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Bayes Inference for the Weibull-Geometric Distribution based on Progressive Hybrid Censored Data

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Abstract: The challenge of estimating the parameters of the Weibull-Geometric distribution using progressively type-I hybrid censored data is addressed in this study. For this, the maximum likelihood and Bayes methods of estimation are applied. The Bayes estimates are calculated using the Markov Chain Monte Carlo (MCMC) method. Through a Monte Carlo simulation investigation, the Bayes estimates of the parameters under two alternative loss functions are investigated and compared to their corresponding maximum likelihood estimates. For illustration, a practical set of data is used.

Keywords: Weibull-Geometric distribution, progressively type-I hybrid censoring data, Markov Chain Monte Carlo technique, Metropolis-Hastings

1 Introduction

Several lifetimes of units put on a life testing experiments may not be detected in reliability theory for various reasons. There are various different kinds of censorship schemes. The two most popular censoring schemes are type-I and type-II. [1] was the first to introduce the hybrid censoring scheme, which is a combination of type-I and type-II censoring schemes. These three traditional censoring schemes only remove the units at the end of the experiment. Such intermediate unit deletions are possible with the progressively hybrid censoring (PHC) scheme. The traditional censoring schemes are included as special cases of this form of censoring. [2] introduced the type-I progressively hybrid censoring scheme (type-I PHC), which is a hybrid of type-I progressive and hybrid censoring schemes. This censoring scheme has gained popularity for evaluating highly trustworthy data and is commonly used in reliability and survival studies, see [3], [4] and [5] for further information on the progressively hybrid censoring scheme.

The type-I PHC can be explained as follows: Suppose that n similar units are put on a life test with a pre-specified progressive censoring scheme $(R_1,R_2,...,R_m)$, $1 \le m \le n$. The experiment is ended at some specified time of termination $T \in (0,\infty)$. At the first failure time $X_{1:m:n}$, R_1 units of the remaining live unites are removed from the test. Similarly, at the second failure time $X_{2:m:n}$, R_2 unites of the remaining live units are also removed from the test, and so on. The termination of the experiment occurs at time $T^* = min\{X_{m:m:n},T\}$. If the $m-th$ failure $X_{m:m:n}$ occurs before the time T, the remaining units $R^*_m = n-(R_1+R_2+...+R_{m-1})-m$ are all removed and the experiment is ended. Otherwise, if the $m-th$ failure $X_{m:m:n}$ does not occur before T, while only J, $0<J<m$, failures occur all the remaining units $R^*_m = n-(R_1+R_2+...+R_J)-J$ are all removed and the experiment terminated. These two cases of termination will be denoted here as Case I and Case II, respectively, where,

Case I: $X_{1:m:n},...,X_{m:m:n}, \quad if \quad X_{m:m:n} < T,$
Case II: $X_{1:m:n},...,X_{J:m:n}, \quad if \quad X_{J:m:n} < T < X_{J+1:m:n}.$

Statistical inference methodologies for estimating the parameters of different lifetime distributions are researched using progressively type-I hybrid censored data; see, for instance, [3], [4], [5] and [6]. The Weibull-Geometric model is highlighted in this study.

2 The Weibull-Geometric Model

[7] was the first one to introduce the Weibull-Geometric (WG) distribution. [8] examined the same distribution

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under the name complementary Weibull-geometric distribution, but with different parametrization. The probability density function (pdf) and cumulative distribution function (cdf) of the WG distribution are as follows:

$$f(x;\alpha,\beta,p) = \alpha\beta^{\alpha}(1-p)x^{\alpha-1}e^{-(\beta x)^{\alpha}}\{1-pe^{-(\beta x)^{\alpha}}\}^{-2}, \quad x>0, \tag{1}$$

and

$$F(x) = (1-e^{-(\beta x)^{\alpha}})(1-pe^{-(\beta x)^{\alpha}})^{-1}, \quad x>0. \tag{2}$$

When $p=0$, the two-parameter Weibull distribution is obtained, as can be seen from (1). For $\alpha=1$, another special form is produced, which is the exponential-geometric (EG) distribution with parameters (β,p). The related survival and failure rate functions of the WG model are given, respectively, by

$$S(x) = [(1-p)e^{-(\beta x)^{\alpha}}](1-pe^{-(\beta x)^{\alpha}})^{-1}, \quad x>0, \tag{3}$$

and

$$H(x) = \frac{f(x)}{S(x)} = \alpha\beta^{\alpha}x^{\alpha-1}\{1-pe^{-(\beta x)^{\alpha}}\}^{-1}, \quad x>0. \tag{4}$$

As seen in Figure 1, the hazard rate function (4) is decreasing for $0<\alpha\leq 1$. For $\alpha>1$, however, it can be either increasing (for small values of p) or upside-down bathtub (for large values of p), as shown in Figure 2.

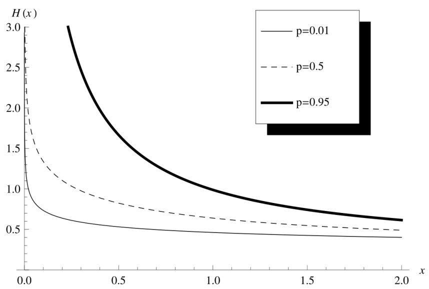

Fig. 1: $(\alpha=0.8,\beta=0.5)$

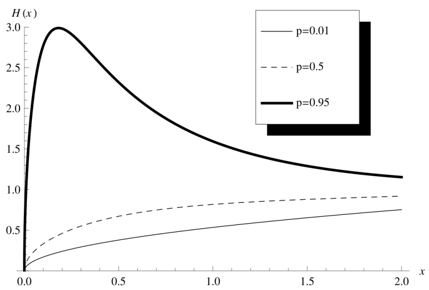

Fig. 2: $(\alpha=1.5,\beta=0.5)$

The WG distribution is a generalization of the exponential-geometric (EG) distribution introduced by [9]. The EG distribution's hazard rate function is monotonic decreasing, whereas the WG distribution's one can take more flexible forms. The WG distribution, unlike the Weibull distribution, can be used for modelling the unimodal failure rates.

[7] fitted WG models to two real data sets to demonstrate the new distribution's flexibility and promise. The first set of data includes the number of consecutive air conditioning system failures for each member of a fleet of 13 Boeing 720 jet aeroplanes. The second data set is a complete set of 100 observations of carbon fibre breaking stress from [10]. Using the EM algorithm, they obtained the estimates of the WG parameters p, β and α. Various characterizations of the WG distribution were offered by [11]. From the half-logistic distribution, the WG distribution, and the long-term WG distribution, [12] generated explicit formulas for the moments of order statistics. [13] used progressively Type-II censored data to estimate the unknown parameters of the WG distribution. [14] used a progressive first-failure censoring approach to estimate the parameters of the WG distribution. Using the one and two-sample prediction techniques, [15] obtained the Bayes point predictors and Bayesian prediction intervals for future observables from the WG model based on progressively Type-II censored data.

The maximum likelihood and Bayes estimation methods are employed in this study to estimate the parameters of the WG distribution based on PHC data. The Bayes estimates are obtained by using MCMC techniques and compared to their corresponding maximum likelihood estimates via Monte Carlo simulation study. An example of the set of practical data that [10] studied is also considered.

3 Maximum Likelihood Estimation

This section deals with computing the maximum likelihood estimates (MLEs) for the parameters of the WG distribution based on type-I PHC data. Let D be the number of failures that occur before time T^* which denotes the terminal time. Then, at time T^* the remaining $R_D^* = n-D-\sum_{i=1}^{D}R_i$ units are all removed and the experiment is stopped, where $D=m$, $T^*=X_{m:m:n}$ in Case I, and $D=J$, $T^*=T$ in Case II. The likelihood function of (α,β,p) can thus be written as

$$L(\alpha,\beta,p;\underline{x}) \propto \prod_{i=1}^{D} f_X(x_i)[1-F_X(x_i)]^{R_i}[1-F_X(T^*)]^{R_D^*}. \tag{5}$$

Assume that $X_{1:m:n},\ldots,X_{D:m:n}$ is a type-I PHC sample from the WG distribution with censoring scheme $(R_1,R_2,\ldots,R_{D-1},R_D^*)$ and stopping time T^*. Here, we will denote the D type-I PHC order statistics by $X_1,X_2,\ldots,X_D$.

Using (1), (2) and (5), the likelihood function is as follows

$$L(\alpha,\beta,p;\underline{x}) \propto (\alpha\beta^{\alpha})^{D}(1-p)^{n}e^{-R_D^*(\beta T^*)^{\alpha}-\sum_{i=1}^{D}(1+R_i)(\beta x_i)^{\alpha}} [1-pe^{-(\beta T^*)^{\alpha}}]^{-R_D^*} \prod_{i=1}^{D} x_i^{\alpha-1}[1-pe^{-(\beta x_i)^{\alpha}}]^{-(2+R_i)} \quad (6)$$

Then, the log-likelihood function is

$$L^* = \ln L(\alpha,\beta,p;\underline{x}) \propto D\ln\alpha + \alpha D\ln\beta + n\ln(1-p) - R_D^*(\beta T^*)^{\alpha} - \sum_{i=1}^{D}(1+R_i)(\beta x_i)^{\alpha} + (\alpha-1)\sum_{i=1}^{D}\ln x_i - \sum_{i=1}^{D}(2+R_i)\ln[1-pe^{-(\beta x_i)^{\alpha}}] - R_D^*\ln[1-pe^{-(\beta T^*)^{\alpha}}] \quad (7)$$

The maximum likelihood estimators can be derived by simultaneously solving the following equations using equation (7)

$$\frac{\partial L^*}{\partial\alpha} = D(\frac{1}{\alpha}+\ln\beta) + \sum_{i=1}^{D}\ln x_i - I_1(T^*)\ln(\beta T^*) - \sum_{i=1}^{D}\lambda_i\ln(\beta x_i) = 0, \quad (8)$$

$$\frac{\partial L^*}{\partial\beta} = \frac{\alpha}{\beta}[D - I_1(T^*) - \sum_{i=1}^{D}\lambda_i] - T^*R_D^* = 0, \quad (9)$$

and

$$\frac{\partial L^*}{\partial p} = \frac{-n}{1-p} - R_D^* e^{-(\beta T^*)^{\alpha}}[1-pe^{-(\beta T^*)^{\alpha}}]^{-1} + \sum_{i=1}^{D}(2+R_i)e^{-(\beta x_i)^{\alpha}}[1-pe^{-(\beta x_i)^{\alpha}}]^{-1} = 0, \quad (10)$$

where,

$$I_1(T^*) = p(\beta T^*)^{\alpha}R_D^* e^{-(\beta T^*)^{\alpha}}[1-pe^{-(\beta T^*)^{\alpha}}]^{-1}, \quad (11)$$

and

$$\lambda_i = (\beta x_i)^{\alpha}[1-pe^{-(\beta x_i)^{\alpha}}]^{-1}[1+pe^{-(\beta x_i)^{\alpha}}+R_i]. \quad (12)$$

A closed form solution for α, β, and p does not exist, as proven by (**??**), (9), and (10). To acquire the MLEs, numerical techniques such as the Newton-Raphson algorithm can be utilised.

4 Bayesian Estimation

Assume that the variables α and β are independent random variables with Gamma prior distributions, denoted by G(.,.), of the forms

$$g_1(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)}\alpha^{a_1-1}e^{-\alpha b_1}, \quad a_1,b_1>0, \quad \alpha>0, \quad (13)$$

and

$$g_2(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2)}\beta^{a_2-1}e^{-\beta b_2}, \quad a_2,b_2>0, \quad \beta>0. \quad (14)$$

[13] were the first to use the Gamma-Gamma prior. Assume also that the parameter p is independent of α and β, and that it has a $U(0,1)$ prior distribution given by

$$g_3(p) = 1, \quad 0 \le p \le 1. \quad (15)$$

As a result, the joint prior distribution of α, β, and p is given by

$$g(\alpha,\beta,p) = \frac{b_1^{a_1}b_2^{a_2}}{\Gamma(a_1)\Gamma(a_2)}e^{-(\alpha b_1+\beta b_2)}\alpha^{a_1-1}\beta^{a_2-1}. \quad (16)$$

From (6) and (16), the joint posterior distribution takes the form

$$q(\alpha,\beta,p|\underline{x}) = K\alpha^{D+a_1-1}\beta^{D\alpha+a_2-1}(1-p)^{n}e^{-(\alpha b_1+\beta b_2)} [1-pe^{-(\beta T^*)^{\alpha}}]^{-R_D^*}e^{-R_D^*(\beta T^*)^{\alpha}-\sum_{i=1}^{D}(1+R_i)(\beta x_i)^{\alpha}} \prod_{i=1}^{D}x_i^{\alpha-1}[1-pe^{-(\beta x_i)^{\alpha}}]^{-(2+R_i)}, \quad \alpha>0, \quad \beta>0, \quad 0\le p\le 1, \quad (17)$$

where, K denotes the normalizing constant given by

$$K^{-1} = \int_0^{\infty}\int_0^{\infty}\int_0^{1} g(\alpha,\beta,p)L(\alpha,\beta,p;\underline{x})\,dp\,d\alpha\,d\beta,$$

and $L(\alpha,\beta,p;\underline{x})$ is the likelihood function given by (6). The Bayes estimator of a function $u(\alpha,\beta,p)$ under a squared error loss function is its posterior mean provided by

$$\hat{u}_{BS}(\alpha,\beta,p) = E(u(\alpha,\beta,p|\underline{x})) = \int_{\alpha}\int_{\beta}\int_{p} u(\alpha,\beta,p)q(\alpha,\beta,p\,|\,\underline{x})\,dp\,d\beta\,d\alpha. \quad (18)$$

The Bayes estimator of $u(\alpha,\beta,p)$ under a Linex loss function is given by

$$\hat{u}_{BL}(\alpha,\beta,p) = -\frac{1}{\xi}ln\Big(E(e^{-\xi u(\alpha,\beta,p)}\,|\,\underline{x})\Big) = -\frac{1}{\xi}ln\Big(\int_{\alpha}\int_{\beta}\int_{p} e^{-\xi u(\alpha,\beta,p)} q(\alpha,\beta,p|\underline{x})\,dp\,d\beta\,d\alpha\Big), \quad (19)$$

where ξ is the asymmetric parameter.

It is evident from Equations (18) and (19) that none of the integrals can be derived analytically, necessitating the employment of numerical integration methods.

The marginal posterior distributions and hence the Bayes estimates are computed from the posterior distribution (17) which includes complicated integrals that cannot be obtained in closed forms. Therefore the MCMC sampling procedure will be used to compute these Bayes estimates. The most two often utilized techniques of the MCMC methods are the Gibbs sampler and the Metropolis-Hastings techniques. The Gibbs sampler technique needs the conditional posterior distributions to be in closed forms that can be simply generated from them. On the other hand, the Metropolis-Hastings technique needs only to use a jumping or a proposal distribution to generate from it instead of some complex distribution. For the algorithm to be efficient, the jumping distribution should be easy to sample from it. These techniques have been established in a number of references, see for example, [16] and [17].

The conditional posterior density functions of α, β, and p, respectively, are provided by (17)

$$q_1(\alpha|\beta,p;\underline{x}) \propto \alpha^{D+a_1-1}\beta^{D\alpha}e^{-R_D^*(\beta T^*)^\alpha-\sum_{i=1}^{D}(1+R_i)(\beta x_i)^\alpha} e^{-\alpha b_1}[1-pe^{-(\beta T^*)^\alpha}]^{-R_D^*} \prod_{i=1}^{D} x_i^{\alpha-1}[1-pe^{-(\beta x_i)^\alpha}]^{-(2+R_i)}, \quad \alpha>0, \tag{20}$$

$$q_2(\beta|\alpha,p;\underline{x}) \propto \beta^{D\alpha+a_2-1}e^{-R_D^*(\beta T^*)^\alpha-\sum_{i=1}^{D}(1+R_i)(\beta x_i)^\alpha} e^{-\beta b_2}[1-pe^{-(\beta T^*)^\alpha}]^{-R_D^*} \prod_{i=1}^{D} x_i^{\alpha-1}[1-pe^{-(\beta x_i)^\alpha}]^{-(2+R_i)}, \quad \beta>0, \tag{21}$$

and

$$q_3(p|\alpha,\beta;\underline{x}) \propto (1-p)^n[1-pe^{-(\beta T^*)^\alpha}]^{-R_D^*} \prod_{i=1}^{D} x_i^{\alpha-1}[1-pe^{-(\beta x_i)^\alpha}]^{-(2+R_i)}, 0\le p\le 1. \tag{22}$$

As can be seen from (20), (21) and (22), these conditional posteriors cannot be reduced to closed forms and therefore we cannot sample directly from them for applying the Gibbs sampler technique. As a result, we'll use the Metropolis algorithm to create samples from these conditional posterior distributions, then construct Bayes estimates using the squared error and Linex loss functions. For this, we employ the Metropolis algorithm, as described by [18] as follows:

1. Start with initial values $\alpha^{(0)}$, $\beta^{(0)}$ and $p^{(0)}$ and set $i=1$.
2. Generate a candidate point α^* from a proposal normal distribution $N(\alpha^{(i-1)},1)$, and calculate the ratio of the posterior density at the candidate α^* and the current $\alpha^{(i-1)}$ points,
$$r_1=\frac{q_1(\alpha^*|\beta^{(i-1)},p^{(i-1)};\underline{x})}{q_1(\alpha^{(i-1)}|\beta^{(i-1)},p^{(i-1)};\underline{x})}.$$
3. Generate u from a U(0,1) distribution, If $u\le r_1$, accept α^* and set $\alpha^{(i)}=\alpha^*$, else set $\alpha^{(i)}=\alpha^{(i-1)}$.
4. Generate a candidate point β^* from a proposal normal distribution $N(\beta^{(i-1)},1)$, and calculate the ratio
$$r_2=\frac{q_2(\beta^*|\alpha^{(i)},p^{(i-1)};\underline{x})}{q_2(\beta^{(i-1)}|\alpha^{(i)},p^{(i-1)};\underline{x})}.$$
5. Generate u from a U(0,1) distribution, If $u\le r_2$, accept β^* and set $\beta^{(i)}=\beta^*$, else set $\beta^{(i)}=\beta^{(i-1)}$.
6. Again for the parameter p, generate a candidate point p^* from a proposal U(0,1) distribution and calculate the ratio
$$r_3=\frac{q_3(p^*|\alpha^{(i)},\beta^{(i)};\underline{x})}{q_3(p^{(i-1)}|\alpha^{(i)},\beta^{(i)};\underline{x})}.$$
7. Generate u from a U(0,1) distribution, If $u\le r_3$, accept p^* and set $p^{(i)}=p^*$, else set $p^{(i)}=p^{(i-1)}$.
8. i=i+1.
9. Repeat steps from 2-8 N times.
10. Under a squared error loss function, calculate the Bayes estimators of α, β, and p as follows
11. Calculate the Bayes estimators of α, β and p under a squared error loss function as follows
$$\hat{\alpha}_{BS}=\frac{1}{N-M}\sum_{i=M+1}^{N}\alpha^{(i)}, \qquad \hat{\beta}_{BS}=\frac{1}{N-M}\sum_{i=M+1}^{N}\beta^{(i)},$$
$$\hat{p}_{BS}=\frac{1}{N-M}\sum_{i=M+1}^{N}p^{(i)},$$
and under a Linex loss function, with the asymmetric parameter ξ, in the forms
$$\hat{\alpha}_{BL}=\frac{-1}{\xi}Ln\left[\frac{\sum_{i=M+1}^{N}e^{-\xi\alpha^{(i)}}}{N-M}\right],$$
$$\hat{\beta}_{BL}=\frac{-1}{\xi}Ln\left[\frac{\sum_{i=M+1}^{N}e^{-\xi\beta^{(i)}}}{N-M}\right],$$
$$\hat{p}_{BL}=\frac{-1}{\xi}Ln\left[\frac{\sum_{i=M+1}^{N}e^{-\xi p^{(i)}}}{N-M}\right],$$
where M is the burn-in period.

It may be noted that we use the maximum likelihood estimates of the parameters as the initial values $\alpha^{(0)}$, $\beta^{(0)}$ and $p^{(0)}$ in step 1.

5 Simulation Study

A Monte Carlo simulation study using a type-I PHC scheme is performed to compare the estimators of the

WG distribution parameters. Under two different loss functions, we examine the performance of the MLEs and their related Bayes estimates of the unknown parameters α, β, and p. Mathematica 7.0 is used to complete all calculations. The following processes are used while comparing the estimates.

1. For given values of the prior parameters, a_1, b_1, a_2 and b_2, generate α, β and p from the prior densities given by (13), (14) and (15), respectively.
2. For given values of sample size n and the progressive scheme $R_i' s$ with the generated values α, β and p in step (1), generate a progressive censored sample of size m using the algorithm described in [19] till the termination of the experiment which occurs at $min\{X_{m:m:n}, T\}$.
3. The maximum likelihood estimates are then obtained by solving the three nonlinear equations given by (8), (9) and (10) numerically.
4. The Bayes estimates under the two different loss functions are then obtained by applying the MCMC technique, as described above.
5. The above steps 2-4 are repeated 500 times and the mean squared errors (MSE) are then computed for the different estimators.

The Bayes estimates are computed based on 10000 MCMC values, where the first 1000 discarded as burn-in. Two different values of the asymmetric parameter $\xi = -3, 3$ are considered to get the corresponding Linex Bayes estimates.
Different combinations of n, m and progressive schemes $R_i, i = 1, \ldots, m$ are considered. The parameter values of α and β are generated from the prior distributions $G(1,2)$ and $G(2,2)$, respectively. The parameter value of p is generated from the $U(0,1)$. The predetermined termination time T is considered to be $T = 0.85$ and $T = 0.5$. The results of this simulation are presented in Tables (1) and (2).

6 Practical Example

[7] fitted an uncensored set of real data, which is previously analyzed by [10], to the WG model. This data set consists of 100 observations of breaking stress of carbon fibers. This data set is listed in Table (3).

Based on this set of data, [7] obtained the estimates of the WG distribution parameters $\widehat{\alpha} = 3.0093$, $\widehat{\beta} = 0.3148$ and $\widehat{p} = 0.3073$ by using an EM algorithm.
In this paper, a type-I PHC data is generated from this original data set to estimate the WG parameters α, β and p. Since we have no prior information about the parameters, the non-informative priors is assumed by setting $a_1 = b_1 = a_2 = b_2 = 0$. For $m = 70$ and a censoring scheme $R = (0^6, 10, 0^{19}, 10, 0^{28}, 10, 0^{14})$, we consider two cases
Case I: $T = 6$ for which $X_{m:m:n} < T$
Case II: $T = 2.7$ for which $X_{J:m:n} < T < X_{J+1:m:n}$,
where $T = 2.7$ is the median of the original data set. The Bayes estimates are computed based on 20000 MCMC samples, where

Table 1: The mean squared errors (MSE) of the ML and Bayes estimates when T=0.85

n	m	scheme		MLE	Bayes estimates		
					SEL	Linex	
						$\xi = 3$	$\xi = -3$
20	15	$(1^5, 0^{10})^*$	α	0.6388	0.5096	0.9203	0.3054
			β	0.2011	0.0453	0.0366	0.1045
			p	0.0574	0.0907	0.1591	0.0392
		$(0^{10}, 1^5)$	α	0.6754	0.5031	0.9079	0.3472
			β	0.2030	0.0429	0.0304	0.1054
			p	0.0696	0.0892	0.1575	0.0383
30	25	$(1^5, 0^{20})$	α	0.3778	0.3307	0.5815	0.2027
			β	0.1693	0.0355	0.0239	0.0763
			p	0.0675	0.0835	0.1502	0.0356
		$(0^{20}, 1^5)$	α	0.3280	0.2983	0.5396	0.1811
			β	0.1511	0.0399	0.0185	0.0893
			p	0.0701	0.0842	0.1511	0.0360
40	30	$(1^{10}, 0^{20})$	α	0.3227	0.2765	0.4662	0.1860
			β	0.1575	0.0407	0.0263	0.0790
			p	0.0683	0.0816	0.1471	0.0352
		$(0^{20}, 0^{10})$	α	0.2904	0.2766	0.4675	0.1838
			β	0.1556	0.0364	0.0204	0.0766
			p	0.0565	0.0797	0.1449	0.0340
50	40	$(1^{10}, 0^{30})$	α	0.2211	0.2218	0.3708	0.1374
			β	0.1354	0.0375	0.0233	0.0717
			p	0.0616	0.0775	0.1415	0.0334
		$(0^{30}, 1^{10})$	α	0.2114	0.2091	0.3516	0.1334
			β	0.1492	0.0379	0.0224	0.0744
			p	0.0627	0.0738	0.1369	0.0313

*The scheme: $(1^5, 0^{10})$ means $n = 20, m = 15$ and $R = (1,1,1,1,1,0,0,0,0,0,0,0,0,0,0)$

Table 2: The mean squared errors (MSE) of the ML and Bayes estimates when T=0.5

n	m	scheme		MLE	Bayes estimates		
					SEL	Linex	
						$\xi = 3$	$\xi = -3$
20	15	$(1^5, 0^{10})$	α	0.1191	0.8553	1.5461	0.4622
			β	0.2349	0.0688	0.1288	0.0934
			p	0.0422	0.0986	0.1669	0.0452
		$(0^{10}, 1^5)$	α	0.9256	0.8021	1.4264	0.4715
			β	0.2286	0.0489	0.0927	0.0809
			p	0.0411	0.0946	0.1629	0.0422
30	25	$(1^5, 0^{20})$	α	0.5456	0.5772	1.0395	0.3370
			β	0.2052	0.0452	0.0660	0.0875
			p	0.0501	0.0866	0.1544	0.0369
		$(0^{20}, 1^5)$	α	0.5872	0.5238	0.9524	0.3581
			β	0.1978	0.0446	0.0589	0.0852
			p	0.0505	0.0869	0.1544	0.0376
40	30	$(1^{10}, 0^{20})$	α	0.4953	0.4717	0.8249	0.3162
			β	0.1874	0.0422	0.0568	0.0823
			p	0.0465	0.0819	0.1492	0.0344
		$(0^{20}, 0^{10})$	α	0.3332	0.4185	0.7396	0.2565
			β	0.1674	0.0372	0.0408	0.0795
			p	0.0402	0.0827	0.1492	0.0355
50	40	$(1^{10}, 0^{30})$	α	0.3096	0.3628	0.6389	0.2299
			β	0.1775	0.0355	0.0414	0.0737
			p	0.0449	0.0808	0.1472	0.0345
		$(0^{30}, 1^{10})$	α	0.3052	0.3117	0.5409	0.2281
			β	0.1567	0.0413	0.0379	0.0828
			p	0.0511	0.0805	0.1461	0.0346

Table 3: Breaking stress of carbon fibers (GPa)

3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47	3.11
4.42	2.41	3.19	3.22	1.69	3.28	3.09	1.87	3.15	4.90
3.75	2.43	2.95	2.97	3.39	2.96	2.53	2.67	2.93	3.22
3.39	2.81	4.20	3.33	2.55	3.31	3.31	2.85	2.56	3.56
3.15	2.35	2.55	2.59	2.38	2.81	2.77	2.17	2.83	1.92
1.41	3.68	2.97	1.36	0.98	2.76	4.91	3.68	1.84	1.59
3.19	1.57	0.81	5.56	1.73	1.59	2.00	1.22	1.12	1.71
2.17	1.17	5.08	2.48	1.18	3.51	2.17	1.69	1.25	4.38
1.84	0.39	3.68	2.48	0.85	1.61	2.79	4.70	2.03	1.80
1.57	1.08	2.03	1.61	2.12	1.89	2.88	2.82	2.05	3.65

Table 4: Estimates of the WG distribution parameters α, β and p based on a real data set

Cases		MLE	MCMC		
			SEL	Linex	
				$\xi = 3$	$\xi = -3$
Case I T=6	α	3.1028	3.1602	2.9779	3.4636
	β	0.3137	0.3021	0.3001	0.3040
	p	0.3570	0.4067	0.3223	0.5031
Case II T=2.7	α	2.7859	2.8629	2.6462	3.1208
	β	0.3085	0.2679	0.2634	0.2712
	p	0.2272	0.4701	0.3759	0.5632
Complete Sample	α	3.0094	3.1328	2.9516	3.4628
	β	0.3148	0.2993	0.2974	0.3012
	p	0.3074	0.4092	0.3218	0.5106

the first 1000 itrations discarded as burn-in. The results are presented in Table (4).

7 Conclusions

The point estimation of the three unknown parameters of the Weibull-Geometric distribution using the type-I PHC scheme was explored in this study. The maximum likelihood and Bayes estimates of these parameters were calculated. It has been discovered that Bayes estimators cannot be produced in explicit forms and must be computed numerically using sophisticated integrals. As a result, the Metropolis-Hastings sampling methodology is used to obtain Bayes estimates under squared error and Linex loss functions using the MCMC method.

The MSE of the Bayes estimates of the WG parameters α, β, and p is substantially smaller than that of their corresponding ML estimates, as shown in Tables (1) and (2). It's also clear that Bayes estimates using the asymmetric (Linex) loss function have lower MSE than Bayes estimates using the symmetric SE loss function. It can also be shown that as sample sizes grow, the mean squared errors decrease.

Table (4) shows that the Bayes estimates based on the PHC scheme are good comparable with the similar estimates derived by Barreto-Souza et al. (2011) using the hale set of data in all circumstances.

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Conflict of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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