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Weak Insertion of a Contra-Continuous Function between two Comparable Contra-$\alpha-$Continuous (Contra-$C-$Continuous) Function

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Abstract: A sufficient condition in terms of lower cut sets are given for the weak insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that kernel of sets are open.

Keywords: Insertion, Strong binary relation, $C-$open set, Semi-preopen set, $\alpha-$open set, Contra-continuous function, Lower cut set

1 Introduction

The concept of a $C-$open set in a topological space was introduced by E. Hatir, T. Noiri and S. Yũksel in [12]. The authors define a set S to be a $C-$*open* set if $S = U \cap A$, where U is open and A is semi-preclosed. A set S is a $C-$*closed* set if its complement (denoted by S^c) is a $C-$open set or equivalently if $S = U \cup A$, where U is closed and A is semi-preopen. The authors show that a subset of a topological space is open if and only if it is an $\alpha-$open set and a $C-$open set or equivalently a subset of a topological space is closed if and only if it is an $\alpha-$closed set and a $C-$closed set. This enables them to provide the following decomposition of continuity: a function is continuous if and only if it is $\alpha-$continuous and $C-$continuous or equivalently a function is contra-continuous if and only if it is contra-$\alpha-$continuous and contra-$C-$continuous.

Recall that a subset A of a topological space (X, τ) is called $\alpha-$*open* if A is the difference of an open and a nowhere dense subset of X [4]. A set A is called $\alpha-$*closed* if its complement is $\alpha-$open or equivalently if A is the union of a closed and a nowhere dense set. Sets which are dense in some regular closed subspace are called *semi-preopen or* $\beta-$*open* [4]. A set is *semi-preclosed or* $\beta-$*closed* if its complement is semi-preopen or $\beta-$open.

In [7] it was shown that a set A is $\beta-$open if and only if $A \subseteq Cl(Int(Cl(A)))$. A generalized class of closed sets was considered by Maki in [19]. He investigated the sets that can be represented as union of closed sets and called them $V-$sets. Complements of $V-$sets, i.e., sets that are intersection of open sets are called $\Lambda-$sets [19].
Recall that a real-valued function f defined on a topological space X is called $A-$continuous [22] if the preimage of every ope subset of $\mathbb{R}$ belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [4, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.
J. Dontchev in [5] introduced a new class of mappings called contra-continuity.S. Jafari and T. Noiri in [13, 14] exhibited and studied among others a new weaker form of this class of mappings called contra-$\alpha-$continuous. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 21].
Hence, a real-valued function f defined on a topological space X is called *contra-continuous* (resp. *contra-$C-$continuous* , *contra-$\alpha-$continuous*) if the preimage of every open subset of $\mathbb{R}$ is closed (resp.

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$C-$closed , $\alpha-$closed) in X[5].
Results of Katětov [15, 16] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that $\Lambda-$sets or kernel of sets are open [19].
If g and f are real-valued functions defined on a space X, we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X. The following definitions are modifications of conditions considered in [17].
A property P defined relative to a real-valued function on a topological space is a *$cc-$property* provided that any constant function has property P and provided that the sum of a function with property P and any contra-continuous function also has property P. If P_1 and P_2 are $cc-$properties, the following terminology is used: A space X has the *weak $cc-$insertion property* for (P_1,P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g \leq h \leq f$.
In this paper, for a topological space whose $\Lambda-$sets or kernel of sets are open, is given a sufficient condition for the weak $cc-$insertion property. Several insertion theorems are obtained as corollaries of these results.

2 The Main Result

Before giving a sufficient condition for insertability of a contra-continuous function, the necessary definitions and terminology are stated.
Definition 2.1. Let A be a subset of a topological space (X,τ). We define the subsets A^Λ and A^V as follows:
$A^\Lambda = \cap\{O : O \supseteq A, O \in (X,\tau)\}$ and $A^V = \cup\{F : F \subseteq A, F^c \in (X,\tau)\}$.
In [6, 18, 20], A^Λ is called the *kernel* of A.
The family of all $\alpha-$open, $\alpha-$closed, $C-$open and $C-$closed will be denoted by $\alpha O(X,\tau)$, $\alpha C(X,\tau)$, $CO(X,\tau)$ and $CC(X,\tau)$, respectively.
We define the subsets $\alpha(A^\Lambda), \alpha(A^V), C(A^\Lambda)$ and $C(A^V)$ as follows:
$\alpha(A^\Lambda) = \cap\{O : O \supseteq A, O \in \alpha O(X,\tau)\}$,
$\alpha(A^V) = \cup\{F : F \subseteq A, F \in \alpha C(X,\tau)\}$,
$C(A^\Lambda) = \cap\{O : O \supseteq A, O \in CO(X,\tau)\}$ and
$C(A^V) = \cup\{F : F \subseteq A, F \in CC(X,\tau)\}$.
$\alpha(A^\Lambda)$ (resp. $C(A^\Lambda)$) is called the $\alpha-$*kernel* (resp. $C-$*kernel*) of A.
The following first two definitions are modifications of conditions considered in [15, 16].
Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x\ \bar{\rho}\ y$ if and only if $y\ \rho\ v$ implies $x\ \rho\ v$ and $u\ \rho\ x$ implies $u\ \rho\ y$ for any u and v in S.
Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:
1) If $A_i\ \rho\ B_j$ for any $i \in \{1,\ldots,m\}$ and for any $j \in \{1,\ldots,n\}$, then there exists a set C in $P(X)$ such that $A_i\ \rho\ C$ and $C\ \rho\ B_j$ for any $i \in \{1,\ldots,m\}$ and any $j \in \{1,\ldots,n\}$.
2) If $A \subseteq B$, then $A\ \bar{\rho}\ B$.
3) If $A\ \rho\ B$, then $A^\Lambda \subseteq B$ and $A \subseteq B^V$.
The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:
Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f,\ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f,\ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.
We now give the following main result:
Theorem 2.1. Let g and f be real-valued functions on the topological space X, in which kernel sets are open, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$, then there exists a contra-continuous function h defined on X such that $g \leq h \leq f$.
Proof. Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$. Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f,t)$ and $G(t) = A(g,t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2), G(t_1)\ \bar{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [16] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2), H(t_1)\ \rho\ H(t_2)$ and $H(t_1)\ \rho\ G(t_2)$. For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$. We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g,t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f,t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$. Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1,t_2) = H(t_2)^V \setminus H(t_1)^\Lambda$. Hence $h^{-1}(t_1,t_2)$ is closed in X, i.e., h is a contra-continuous function on X. ■
The above proof used the technique of theorem 1 in [15].

3 Applications

The abbreviations $c\alpha c$ and cCc are used for contra-$\alpha-$continuous and contra-$C-$continuous, respectively. Before stating the consequences of theorems 2.1 we suppose that X is a topological space whose kernel sets are open.

Corollary 3.1. If for each pair of disjoint $\alpha-$open (resp. $C-$open) sets G_1, G_2 of X, there exist closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X has the weak $cc-$insertion property for $(c\alpha c, c\alpha c)$ (resp. (cCc, cCc)).

Proof. Let g and f be real-valued functions defined on X, such that f and g are $c\alpha c$ (resp. cCc), and $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $\alpha(A^{\Lambda}) \subseteq \alpha(B^{V})$ (resp. $C(A^{\Lambda}) \subseteq C(B^{V})$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is an $\alpha-$open (resp. $C-$open) set and since $\{x \in X : g(x) < t_2\}$ is an $\alpha-$closed (resp. $C-$closed) set, it follows that $\alpha(A(f,t_1)^{\Lambda}) \subseteq \alpha(A(g,t_2)^{V})$ (resp. $C(A(f,t_1)^{\Lambda}) \subseteq C(A(g,t_2)^{V})$). Hence $t_1 < t_2$ implies that $A(f,t_1)\ \rho\ A(g,t_2)$. The proof follows from Theorem 2.1. ■

Corollary 3.2. If for each pair of disjoint $\alpha-$open (resp. $C-$open) sets G_1, G_2, there exist closed sets F_1 and F_2 such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then every contra-$\alpha-$continuous (resp. contra-$C-$continuous) function is contra-continuous.

Proof. Let f be a real-valued contra-$\alpha-$continuous (resp. contra-$C-$continuous) function defined on X. Set $g = f$, then by Corollary 3.1, there exists a contra-continuous function h such that $g = h = f$.■

Corollary 3.3. If for each pair of disjoint subsets G_1, G_2 of X, such that G_1 is $\alpha-$open and G_2 is $C-$open, there exist closed subsets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X have the weak $cc-$insertion property for $(c\alpha c, cCc)$ and $(cCc, c\alpha c)$.

Proof. Let g and f be real-valued functions defined on X, such that g is $c\alpha c$ (resp. cCc) and f is cCc (resp. $c\alpha c$), with $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $C(A^{\Lambda}) \subseteq \alpha(B^{V})$ (resp. $\alpha(A^{\Lambda}) \subseteq C(B^{V})$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a $C-$open (resp. $\alpha-$open) set and since $\{x \in X : g(x) < t_2\}$ is an $\alpha-$closed (resp. $C-$closed) set, it follows that $C(A(f,t_1)^{\Lambda}) \subseteq \alpha(A(g,t_2)^{V})$ (resp. $\alpha(A(f,t_1)^{\Lambda}) \subseteq C(A(g,t_2)^{V})$). Hence $t_1 < t_2$ implies that $A(f,t_1)\ \rho\ A(g,t_2)$. The proof follows from Theorem 2.1. ■

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