\alpha,n] := {}_1F_2\left(\frac{\alpha+1}{2};\frac{\alpha+3}{2},1\mp\frac{1}{2};-\frac{\pi^2n^2}{4}\right), \quad (n\in\mathbb{Z}). \quad (8)
$$

We denote this function by $T_{\pm}[\alpha,n]$ for simplification. These functions stand out among other the generalized hypergeometric functions by the power-law form of its Fourier transform.

3 Fourier series transform of generalized hypergeometric function

Let us consider the Fourier series transform $\mathscr{F}_\Delta$ that is defined by the equation

$$
\mathscr{F}_\Delta\{f[n]\} := \sum_{n=-\infty}^{+\infty} e^{-ikn}\,f[n], \quad (9)
$$

and the inverse Fourier series transform $\mathscr{F}_\Delta^{-1}$ in the form

$$
\mathscr{F}_\Delta^{-1}\{\hat{f}(k)\} := \frac{1}{2\pi}\int_{-\pi}^{+\pi} dk\,e^{ikn}\,\hat{f}(k). \quad (10)
$$

Note that we use the minus sign in the exponent of (9) instead of plus that is usually used.

The Fourier series transforms of these generalized hypergeometric functions with $n\in\mathbb{N}$ are given by the following proposition.

Proposition 1. *The Fourier series transforms of the generalized hypergeometric functions (8) have the forms*

$$
\mathscr{F}_\Delta\{T_+[\alpha,n]\} = \sum_{n=-\infty}^{+\infty} e^{-ikn}\,{}_1F_2\left(\frac{\alpha+1}{2};\frac{\alpha+3}{2},\frac{1}{2};-\frac{\pi^2n^2}{4}\right) = \frac{\alpha+1}{\pi^\alpha}|k|^\alpha, \quad (11)
$$

$$
\mathscr{F}_\Delta\{n\,T_-[\alpha,n]\} = \sum_{n=-\infty}^{+\infty} e^{-ikn}\,n\,{}_1F_2\left(\frac{\alpha+1}{2};\frac{\alpha+3}{2},\frac{3}{2};-\frac{\pi^2n^2}{4}\right) = -\,i\,\frac{\alpha+1}{\pi^\alpha}\operatorname{sgn}(k)\,|k|^{\alpha-1}, \quad (12)
$$

where $\alpha>-1$.

Proof. To prove (11) and (12), we use the equations

$$
\mathscr{F}_\Delta^{-1}\left(|k|^\alpha\right) = \frac{1}{\pi}\int_0^{\pi} k^\alpha\cos(nk)\,dk = \frac{\pi^\alpha}{\alpha+1}\,{}_1F_2\left(\frac{\alpha+1}{2};\frac{\alpha+3}{2},\frac{1}{2};-\frac{\pi^2n^2}{4}\right), \quad (13)
$$

where $\alpha>-1$ and

$$
\mathscr{F}_\Delta^{-1}\left(i\operatorname{sgn}(k)\,|k|^\alpha\right) = -\frac{1}{\pi}\int_0^{\pi} k^\alpha\sin(nk)\,dk = -\frac{\pi^{\alpha+1}n}{\alpha+2}\,{}_1F_2\left(\frac{\alpha+2}{2};\frac{\alpha+4}{2},\frac{3}{2};-\frac{\pi^2n^2}{4}\right), \quad (14)
$$

where $\alpha>-2$. Applying the Fourier series transform $\mathscr{F}_\Delta$, which is defined by (9), to equations (13) and (14), we get expression (11) and (12). □

Remark 1. The Fourier series transforms of the functions $T_{\pm}[\alpha,n]$ have been represented in the form

$$
\mathscr{F}_\Delta\left(T_+[\alpha,n]\right) = \sum_{n=-\infty}^{+\infty} e^{-ikn}\,T_+[\alpha,n] = 2\sum_{n=1}^{\infty} T_+[\alpha,n]\cos(kn) + T_+[\alpha,0], \quad (15)
$$

$$\mathscr{F}_\Delta\Big(n\,T_-[\alpha,n]\Big) = \sum_{n=-\infty}^{+\infty} e^{-ikn}\,n\,T_-[\alpha,n] = -2i\sum_{n=1}^{\infty} n\,T_-[\alpha,n]\,\sin(kn). \quad (16)$$

Note that

$$T_\pm[\alpha,0] = 1. \quad (17)$$

Proposition 1 of the Fourier transforms of the generalized hypergeometric functions (8) will be used to prove identities for these functions.

4 Identities with generalized hypergeometric functions

One of the main results of this paper are the following propositions for the generalized hypergeometric functions (8).

Proposition 2. *The following identities with the generalized hypergeometric functions hold*

$$\sum_{k=-\infty}^{+\infty} {}_1F_2\left(\frac{\alpha+1}{2};\frac{\alpha+3}{2},\frac{1}{2};-\frac{\pi^2k^2}{4}\right) {}_1F_2\left(\frac{\beta+1}{2};\frac{\beta+3}{2},\frac{1}{2};-\frac{\pi^2(k-n)^2}{4}\right) = \frac{(\alpha+1)(\beta+1)}{\alpha+\beta+1}\, {}_1F_2\left(\frac{\alpha+\beta+1}{2};\frac{\alpha+\beta+3}{2},\frac{1}{2};-\frac{\pi^2n^2}{4}\right), \quad (18)$$

where $\alpha > -1$, $\beta > -1$, $\alpha+\beta > -1$.

Proof. Using relation (11), the Fourier series transforms of the left side and the right side of equation (18) gives

$$\frac{\alpha+1}{\pi^\alpha}|k|^\alpha\,\frac{\beta+1}{\pi^\beta}|k|^\beta = \frac{(\alpha+1)(\beta+1)}{\alpha+\beta+1}\,\frac{\alpha+\beta+1}{\pi^{\alpha+\beta}}|k|^{\alpha+\beta}. \quad (19)$$

As a result we obtain the identity. □

Remark 2. Using notations (8), the suggested identity (18) can be written in the form

$$\sum_{k=-\infty}^{+\infty} T_+[\alpha,k]\,T_+[\beta,k-n] = \frac{(\alpha+1)(\beta+1)}{\alpha+\beta+1}\,T_+[\alpha+\beta,n], \quad (20)$$

where $\alpha > -1$, $\beta > -1$, $\alpha+\beta > -1$.

There are other identities that involving the generalized hypergeometric functions (8). Using notations $T_\pm[\alpha,n]$, these identities are represented by the following proposition.

Proposition 3. *The generalized hypergeometric functions (8) satisfy the following identities*

$$\sum_{k=-\infty}^{+\infty} k\,(k-n)\,T_-[\alpha,k]\,T_-[\beta,k-n] = -\frac{(\alpha+1)(\beta+1)}{\pi^2(\alpha+\beta-1)}\,T_+[\alpha+\beta-2,n], \quad (21)$$

where $\alpha > -1$, $\beta > -1$, $\alpha+\beta-2 > -1$.

$$\sum_{k=-\infty}^{+\infty} k\,T_-[\alpha,k]\,T_+[\beta,k-n] = \frac{(\alpha+1)(\beta+1)}{(\alpha+\beta+1)}\,T_-[\alpha+\beta,n], \quad (22)$$

where $\alpha > -1$, $\beta > -1$, $\alpha+\beta > -1$.

$$\sum_{k=-\infty}^{+\infty} (k-n)\,T_+[\alpha,k]\,T_-[\beta,k-n] = \frac{(\alpha+1)(\beta+1)}{(\alpha+\beta+1)}\,T_-[\alpha+\beta,n], \quad (23)$$

where $\alpha > -1$, $\beta > -1$, $\alpha+\beta > -1$.

Proof. The proof is similar to the proof of Proposition 2. To prove the identities (21)-(23), we used relations (11) and (12) of the Fourier series transform of the functions (8). As a result, we get the identities. □

5 Simple examples of identities

For $\alpha = 2$, the generalized hypergeometric functions $T_\pm[\alpha,n]$ have the simple forms

$$T_-[2,n] = {}_1F_2\left(\frac{3}{2};\frac{5}{2},\frac{3}{2};-\frac{\pi^2n^2}{4}\right) = -\frac{3}{\pi^2 n}\,\frac{(-1)^n}{n} \quad (24)$$

and

$$T_+[2,n] = {}_1F_2\left(\frac{3}{2};\frac{5}{2},\frac{1}{2};-\frac{\pi^2n^2}{4}\right) = -\frac{3}{\pi^2}\,\frac{2(-1)^n}{n^2} \quad (25)$$

where $n \neq 0$. For $n = 0$, we have (17). Then, equality (21) can be represented by equations of the following proposition.

Proposition 4. *For* $\alpha = \beta = 2$, *identity (21) with the functions (24) and (24) has the form*

$$\sum_{k=-\infty}^{+\infty} k\,(k-n)\,T_-[2,k]\,T_-[2,k-n] = -\frac{3}{\pi^2}\,T_+[2,n] \quad (26)$$

that can be written as

$$\sum_{\substack{k=-\infty \\ k\neq 0, k\neq n}}^{\infty} \frac{1}{k\,(k-n)} = \frac{2}{n^2} \quad (n \neq 0). \tag{27}$$

For $n=0$, we have

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{3}. \tag{28}$$

Proof. Equality (27) can be proved directly by splitting sum on three sums

$$\sum_{\substack{k=-\infty \\ k\neq 0, k\neq n}}^{\infty} \frac{1}{k\,(k-n)} = \sum_{k=n+1}^{\infty} \frac{1}{k\,(k-n)} + \sum_{k=1}^{n-1} \frac{1}{k\,(k-n)} + \sum_{k=-1}^{-\infty} \frac{1}{k\,(k-n)}. \tag{29}$$

Using the fractions decomposition into prime factors, we get

$$\sum_{\substack{k=-\infty \\ k\neq 0, k\neq n}}^{\infty} \frac{1}{k\,(k-n)} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{n}\sum_{k=1}^{n-1} \frac{2}{k} + \frac{1}{n}\sum_{k=1}^{n} \frac{1}{k}. \tag{30}$$

As a result, we get

$$\sum_{\substack{k=-\infty \\ k\neq 0, k\neq n}}^{\infty} \frac{1}{k\,(k-n)} = \frac{1}{n}\frac{2}{n}. \tag{31}$$

For $n=0$, we have

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k^2} = 2\sum_{k=1}^{\infty} \frac{1}{k^2} = 2\frac{\pi^2}{6} = \frac{\pi^2}{3}. \tag{32}$$

□

Remark 3. The identities (20), (22), (23) with $\alpha = \beta = 2$ have the forms

$$\sum_{k=-\infty}^{+\infty} T_+[2,k]\, T_+[2,k-n] = \frac{9}{5}\, T_+[4,n], \tag{33}$$

$$\sum_{k=-\infty}^{+\infty} k\, T_-[2,k]\, T_+[2,k-n] = \frac{9}{5}\, T_-[4,n], \tag{34}$$

$$\sum_{k=-\infty}^{+\infty} (k-n)\, T_+[2,k]\, T_-[2,k-n] = \frac{9}{5}\, T_-[4,n], \tag{35}$$

where $T_\pm[2,n]$ are presented in (24), (25), and the functions $T_\pm[4,n]$ are defined by the equations

$$T_-[4,n] = {}_1F_2\left(\frac{5}{2};\frac{7}{2},\frac{3}{2};-\frac{\pi^2 n^2}{4}\right) = -\frac{5}{\pi^4 n}\left(\frac{6\,(-1)^n}{n^3} - \frac{(-1)^n \pi^2}{n}\right) \quad (n \neq 0), \tag{36}$$

and

$$T_+[4,n] = {}_1F_2\left(\frac{5}{2};\frac{7}{2},\frac{1}{2};-\frac{\pi^2 n^2}{4}\right) = \frac{5}{\pi^4}\left(\frac{4\pi^2(-1)^n}{n^2} - \frac{24\,(-1)^n}{n^4}\right) \quad (n \neq 0), \tag{37}$$

and $T_\pm[4,0] = 1$.

Remark 4. Using equation 2.5.3.5 of [14] we can give the representation of the functions (8) with $\alpha = m \in \mathbb{N}$ in the form

$$T_+[m,n] = \frac{m+1}{\pi^{m+1}} \frac{(-1)^{n+2}}{n^{m+1}} \sum_{k=0}^{[(m-1)/2]} \frac{(-1)^k\, m!}{(m-2k-1)!} (\pi n)^{m-2k-1} + \frac{(-1)^{[(m+1)/2]}\, m!}{n^{m+1}} \Big(2[(m+1)/2] - m\Big), \tag{38}$$

and

$$T_-[m,n] = \frac{m+1}{\pi^{m+1} n} \frac{(-1)^{n+1}}{n^m} \sum_{k=0}^{[(m-1)/2]} \frac{(-1)^k\, (m-1)!}{(m-2k-1)!} (\pi n)^{m-2k-1} + \frac{(-1)^{[(m-1)/2]}\, (m-1)!}{n^m} \Big(2[(m-1)/2] - m + 2\Big), \tag{39}$$

where $n \in \mathbb{N}$ and $[z]$ is the integer part of the value z.

6 Integral identities with ${}_1F_2$ functions

Let us consider the generalized hypergeometric function in the form

$${}_1F_2\left(\alpha;\beta,1\pm\frac{1}{2};-\frac{x^2}{4}\right), \tag{40}$$

where $\alpha > 0$, $\beta > 0$ and $x \in \mathbb{R}$.

Proposition 5. *The following identities with the generalized hypergeometric functions (40) hold*

$$\int_{-\infty}^{+\infty} dy\; {}_1F_2\left(\alpha_1;\beta_1,\frac{1}{2};-\frac{y^2}{4}\right) {}_1F_2\left(\alpha_2;\beta_2,\frac{1}{2};-\frac{(x-y)^2}{4}\right) = g_1(\alpha_1,\alpha_2,\beta_1,\beta_2)\; {}_1F_2\left(\alpha_1+\alpha_2-\frac{1}{2};\beta_1+\beta_2-\frac{3}{2},\frac{1}{2};-\frac{x^2}{4}\right), \tag{41}$$

where

$$g_1(\alpha_1,\alpha_2,\beta_1,\beta_2) := \frac{2\pi\,\Gamma(\beta_1)\,\Gamma(\beta_1)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(\beta_1-\alpha_1)} \frac{\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-1)\,\Gamma(\alpha_1+\alpha_2-1/2)}{\Gamma(\beta_2-\alpha_2)\,\Gamma(\beta_1+\beta_2-3/2)}, \tag{42}$$

and $\beta_1 > \alpha_1 > 0$, $\beta_2 > \alpha_2 > 0$, $\beta_1+\beta_2-3/2 > \alpha_1+\alpha_2-1/2 > 0$.

Proof. To prove identity (41), the Fourier transform is applied to equation (41). Using equation (5) of Section 1.14 of [13], which has the form

$$\int_0^{\infty} dx \cos(kx)\, {}_1F_2\left(\alpha;\beta,\frac{1}{2};-\frac{x^2}{4}\right) = \frac{\pi\,\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\,k^{2\alpha-1}\,(1-k^2)^{\beta-\alpha-1} \qquad (43)$$

for $0<k<1$,

$$\int_0^{\infty} dx \cos(kx)\, {}_1F_2\left(\alpha;\beta,\frac{1}{2};-\frac{x^2}{4}\right) = 0 \qquad (44)$$

for $1<k<\infty$, where $\beta>\alpha>0$, we obtain the identity. □

Proposition 6. *The following identities with the generalized hypergeometric functions (40) hold*

$$\int_{-\infty}^{+\infty} dy\, x\,(x-y)\, {}_1F_2\left(\alpha_1;\beta_1,\frac{3}{2};-\frac{y^2}{4}\right) {}_1F_2\left(\alpha_2;\beta_2,\frac{3}{2};-\frac{(x-y)^2}{4}\right) = g_2(\alpha_1,\alpha_2,\beta_1,\beta_2)\; {}_1F_2\left(\alpha_1+\alpha_2-\frac{3}{2};\beta_1+\beta_2-\frac{5}{2},\frac{1}{2};-\frac{x^2}{4}\right), \qquad (45)$$

where

$$g_2(\alpha_1,\alpha_2,\beta_1,\beta_2) := -\frac{2\pi\,\Gamma(\beta_1)\,\Gamma(\beta_1)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(\beta_1-\alpha_1)}\,\frac{\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-1)\,\Gamma(\alpha_1+\alpha_2-3/2)}{\Gamma(\beta_2-\alpha_2)\,\Gamma(\beta_1+\beta_2-5/2)}, \qquad (46)$$

and $\beta_1 > \alpha_1 > 1/2$, $\beta_2 > \alpha_2 > 1/2$, $\beta_1+\beta_2-5/2 > \alpha_1+\alpha_2-3/2 > 1/2$.

Proof. To prove identity (45), the Fourier transform is applied to equation (45). Using equation (7) of Section 2.14 of [13], which has the form

$$\int_0^{\infty} dx \sin(kx)\, x\, {}_1F_2\left(\alpha;\beta,\frac{3}{2};-\frac{x^2}{4}\right) = \frac{\pi\,\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\,k^{2\alpha-2}\,(1-k^2)^{\beta-\alpha-1} \qquad (47)$$

for $0<k<1$, and

$$\int_0^{\infty} dx \sin(kx)\, x\, {}_1F_2\left(\alpha;\beta,\frac{3}{2};-\frac{x^2}{4}\right) = 0 \qquad (48)$$

for $1<k<\infty$, where $\beta>\alpha>1/2$, we obtain the identity. □

Proposition 7. *The following identities with the functions (40) hold*

$$\int_{-\infty}^{+\infty} dy\, x\, {}_1F_2\left(\alpha_1;\beta_1,\frac{3}{2};-\frac{y^2}{4}\right) {}_1F_2\left(\alpha_2;\beta_2,\frac{1}{2};-\frac{(x-y)^2}{4}\right) = g_1(\alpha_1,\alpha_2,\beta_1,\beta_2)\; {}_1F_2\left(\alpha_1+\alpha_2-\frac{1}{2};\beta_1+\beta_2-\frac{3}{2},\frac{3}{2};-\frac{x^2}{4}\right), \qquad (49)$$

where $g_1(\alpha_1,\alpha_2,\beta_1,\beta_2)$ *is defined by (42), and* $\beta_1 > \alpha_1 > 1/2$, $\beta_2 > \alpha_2 > 0$, $\beta_1+\beta_2-3/2 > \alpha_1+\alpha_2-1/2 > 1/2$.

Proof. To prove identity (49), the Fourier transform is applied to equation (49). Using equations (43) and (47), we obtain the identity. □

Remark 5. The suggested identities for special type of the generalized hypergeometric functions ${}_1F_2(a;b,c;z)$ can be used in fractional calculus and fractional dynamics of media and systems with power-law nonlocality. An importance of these functions for fractional dynamics is based on the power-law form of the Fourier transform of considered hypergeometric functions.

References

[1] W.N. Bailey, Generalised Hypergeometric Series, Cambridge University Press, 1935.
[2] L.J. Slater, Generalized Hypergeometric Functions, Cambridge University Press, 1966.
[3] A.M. Mathai and R.K. Saxena, Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences, Springer, Berlin, Heidelberg, 1973.
[4] S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives Theory and Applications, Gordon and Breach, New York, 1993.
[5] V. Kiryakova, Generalized Fractional Calculus and Applications, Longman, Harlow and Wiley, New York, 1994.
[6] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.
[7] H.M. Srivastava, Special Functions in Fractional Calculus and Related Fractional Differintegral Equations, World Scientific, Singapore, 2014.
[8] J. Tenreiro Machado, V. Kiryakova and F. Mainardi, Communications in Nonlinear Science and Numerical Simulation. **16**, 1140-1153 (2011).
[9] V.E. Tarasov, Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media, Springer, New York, 2011.
[10] V.E. Tarasov, Advances in High Energy Physics. **2014**, 957863 (2014).
[11] V.E. Tarasov, Journal of Statistical Mechanics: Theory and Experiment. **2014**, P09036 (2014).

[12] V.E. Tarasov, Physica A: Statistical Mechanics and its Applications. **421**, 330-342. (2015).
[13] A. Erdelyi, Tables of integral transforms. Volume 1, McGraw-Hill, New York, 1954.
[14] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, Integrals and Series, Vol. 1: Elementary Functions, Gordon and Breach, New York, 1986.
[15] A. Erdelyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, Higher Transcendental Functions. Volume 1, Krieeger, Melbourne, Florida, 1981.
[16] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, Integrals and Series, Vol. 2: Special Functions, Gordon and Breach, New York, 1990.
[17] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, Integrals and Series, Vol. 3: More Special Functions, Gordon and Breach, New York, 1989.

Vasily Tarasov received his Ph.D. in Theoretical Physics from the Lomonosov Moscow State University in 1995. He was a Research Associate at the Skobeltsyn Institute of Nuclear Physics, Moscow State University for six years, and then became a Senior Research Associate at the Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University. Vasily is an Associate Professor at the Applied Mathematics and Physics Department of Moscow Aviation Institute. He is a member of Organizing Committee of International Workshop on High Energy Physics and Quantum Field Theory. Vasily E. Tarasov was Senior Visitor at Courant Institute of Mathematical Sciences, New York University in 2005-2009. V.E. Tarasov is an Associate Editor of ”Communications in Nonlinear Science and Numerical Simulations” (Elsevier); ”Journal of Applied Nonlinear Dynamics” (LH Scientific Publishing). V.E. Tarasov is a member of Editorial Boards of the journals: ”Fractional Differential Calculus” (Ele-Math); ”International Journal of Applied and Computational Mathematics” (Springer); ”Discontinuity, Nonlinearity, and Complexity” (LH Scientific Publishing); ”Physics International” (Science Publications). He has published about 160 scientific works, among which 8 books, 4 chapters in books and about 140 papers in refereed journals.