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Note On Some Identities of New Combinatorial Integers

Adem Kılıçman[1], Cenap Özel[2] and Erol Yılmaz[3]

[1]Department of Mathematics and Institute for Mathematical Research

University Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia

[2,3]Department of Mathematics, Abant Izzet Baysal University

Golkoy Kampusu, Bolu 14280, Turkey

Emails Address: akilicman@putra.upm.edu.my

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The binomial numbers $\binom{m}{n}$ are very important in several applications and satisfy several number of identities. The purpose of this paper is to introduce a new combinatorial integer $\binom{m,n}{j}$ and obtain some algebraic identities by means of double combinatorial argument. Further several arithmetic properties of this type of integers are proven and some interesting identities are also provided.

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1 Introduction

The binomial coefficients, denoted by $\binom{n}{k}$, play an important role in combinatorics and these numbers appear as coefficients in the expansion of the binomial expression $(x+y)^n$. That is,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{1.1}$$

and this identity is known as Binomial theorem. The binomial coefficients are also known as combinations or combinatorial numbers. In fact the equation (1.1) has very close relation to the to the discussion of prime numbers. Further, primes come up in many different places in the mathematical literature, and there are a lot of discussions to distinguish primes from the composites.

In the literature the well known and the most amazing properties of prime numbers, discovered by Fermat that, if n is prime, then n divides $a^n - a$ for all integers a. That is

$a^n \equiv a \pmod{n}$, for all integers a and n. Now if n is a prime number then it was proved that $(x+y)^n \equiv x+y \equiv x^n + y^n \pmod{n}$ for all integers x, y and primes n. Then we state the following related theorem which was proved in [1].

Theorem 1.1. Integer n is prime if and only if $(x+1)^n \equiv x^n + 1 \pmod{n}$ in $Z[x]$.

The arithmetical properties of binomial coefficients have also been studied by many authors, for example see [3]. The sequence of middle binomial coefficients $(a_n) = \binom{2n}{n}$ known also as central binomial coefficient is an particular interest to many people and have the following generating function

$$\frac{1}{\sqrt{1-4x}} = 1 + 2x + 6x^2 + 20x^3 + 70x^4 + 252x^5 + \dots. \tag{1.2}$$

The middle binomial coefficients also play an significant role in Erdös Conjecture that is widely known as square-free integers, see [2]. In constructing the properties and identities of some special numbers, binomial coefficients are frequently involved. Note that the definition of the binomial coefficients was extended in [3] where n can be a complex number. However there are still many properties and identities that one can establish by using the binomial coefficients. In the next we introduce the combinatorial integers $\binom{n,m}{j}$ which is useful for the calculations in cohomology.

2 Identities on Combinatorial Integers

Now recall the equation (1.1) and consider to multiply by $(x-y)$ then we have

$$(x+y)^n(x-y) = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}(x-y) = \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} - \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k+1}. \tag{2.1}$$

More general,

$$\begin{aligned}(x+y)^n(x-y)^m &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}(x-y)^m \\ &= \sum_{k=0}^{n} \binom{n}{k} \left(\sum_{j=0}^{m} \binom{m}{j} x^j(-y)^{m-j} \right) x^k y^{n-k} \\ &= \sum_{k=0}^{n} \sum_{j=0}^{m} (-1)^{m-j} \binom{n}{k} \binom{m}{j} x^{k+j} y^{m+n-j-k} \end{aligned} \tag{2.2}$$

where $m, n \geq 0$. Now we let $P_n(x)$ denote the Legendre polynomials of nth order. Then the function

$$P(x,y) = \sum_{k=0}^{n} P_k(x) y^k = (1 - 2xy + y^2)^{-1/2}$$

is the generating function for Legendre polynomials. Then we can easily have

$$P(1+2x,y)=(1-y)^{-1}[1-4xy(1-y)^{-2}]^{-1/2},$$

where $Q_n(x)=P_n(1+2x), Q(x,y)=P(1+2x,y)$ and

$$\begin{aligned} P(1+2x) &= \lim_{n\to\infty} P_n(1+2x) \\ Q(x,y) &= \lim_{n\to\infty} P_n(1+2x,y)=P(1+2x,y). \end{aligned}$$

Now we have two expansions

$$\begin{aligned} Q(x,y) &= (1-y))^{-1}\left[1-4xy(1-y)^{-2}\right]^{-1/2}, \\ &= \sum_{k=0}\binom{2k}{k}x^k y^k(1-y)^{-2k-1} \\ &= \sum_{n=0} y^n \sum_{k=0}\binom{n+k}{2k}\binom{2k}{k}x^k, \end{aligned}$$

$$\begin{aligned} Q(x,y) &= (1-(1+2x)y))^{-1}\left[1-4(x+x^2)y^2(1-y-2xy)^{-2}\right]^{-1/2}, \\ &= \sum_{n=0} y^n \sum_{k=0}\binom{n}{2k}\binom{2k}{k}(1+2x)^{n-2k}(x+x^2)^k, \qquad (2.3) \end{aligned}$$

so that

$$Q_n(x)=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}x^k=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{2k}\binom{2k}{k}(1+2x)^{n-2k}(x+x^2)^k.$$

Then

$$Q_n(x)=\sum_{k=0}^{n} q_{n,k}x^k(1+x)^{n-k} \quad \text{where}$$

$$\begin{aligned} q_{n,k} &= \sum_{j=0}^{k}\binom{n}{2j}\binom{2j}{j}\binom{n-2j}{k-j}, \\ &= \binom{n}{k}\sum_{j=0}^{k}\binom{k}{k-j}\binom{n-k}{j}=\binom{n}{k}^2 \quad \text{(by Vandermonde convolution)} \end{aligned}$$

so that

$$Q_n(x)=\sum_{k=0}^{n}\binom{n}{k}^2 x^k(1+x)^{n-k}$$

see [4]. Now by making substitutions for x we can obtain several identities, for example, if we replace $x=-\frac{1}{2}$ in equation (2.3) then it follows that

$$Q\left(-\frac{1}{2},y\right)=\left(1+y^2\right)^{-1/2}=\sum_{n=0}(-1)^n\binom{2n}{n}2^{-2n}y^{2n},$$

then we obtain the following identities

$$(-1)^n\binom{2n}{n} = \sum_{k=0}^{2n}(-1)^k\binom{2n+k}{2k}\binom{2k}{k}2^{2n-k} = \sum_{k=0}^{2n}\binom{2n}{k}^2(-1)^k.$$

Now by using the right hand side of the equation (2.2) and symmetric properties of the binomial coefficients respectively we can introduce a new and interesting integer sequence $\binom{n,m}{j}$ that will simplify the calculation and we will call it twin pairs binomial coefficients. In particular we can represent

$$(1+x)^n(1-x)^m = \sum_{j=0}^{n+m}\binom{n,m}{j}x^j.$$

Note that if we consider $x=1$ and $x=-1$ then it is easy to obtain

$$\sum_{j=0}^{n+m}\binom{n,m}{j} = \sum_{j=0}^{n+m}(-1)^j\binom{n,m}{j} = 0.$$

In fact, this type of sequences play an important role for the computation in cohomology, see [5]. Through out this study we call it *twin pairs binomial coefficients* as follows:

$$\binom{m,n}{j} = \sum_{k=0}^{j}(-1)^k\binom{m}{k}\binom{n}{j-k}$$

where $n,m \geq 0$ integers and $0 \leq j \leq m+n$, and of course

$$\binom{m}{k} = \begin{cases} \frac{m!}{k!(m-k)!} & \text{if } m \geq k, \\ 0 & \text{if } m < k. \end{cases}$$

Note that the generating function of the integer sequence of $\binom{m,n}{j}$ is the function $(1+x)^n(1-x)^m$. By definition of $\binom{m,n}{j}$ we can easily show that

$$\begin{aligned} \binom{m,n}{j} &= \sum_{k=0}^{j}(-1)^k\binom{m}{k}\binom{n}{j-k} = \sum_{k=0}^{j}(-1)^k\binom{n}{j-k}\binom{m}{k} \\ &= \sum_{k=0}^{j}(-1)^{k-j}\binom{n}{k}\binom{m}{j-k} \\ &= \begin{cases} \binom{n,m}{j} & \text{if } j \text{ even}, \\ -\binom{n,m}{j} & \text{if } j \text{ odd}, \end{cases} \end{aligned}$$

and hence $\binom{n,n}{j} = 0$ whenever j is odd. The calculation of the twin pairs binomial coefficients also play a crucial role in the calculation of generators for ideals. Further if we let G be a compact semi-simple Lie group, LG the space of smooth loops on the group G and T is the maximal torus of G, then twin pairs of binomial coefficients will also be very useful during the determination of the rank for each module which is graded by integral cohomology algebra of finite dimensional flag manifolds LG/T and ΩG in the local coefficient ring $Z[\frac{1}{2}]$ for $G = A_2$, see [5]. Now the symmetry and anti-symmetry property can be given in the following theorem.

Theorem 2.1. *Let n be a non-negative integer. For $k = 0, 1, 2, \ldots, n$ we have*

$$\binom{k, n-k}{k} = \begin{cases} \binom{n-k,k}{n-k} & \textit{if } n \textit{ even}, \\ -\binom{n-k,k}{n-k} & \textit{if } n \textit{ odd}. \end{cases}$$

Proof. By definition, for $k = 0, 1, 2, \ldots, n$ we have

$$\begin{aligned} \binom{k, n-k}{k} &= \sum_{i=0}^{k} (-1)^i \binom{k}{i}\binom{n-k}{k-i} = \sum_{i=0}^{k} (-1)^i \binom{n-k}{k-i}\binom{k}{i} \\ &= \sum_{i=0}^{k} (-1)^i \binom{n-k}{n+i-2k}\binom{k}{k-i} \\ &= \sum_{i=n-2k}^{n-k} (-1)^{i-n+2k} \binom{n-k}{i}\binom{k}{n-k-i}, \\ &= \begin{cases} \displaystyle\sum_{i=n-2k}^{n-k} (-1)^i \binom{n-k}{i}\binom{k}{n-k-i} & \text{if} \quad n \text{ even} \\ \displaystyle -\sum_{i=n-2k}^{n-k} (-1)^i \binom{n-k}{i}\binom{k}{n-k-i} & \text{if} \quad n \text{ odd.} \end{cases} \end{aligned}$$

Since for $i < n - 2k$, we have $n - k - i > k$ so it follows that $\binom{k}{n-k-i} = 0$ where $i = 0, 1, \ldots, n - 2k - 1$. Therefore we have

$$\binom{k, n-k}{k} = \begin{cases} \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i}\binom{k}{n-k-i} & \text{if} \quad n \text{ even} \\ -\sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i}\binom{k}{n-k-i} & \text{if} \quad n \text{ odd.} \end{cases}$$

Hence we have the desired result. $\square$

We note that the twin pairs binomial coefficients have also similar properties to the binomial coefficients. In particular

$$\begin{array}{lccccccccccc}
n=0 & & & & & & \binom{0,0}{0} & & & & & \\
n=1 & & & & & \binom{0,1}{0} & & \binom{1,0}{1} & & & & \\
n=2 & & & & \binom{0,2}{0} & & \binom{1,1}{1} & & \binom{2,0}{2} & & & \\
n=3 & & & \binom{0,3}{0} & & \binom{1,2}{1} & & \binom{2,1}{2} & & \binom{3,0}{3} & & \\
n=4 & & \binom{0,4}{0} & & \binom{1,3}{1} & & \binom{2,2}{2} & & \binom{3,1}{3} & & \binom{4,0}{4} & \\
n=5 & \binom{0,5}{0} & & \binom{1,4}{1} & & \binom{2,3}{2} & & \binom{3,2}{3} & & \binom{4,1}{4} & & \binom{5,0}{5}
\end{array}$$

Similar to triangular properties in the single binomial coefficients form we also we have the following identities:

Theorem 2.2. *Let r, s, l, p be non-negative integers. Then*

$$(\textit{right shifting property}) \quad \binom{r,s}{l} = \binom{r,s-1}{l} + \binom{r,s-1}{l-1}, \tag{2.4}$$

$$(\textit{left shifting property}) \quad \binom{r,s}{l} = \binom{r-1,s}{l} - \binom{r-1,s}{l-1}, \tag{2.5}$$

$$(\textit{right shifting expansion}) \quad \binom{r,s}{l} = \sum_{i=0}^{l} \binom{r,s-i-1}{l-i}, \tag{2.6}$$

$$(\textit{Vandermonde convolution}) \quad \binom{r,s}{l} = \sum_{i=0}^{p} \binom{r,s-p}{l-i}\binom{p}{i}, \tag{2.7}$$

$$2\binom{r-1,s-1}{l} = \binom{r-1,s}{l} + \binom{r,s-1}{l} \quad \textit{and} \tag{2.8}$$

$$2\binom{r-1,s-1}{l-1} = \binom{r-1,s}{l} - \binom{r,s-1}{l}. \tag{2.9}$$

Proof. First we shall prove that equation (2.4) holds. Then

$$\begin{aligned}
\binom{r,s}{l} &= \sum_{i=0}^{l}(-1)^i\binom{r}{i}\binom{s}{l-i} = \sum_{i=0}^{l}(-1)^i\binom{r}{i}\left[\binom{s-1}{l-i} + \binom{s-1}{l-i-1}\right] \\
&= \sum_{i=0}^{l}(-1)^i\binom{r}{i}\binom{s-1}{l-i} + \sum_{i=0}^{l}(-1)^i\binom{r}{i}\binom{s-1}{l-1-i}.
\end{aligned}$$

Since $\binom{s-1}{-1} = 0$, then

$$\binom{r,s}{l} = \sum_{i=0}^{l}(-1)^i\binom{r}{i}\binom{s-1}{l-i} + \sum_{i=0}^{l-1}(-1)^i\binom{r}{i}\binom{s-1}{l-1-i} = \binom{r,s-1}{l} + \binom{r,s-1}{l-1}.$$

Let l be even. Then we have

$$\begin{aligned}\binom{r,s}{l} &= \binom{s,r}{l} = \binom{s,r-1}{l} + \binom{s,r-1}{l-1}\\ &= \binom{r-1,s}{l} - \binom{r-1,s}{l-1} \text{this is the equation (2.5).}\end{aligned}$$

Now let l be odd. Then we have the equation (2.6) as follows:

$$\begin{aligned}\binom{r,s}{l} &= -\binom{s,r}{l} = -\binom{s,r-1}{l} - \binom{s,r-1}{l-1}\\ &= \binom{r-1,s}{l} - \binom{r-1,s}{l-1}.\end{aligned}$$

If we take the sum (difference) of both sides of equations (2.4) and (2.5), then we obtain equations (2.8) and (2.9). Equations (2.6) and (2.7) can be also obtained from equation (2.4). □

Theorem 2.3. *Let r, s, l be non-negative integers. Then*

$$s\binom{r,s-1}{l-1} = (l-r)\binom{r,s}{l} + r\binom{r-1,s}{l}, \tag{2.10}$$

$$r\binom{r-1,s}{l-1} = -(l-s)\binom{r,s}{l} - s\binom{r,s-1}{l}, \tag{2.11}$$

$$(r+s-l)\binom{r,s}{l} = r\binom{r-1,s}{l} + s\binom{r,s-1}{l}. \tag{2.12}$$

Proof. Let us begin the proof of the first equation (2.10). Then

$$\begin{aligned}&(l-r)\binom{r,s}{l} + r\binom{r-1,s}{l}\\ &= (l-r)\sum_{i=0}^{l}(-1)^i\binom{r}{i}\binom{s}{l-i} + r\sum_{i=0}^{l}(-1)^i\binom{r-1}{i}\binom{s}{l-i}\end{aligned}$$

$$= \sum_{i=0}^{l}(-1)^i\left[(l-r)\frac{r!}{i!(r-i)!}\frac{s!}{(l-i)!(s-l+i)!} + \frac{r!}{i!(r-i-1)!}\frac{s!}{(l-i)!(s-l+i)!}\right]$$

$$= \sum_{i=0}^{l}(-1)^i(l-r+r-i)\frac{r!s!}{i!(r-i)!(l-i)!(s-l+i)!}$$

$$= s\sum_{i=0}^{l}(-1)^i\frac{r!}{i!(r-i)!}\frac{(s-1)!}{(l-i-1)!(s-l+i)!} = s\sum_{i=0}^{l}(-1)^i\binom{r}{i}\binom{s-1}{l-1-i}$$

$$= s\sum_{i=0}^{l-1}(-1)^i\binom{r}{i}\binom{s-1}{l-1-i}$$

since $\binom{s-1}{-1} = 0$. Therefore we have

$$(l-r)\binom{r,s}{l} + r\binom{r-1,s}{l} = s\binom{r,s-1}{l-1}.$$

If l is odd then we have

$$\begin{aligned} r\binom{r-1,s}{l-1} &= r\binom{s,r-1}{l-1} = (l-s)\binom{s,r}{l} + s\binom{s-1,r}{l} \\ &= -(l-s)\binom{r,s}{l} - s\binom{r,s-1}{l}. \end{aligned}$$

If l is even then we have

$$\begin{aligned} r\binom{r-1,s}{l-1} &= -r\binom{s,r-1}{l-1} = -(l-s)\binom{s,r}{l} - s\binom{s-1,r}{l} \\ &= -(l-s)\binom{r,s}{l} - s\binom{r,s-1}{l}. \end{aligned}$$

By using the equations (2.10) and (2.11) we obtain the equation (2.12) as follows

$$\begin{aligned} r\binom{r-1,s}{l} + s\binom{r,s-1}{l} &= s\binom{r,s-1}{l-1} - (l-r)\binom{r,s}{l} + (s-l)\binom{r,s}{l} - r\binom{r-1,s}{l} \\ &= 2(r+s-l)\binom{r,s}{l} - s\binom{r,s-1}{l} - r\binom{r-1,s}{l} \end{aligned}$$

and hence we have

$$2\left\{r\binom{r-1,s}{l} + s\binom{r,s-1}{l}\right\} = 2(r+s-l)\binom{r,s}{l}.$$

□

Lemma 2.1. *Let n be a non-negative integer and $k = 0, 1, 2, \ldots, n$ then*

$$\sum_{j=0}^{n}\binom{k,n-k}{j} = \begin{cases} 2^n & \textit{if} \quad k = 0 \\ 0 & \textit{if} \quad k \neq 0. \end{cases}$$

Proof. For $k = 0$,

$$\sum_{j=0}^{n}\binom{k,n-k}{j} = \sum_{j=0}^{n}\binom{n}{j} = 2^n.$$

Let $k \neq 0$. Since

$$(1+x)^{n-k}(1-x)^k = \sum_{j=0}^{n}\binom{k,n-k}{j}x^j,$$

for $x = 1$, then we have

$$0 = \sum_{j=0}^{n}\binom{k,n-k}{j}.$$

□

Similarly we have the following result.

Lemma 2.2. *Let n be a non-negative integer. For $k = 0, 1, 2, \ldots, n$ we have*

$$\sum_{j=0}^{n} (-1)^j \binom{k, n-k}{j} = \begin{cases} 2^n & \text{if} \quad k = n \\ 0 & \text{if} \quad k \neq n. \end{cases}$$

Thus we can state this result as the following corollary.

Corollary 2.1. *(Twin pairs) Let n be a non-negative integer. Then we have*

$$\binom{2n, 2n}{2n} = \sum_{j=0}^{2n} (-1)^j \binom{2n}{j}^2 = (-1)^n \binom{2n}{n}.$$

Note that in particular we can easily show that

$$\binom{n, n}{n} = \begin{cases} (-1)^{\frac{n}{2}} \binom{n}{\frac{n}{2}} & \text{if} \quad n \text{ even}, \\ 0 & \text{if} \quad n \text{ odd}. \end{cases}$$

Now let us consider addition of the twin binomial coefficients as follows:

$$\begin{array}{llllllr}
\binom{0,0}{0} & & & & & & = 1 \\
\binom{0,1}{0} & + \binom{1,0}{1} & & & & & = 0 \\
\binom{0,2}{0} & + \binom{1,1}{1} & + \binom{2,0}{2} & & & & = 2^1 \\
\binom{0,3}{0} & + \binom{1,2}{1} & + \binom{2,1}{2} & + \binom{3,0}{3} & & & = 0 \\
\binom{0,4}{0} & + \binom{1,3}{1} & + \binom{2,2}{2} & + \binom{3,1}{3} & + \binom{4,0}{4} & & = 2^2 \\
\binom{0,5}{0} & + \binom{1,4}{1} & + \binom{2,3}{2} & + \binom{3,2}{3} & + \binom{4,1}{4} & + \binom{5,0}{5} & = 0
\end{array}$$

and so on. Thus in general case we have the following theorem.

Theorem 2.4. *(Diagonal formula) Let n be a non-negative integer. Then we have*

$$\sum_{k=0}^{n} \binom{k, n-k}{k} = \begin{cases} 2^{\frac{n}{2}} & \text{if} \quad n \text{ even}, \\ 0 & \text{if} \quad n \text{ odd}. \end{cases}$$

Proof: We prove this in two cases:

Case 1. Let n be odd. Then by using the theorem (2.1) we have

$$\sum_{k=0}^{n} \binom{k, n-k}{k} = -\sum_{k=0}^{n} \binom{n-k, k}{n-k}$$

then by making substitution $n-k=j$ we have

$$\sum_{k=0}^{n}\binom{k,n-k}{k}=-\sum_{j=0}^{n}\binom{j,n-j}{j}=-\sum_{k=0}^{n}\binom{k,n-k}{k}$$
$$\Rightarrow 2\sum_{k=0}^{n}\binom{k,n-k}{k}=0\Rightarrow\sum_{k=0}^{n}\binom{k,n-k}{k}=0.$$

Case 2. If n is even integer then we will show that $\sum_{k=0}^{n}\binom{k,n-k}{k}=2^{\frac{n}{2}}$. The proof will be done by using the induction.
Case is trivial for $n=0$ since we have $\binom{0,0}{0}=1=2^{\frac{0}{2}}=2^0$.
For $0\le j\le n$ with j is even and assume that $\sum_{k=0}^{j}\binom{k,j-k}{k}=2^{\frac{j}{2}}$.
Our aim is now to show $\sum_{k=0}^{n+2}\binom{k,n+2-k}{k}=2^{\frac{n+2}{2}}$.
Now if $j=n$ since $\sum_{k=0}^{n}\binom{k,n-k}{k}=2^{n/2}$ then it follows that $\sum_{k=0}^{n+1}\binom{k,n-k}{k}=0$. Now

$$\begin{aligned}
\sum_{k=0}^{n+2}\binom{k,n+2-k}{k} &= 1+\sum_{k=1}^{n+1}\binom{k,n+1-k}{k}+\binom{k,n+1-k}{k-1}+1\\
&= \sum_{k=0}^{n+1}\binom{k,n+1-k}{k}+\sum_{k=1}^{n+1}\binom{k,n+1-k}{k-1}\\
&= 2+\sum_{k=0}^{n}\left[\binom{k,n-k}{k}-\binom{k,n-k}{k-1}\right]\\
&= 1+\sum_{k=0}^{n}\binom{k,n-k}{k}-\sum_{k=1}^{n}\binom{k,n-k}{k-1}\ \text{on using shifting properties}\\
&= 1+2^{\frac{n}{2}}-\sum_{k=0}^{n-1}\binom{k+1,n-k-1}{k}\\
&= 1+2^{\left(\frac{n}{2}\right)}+2^{\left(\frac{n-2}{2}\right)}+2^{\left(\frac{n-4}{2}\right)}+\ldots+2^{\left(\frac{n-2i}{2}\right)}+\sum_{k=1}^{n-2i}\binom{k,n-2i-k}{k-1}
\end{aligned}$$

for $0\le i\le\frac{n}{2}$. On using the geometric series summation we obtain

$$\sum_{k=0}^{n+2}\binom{k,n+2-k}{k}=2^{\left(\frac{n+2}{2}\right)}$$

that completes the proof.

Theorem 2.5. *(Orthogonality formula) Let n be a non-negative integer. For $i, j = 0, 1, 2, \ldots, n$ we have*

$$\frac{1}{2^n}\sum_{k=0}^{n}\binom{k, n-k}{i}\binom{j, n-j}{k} = \delta_{ij} = \begin{cases} 1 & \textit{if} \quad i = j, \\ 0 & \textit{if} \quad i \neq j. \end{cases}$$

Proof: Similar to the previous theorem we also use the induction methods on n. The case is very obvious for $n = 1$. Suppose that

$$\frac{1}{2^n}\sum_{k=0}^{n}\binom{k, n-k}{i}\binom{j, n-j}{k} = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

is true. Then we will show that

$$\frac{1}{2^{n+1}}\sum_{k=0}^{n+1}\binom{k, n-k+1}{i}\binom{j, n-j+1}{k} = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

There are two different cases:

(i) Let $i = j$ then it follows that

$$\begin{aligned}
&\sum_{k=0}^{n+1}\binom{k, n-k+1}{i}\binom{j, n-j+1}{k} = \binom{n+1}{i} + \binom{n+1}{i} \\
+ &\sum_{k=1}^{n}\left(\left[\binom{k, n-k}{i} + \binom{k, n-k}{i-1}\right]\left[\binom{i, n-i}{k} + \binom{i, n-i}{k-1}\right]\right) \\
= \; & 2\binom{n+1}{i} + \sum_{k=0}^{n}\binom{k, n-k}{i}\binom{i, n-i}{k} + \sum_{k=1}^{n}\binom{k, n-k}{i}\binom{i, n-i}{k-1} \\
+ &\sum_{k=1}^{n}\binom{k, n-k}{i-1}\binom{i, n-i}{k} + \sum_{k=1}^{n}\binom{k, n-k}{i-1}\binom{i, n-i}{k-1} \\
= \; & \binom{n}{i-1} + 2^n + \sum_{k=0}^{n}\binom{k, n-k}{i}\binom{i, n-i}{k} \\
- &\sum_{k=0}^{n}\binom{k, n-k}{i-1}\binom{i, n-i}{k} + (-1)^{2i-1}\binom{n}{i-1} \\
= \; & 2^n + 2^n = 2(2^n) = 2^{n+1}
\end{aligned}$$

(ii) Let $i \neq j$.

Then there are two subcases:

Case 1. If $i - 1 \neq j$. Then we have

$$
\begin{aligned}
& \sum_{k=0}^{n+1} \binom{k, n-k+1}{i} \binom{j, n-j+1}{k} = \binom{n+1}{i} + (-1)^{i+j} \binom{n+1}{i} \\
+ \quad & \sum_{k=1}^{n} \left[\binom{k, n-k}{i} + \binom{k, n-k}{i-1} \right] \left[\binom{j, n-j}{k} + \binom{j, n-j}{k-1} \right] \\
= \quad & \binom{n+1}{i} + (-1)^{i+j} \binom{n+1}{i} - \binom{n}{i} \\
+ \quad & \sum_{k=0}^{n-1} \binom{k+1, n-k-1}{i} \binom{j, n-j}{k} \\
- \quad & \binom{n}{i-1} + \sum_{k=0}^{n-1} \binom{k+1, n-k-1}{i-1} \binom{j, n-j}{k} \\
= \quad & (-1)^{i+j} \binom{n+1}{i} - (-1)^{i+j} \binom{n}{i} + (-1)^{i+j-1} \binom{n+1}{i-1} \\
= \quad & 0
\end{aligned}
$$

on noting that $\sum_{k=0}^{n} \binom{k, n-k}{i} \binom{j, n-j}{k} = 0.$

Case 2. If $j = i - 1$ then the proof is similar to the case 1 on noting that

$$\sum_{k=0}^{n} \binom{k, n-k}{i} \binom{j, n-j}{k} = 0$$

and

$$\sum_{k=0}^{n} \binom{k, n-k}{i-1} \binom{j, n-j}{k} = 2^n.$$

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Adem Kilicman is a full professor at the Department of Mathematics, Faculty of Science, University Putra Malaysia. He received his B.Sc. and M.Sc. degree from Department of Mathematics, Hacettepe University, Turkey and Ph.D from Leicester University, UK. He has joined University Putra Malaysia in 1997 since then working with Faculty of Science and he is also active member of Institute for Mathematical Research, University Putra Malaysia. His research areas includes Functional Analysis and Topology.

C. Özel has got PhD from University of Glasgow, Scotland, 1998. He is the recipient of Sedat Simavi Foundation National Science Laudable prize from Turkish Journalists Society 2005, and Young Outstanding Scientist prize from Turkish Science Academy 2007. His interests are Algebraic Topology especially complex cobordism of infinite dimensional manifolds, Lie theory, Quantum Mechanics and Mathematical Physics. He is a member of Abant Izzet Baysal University AIBU, Bolu Turkey since 1998.

Erol Yilmaz graduated from Istanbul Technical University in 1992 with B.Sc degree in Mathematical engineering. He received hit M.Sc and Ph.D from The University of Texas at Arlington in 1996 and 1999 respectively. His researches include the lifting problem of polynomial ideals, computing primary decompositions, radicals and envelopes of submodules of Noetherian modules, computational algebraic approach to center and cyclicity problems of dinamical systems, and problems about homogeneous ideals and syzygies. He generally used Gr?bner Basis Techniques in his works. Erol Yilmaz is member of Mathematic department of Abant Izzet Baysal University, Bolu, Turkey since 2000.