Information Sciences Letters
An International Journal

http://dx.doi.org/10.18576/isl/080305

Common Fixed Point Results for Weakly Compatible Mappings Under Implicit Relations in Complex Valued G-Metric Spaces

R. A. Rashwan[1], *H. A. Hammad*[2,*] *and M. G. Mahmoud*[3]

[1]Department of Mathematics, Faculty of Science, Assuit University, Assuit 71516, Egypt
[2]Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt
[3]Department of Mathematics, Faculty of Science, South Valley University, Luxor 85844, Egypt

Received: 15 Jul. 2019, Revised: 25 Aug. 2019, Accepted: 28 Aug. 2019
Published online: 1 Sep. 2019

Abstract: The aim of this paper is to establish some common fixed point theorems using four weakly compatible mappings in complex valued G-metric spaces. New contraction conditions in such space are derived as a corollary. Finally, illustrative example is provided to validate our results.

Keywords: Complex valued G-metric spaces, common fixed point theorems, weakly compatible mappings, an implicit relation.

1 Introduction

The study of fixed point theorem in nonlinear analysis has a lot set of applications as determining the existence and uniqueness of solutions to many mathematical equations in mathematical science, engineering and in other fields. The existence and uniqueness of fixed points in different metric spaces is very famous problem. Banach's contraction principle play an important role as the most widely used fixed point theorem in all analysis.

In 1997, Popa [8]gave the definition of an implicit relation, which is cover several well known contractions of the existing literature. So, many authors showed several fixed point results under this concept (see for example [5,9,10,12]. In fact, the force of implicit relations lies in their unifying power besides being general enough to a multitude yield new contraction.

In 2011, Azam et al. [1]introduced the concept of complex valued metric space which is more general than classical metric space and established some fixed point results for mappings involving rational expressions. Subsequently, many authors have studied the existence and uniqueness of the fixed points and common fixed points of self mapping in complex valued metric spaces (see for example [4,13,14,14,15,17,18,19]).

In 2013, Kang et al. [6]initiated the concept of complex valued G-metric spaces as a generalization of the notion of a complex valued metric space on G-metric space. They obtained some existing results concerned with common fixed point results in this space.

In this paper, we use the idea of an implicit relation in complex valued G-metric spaces to prove some common fixed point theorems for four weakly compatible mappings. Next, we give a new contraction conditions as a corollary. Finally, an example is obtained to support our results.

2 Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ iff } \mathrm{Re}(z_1) \leq \mathrm{Re}(z_2) \text{ and } \mathrm{Im}(z_1) \leq \mathrm{Im}(z_2).$$

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(C$_1$) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,
(C$_2$) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,
(C$_3$) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,
(C$_4$) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

* Corresponding author e-mail: h_elmagd89@yahoo.com

In particular, we write $z_1 \lessapprox z_2$ if $z_1 \neq z_2$ and one of (C_1), (C_2) and (C_3) is satisfied and we write $z_1 \prec z_2$ if only (C_3) is satisfied.

Definition 2.1 [1] Let X be a nonempty set. A mapping $d: X \times X \to \mathbb{C}$ is called a complex valued metric on X if the following conditions hold:
(M_1) $0 \precsim d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0 \Leftrightarrow x = y$,
(M_2) $d(x,y) = d(y,x)$ for all $x,y \in X$,
(M_3) $d(x,y) \precsim d(x,z) + d(z,y)$ for all $x,y,z \in X$.
In this case, we say that (X,d) is called a complex valued metric space.

Example 2.1 [3] Let $X = \mathbb{C}$. Define a mapping $d: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by

$$d(z_1,z_2) = e^{ik}|z_1 - z_2| \quad \forall\, z_1, z_2 \in \mathbb{C},$$

where $k \in [0, \pi/2]$. Then (X,d) is called a complex valued metric space.

Definition 2.2 [1] Let $\{x_r\}$ be a sequence in a complex valued metric space (X,d) and $x \in X$. Then

(i) x is called the limit of $\{x_r\}$ if for every $\varepsilon > 0$ there exist $r_0 \in \mathbb{N}$ such that $d(x_r, x) \prec \varepsilon$ for all $r > r_0$ and we can write $\lim_{r\to\infty} x_r = x$.

(ii) $\{x_r\}$ is called a Cauchy sequence if for every $\varepsilon > 0$ there exist $r_0 \in \mathbb{N}$ such that $d(x_r, x_{r+s}) \prec \varepsilon$ for all $r > r_0$ and $s \in \mathbb{N}$.

(iii) (X,d) is said to be a complete complex valued metric space if every Cauchy sequence is convergent in (X,d).

Lemma 2.1 [1] Let (X,d) be a complex valued metric space and $\{x_r\}$ be a sequence in X. Then $\{x_r\}$ converges to x iff $|d(x_r,x)| \to 0$ as $r \to \infty$.

Lemma 2.2 [1] Let (X,d) be a complex valued metric space. Then a sequence $\{x_r\}$ in X is a Cauchy sequence iff $|d(x_r, x_{r+s})| \to 0$ as $r \to \infty$ and $s \in \mathbb{N}$.

Definition 2.3 [6] Let X be a nonempty set and $G: X \times X \times X \longrightarrow \mathbb{C}$ be a function satisfying:

(CG_1) $G(x,y,z) = 0$ if $x = y = z$,
(CG_2) $0 \prec G(x,x,y)$ for all $x,y \in X$ with $x \neq y$,
(CG_3) $G(x,x,y) \precsim G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$,
(CG_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$ (symmetry in all three variables),
(CG_5) $G(x,y,z) \precsim G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then, the function G is called a complex valued generalized metric or a complex valued G-metric on X and the pair (X,G) is called a complex valued G-metric space.

Example 2.2 [7] Let $X = \mathbb{C}$ and $G: X \times X \times X \longrightarrow \mathbb{C}$ be a complex valued G-metric defined by

$$G(z_1,z_2,z_3) = (|x_1 - x_2| + |x_2 - x_3| + |x_3 - x_1|) + i(|y_1 - y_2| + |y_2 - y_3| + |y_3 - y_1|),$$

where $z_k = x_k + iy_k \in \mathbb{C}$ for any $k \in \{1,2,3\}$.

Then (X,G) is a complex valued $G-$metric space.

Proposition 2.1 [16] Let (X,G) be a complex valued G-metric space. Then, for any $x,y,z \in X$, it follows that:

(1) $G(x,y,z) \precsim G(x,x,y) + G(x,x,z)$,
(2) $G(x,y,y) \precsim 2G(y,y,x)$,
(3) $G(x,y,z) \precsim G(x,a,z) + G(a,y,z)$,
(4) $G(x,y,z) \precsim \frac{2}{3}(G(x,y,a) + G(x,a,z) + G(a,y,z))$,
(5) $G(x,y,z) \precsim G(x,a,a) + G(y,a,a) + G(z,a,a)$.

Definition 2.4 [6] Let (X,G) be a complex valued $G-$metric space and $\{x_n\}$ be a sequence in X, we say that $\{x_n\}$ is a complex valued $G-$convergent to x if for every $c \in \mathbb{C}$ with $0 \prec c$, there exists $k \in \mathbb{N}$ such that $G(x, x_n, x_m) \prec c$ for all $n,m \geq k$. We refer to x as the limit of the sequence $\{x_n\}$ and we write $x_n \longrightarrow x$.

Definition 2.5 [6] Let (X,G) be a complex valued G-metric space. Then a sequence $\{x_n\}$ is called complex valued $G-$Cauchy if for every $c \in \mathbb{C}$ with $0 \prec c$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_l) \prec c$ for all $n,m,l \geq k$.

Definition 2.6 [6] A complex valued $G-$metric space (X,G) is said to be complex valued $G-$complete if every complex valued $G-$Cauchy sequence is complex valued $G-$convergent in (X,G).

Definition 2.7 [2]Let X be a non-empty set and (S,T) be a pair of self-mappings on X. Then (S,T) is said to be weakly compatible if

$$Sx = Tx \Rightarrow STx = TSx \quad \forall x \in X.$$

3 Main Results

We begin with the following definition:

An implicit relation. Let $\digamma$ be the set of all complex valued lower semi-continuous functions $F : \mathbb{C}^6 \to \mathbb{C}$ satisfying the following conditions:

(F_1) F is non-increasing in the 5^{th} and 6^{th} variable,

(F_2) for $u,v \succsim 0$, there exists $q \in [0,1)$ such that $|u| \le q|v|$ if $F(u,v,u,v,0,u+v) \precsim 0$,

(F_3) for $u,u^{'} \succ 0$, there exists $q \in [0,1)$ such that $|u| \le q\left|u^{'}\right|$ if $F(u,u,0,0,u^{'},u) \precsim 0$.

Now, we present our first main theorems in complex valued $G-$metric space.

Theorem 3.1 Let S,T,P and Q be four self-mappings on a complete complex valued $G-$metric space (X,G) such that $S(X) \subseteq P(X)$ and $T(X) \subseteq Q(X)$. Assume that there exists $H_1,H_2 \in \digamma$ such that for all $x,y \in X$, $x \neq y$,

$$\begin{cases} H_1\left(G(Tx,Tx,Sy),G(Px,Px,Qy),G(Px,Tx,Tx),\right.\\ \left.G(Qy,Sy,Sy),G(Px,Sy,Sy),G(Qy,Tx,Tx)\right) \precsim 0, \\ \\ H_2\left(G(Sx,Sx,Ty),G(Qx,Qx,Py),G(Qx,Sx,Sx),\right.\\ \left.G(Py,Ty,Ty),G(Qx,Ty,Ty),G(Py,Sx,Sx)\right) \precsim 0. \end{cases} \quad (1)$$

If $P(X) \cup Q(X)$ is complete subspace of X, then the pairs (S,Q) and (T,P) have a unique common point of coincidence. Moreover, if the pairs (S,Q) and (T,P) are weakly compatible, then the four mappings have a unique common fixed point.

Proof. Let x_0 be arbitrary point in X. Since $S(X) \subseteq P(X)$ and $T(X) \subseteq Q(X)$, then we can define the sequence $\{x_n\}$ in X such that,

$$\begin{cases} y_{2n+1} = Px_{2n+1} = Sx_{2n}, \\ y_{2n+2} = Qx_{2n+2} = Tx_{2n+1}. \end{cases} \quad (2)$$

Since $\{y_n\} \subseteq P(X) \cup Q(X)$. Now, we show that $\{y_n\}$ is a Cauchy sequence. Taking $x = x_{2n+1}$ and $y = x_{2n}$ in (H_1), we have

$$H_1\left(G(Tx_{2n+1},Tx_{2n+1},Sx_{2n}),G(Px_{2n+1},Px_{2n+1},Qx_{2n}),G(Px_{2n+1},Tx_{2n+1},Tx_{2n+1}),G(Qx_{2n},Sx_{2n},Sx_{2n}),G(Px_{2n+1},Sx_{2n},Sx_{2n}),G(Qx_{2n},Tx_{2n+1},Tx_{2n+1})\right) \precsim 0.$$

This tends to

$$H_1\left(G(y_{2n+2},y_{2n+2},y_{2n+1}),G(y_{2n+1},y_{2n+1},y_{2n}),G(y_{2n+1},y_{2n+2},y_{2n+2}),G(y_{2n},y_{2n+1},y_{2n+1}),G(y_{2n+1},y_{2n+1},y_{2n+1}),G(y_{2n},y_{2n+2},y_{2n+2})\right) \precsim 0,$$

that is,

$$H_1\left(G(y_{2n+2},y_{2n+2},y_{2n+1}),G(y_{2n+1},y_{2n+1},y_{2n}),G(y_{2n+2},y_{2n+2},y_{2n+1}),G(y_{2n+1},y_{2n+1},y_{2n}),0,G(y_{2n},y_{2n+1},y_{2n+1})+G(y_{2n+1},y_{2n+2},y_{2n+2})\right) \precsim 0.$$

By (F_1) and (CG_4), we get

$$H_1\left(G(y_{2n+2},y_{2n+2},y_{2n+1}),G(y_{2n+1},y_{2n+1},y_{2n}),G(y_{2n+2},y_{2n+2},y_{2n+1}),G(y_{2n+1},y_{2n+1},y_{2n}),0,G(y_{2n+2},y_{2n+2},y_{2n+1})+G(y_{2n+1},y_{2n+1},y_{2n})\right) \precsim 0.$$

By (F_2), we obtain

$$|G(y_{2n+2},y_{2n+2},y_{2n+1})| \le q\,|G(y_{2n+1},y_{2n+1},y_{2n})|.$$

Similarly, by taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in (H_2), we have successively

$$H_2\left(G(Sx_{2n+2},Sx_{2n+2},Tx_{2n+1}),G(Qx_{2n+2},Qx_{2n+2},Px_{2n+1}),G(Qx_{2n+2},Sx_{2n+2},Sx_{2n+2}),G(Px_{2n+1},Tx_{2n+1},Tx_{2n+1}),G(Qx_{2n+2},Tx_{2n+1},Tx_{2n+1}),G(Px_{2n+1},Sx_{2n+2},Sx_{2n+2})\right) \precsim 0,$$

that is, $H_2\left(G(y_{2n+3},y_{2n+3},y_{2n+2}),G(y_{2n+2},y_{2n+2},y_{2n+1}),G(y_{2n+2},y_{2n+3},y_{2n+3}),G(y_{2n+1},y_{2n+2},y_{2n+2}),G(y_{2n+2},y_{2n+2},y_{2n+2}),G(y_{2n+1},y_{2n+3},y_{2n+3})\right) \precsim 0.$

This implies that

$$H_2\left(G(y_{2n+3},y_{2n+3},y_{2n+2}),G(y_{2n+2},y_{2n+2},y_{2n+1}),\right.$$
$$G(y_{2n+3},y_{2n+3},y_{2n+2}),G(y_{2n+2},y_{2n+2},y_{2n+1}),0,$$
$$\left.G(y_{2n+1},y_{2n+2},y_{2n+2})+G(y_{2n+2},y_{2n+3},y_{2n+3})\right)$$
$$\precsim 0.$$

By (F_1) and (CG_4), we obtain

$$H_2\left(G(y_{2n+3},y_{2n+3},y_{2n+2}),G(y_{2n+2},y_{2n+2},y_{2n+1}),\right.$$
$$G(y_{2n+3},y_{2n+3},y_{2n+2}),G(y_{2n+2},y_{2n+2},y_{2n+1}),0,$$
$$\left.G(y_{2n+2},y_{2n+2},y_{2n+1})+G(y_{2n+3},y_{2n+3},y_{2n+2})\right)$$
$$\precsim 0.$$

By (F_2), we have

$$|G(y_{2n+3},y_{2n+3},y_{2n+2})| \le q\,|G(y_{2n+2},y_{2n+2},y_{2n+1})|.$$

Consequently,

$$|G(y_{n+1},y_{n+1},y_n)| \le q\,|G(y_n,y_n,y_{n-1})| \le \ldots \le q^n\,|G(y_1,y_1,y_0)|.$$

Also, for any $n>m$, we get

$$|G(y_n,y_n,y_m)| \le |G(y_{m+1},y_{m+1},y_m)| + |G(y_{m+2},y_{m+2},y_{m+1})| + \ldots + |G(y_n,y_n,y_{n-1})|$$
$$\le \left(q^m+q^{m+1}+\ldots+q^{n-1}\right)|G(y_1,y_1,y_0)|$$
$$\le \frac{q^m}{1-q}|G(y_1,y_1,y_0)| \longrightarrow 0 \text{ as } m\longrightarrow\infty.$$

This shows that $\{y_n\}$ is a Cauchy sequence in X. Since (X,G) is complete, then there exists $u\in X$ such that $y_n\longrightarrow u$ as $n\longrightarrow\infty$. Then from (2), we obtain

$$\lim_{n\longrightarrow\infty} Sx_{2n} = \lim_{n\longrightarrow\infty} Px_{2n+1} = \lim_{n\longrightarrow\infty} Qx_{2n+2} = \lim_{n\longrightarrow\infty} Tx_{2n+1} = u. \quad (3)$$

Since $S(X)\subseteq P(X)$, if $u\in P(X)$, then there exists $v\in X$ such that

$$Pv = u. \quad (4)$$

We will prove that $Tv=Pv$. By taking $x=v$ and $y=x_{2n}$ in (H_1), we have

$$H_1\left(G(Tv,Tv,Sx_{2n}),G(Pv,Pv,Qx_{2n}),\right.$$
$$G(Pv,Tv,Tv),G(Qx_{2n},Sx_{2n},Sx_{2n})$$
$$\left.,G(Pv,Sx_{2n},Sx_{2n}),G(Qx_{2n},Tv,Tv)\right)\precsim 0.$$

Taking $n\longrightarrow\infty$ and using (3) and (4), we have

$$H_1\left(G(Tv,Tv,u),G(u,u,u),G(u,Tv,Tv),\right.$$
$$\left.G(u,u,u),G(u,u,u),G(u,Tv,Tv)\right)\precsim 0.$$

This implies that

$$H_1\left(G(Tv,Tv,u),0,G(Tv,Tv,u),0,0,G(Tv,Tv,u)\right)\precsim 0.$$

By (F_2), we get $G(Tv,Tv,u)=0$ which implies $Tv=u$. Hence

$$Tv = Pv = u. \quad (5)$$

Then, u is a point of coincidence of the pair (T,P).

Since $T(X)\subseteq Q(X)$, there exists $w\in X$ such that

$$Qw = u. \quad (6)$$

We will prove that $Sw=Qw$. By taking $x=w$ and $y=x_{2n+1}$ in (H_2), we have

$$H_2\left(G(Sw,Sw,Tx_{2n+1}),G(Qw,Qw,Px_{2n+1}),\right.$$
$$G(Qw,Sw,Sw),G(Px_{2n+1},Tx_{2n+1},Tx_{2n+1}),$$
$$\left.G(Qw,Tx_{2n+1},Tx_{2n+1}),G(Px_{2n+1},Sw,Sw)\right)$$
$$\precsim 0.$$

Taking $n\longrightarrow\infty$ and using (3) and (6), we get

$$H_2\left(G(Sw,Sw,u),0,G(Sw,Sw,u),0,0,G(Sw,Sw,u)\right)\precsim 0.$$

By (F_2), we obtain $G(Sw,Sw,u)=0$ which implies $Sw=u$. Hence

$$Sw = Qw = u. \quad (7)$$

Then, u is a point of coincidence of the pair (S,Q).

Hence, $u\in X$ is a common point of coincidence for the four mappings.

To prove the uniqueness of a point of coincidence. Suppose that $u^*\neq u$ be another point of coincidence of the four mappings. Then, there exists v^*,w^* such that $Tv^*=Pv^*=u^*$ and $Sw^*=Qw^*=u^*$. Putting $x=v^*$ and $y=w$ in (H_1), one can write

$$H_1\left(G(Tv^*,Tv^*,Sw),G(Pv^*,Pv^*,Qw),\right.$$
$$G(Pv^*,Tv^*,Tv^*),G(Qw,Sw,Sw),$$
$$\left.G(Pv^*,Sw,Sw),G(Qw,Tv^*,Tv^*)\right)\precsim 0.$$

This implies that

$$H_1\left(G(u^*,u^*,u),G(u^*,u^*,u),G(u^*,u^*,u^*),\right.$$
$$\left.G(u,u,u),G(u^*,u,u),G(u,u^*,u^*)\right)\precsim 0,$$

that is,

$$H_1\left(G(u^*,u^*,u),G(u^*,u^*,u),0,0,\right.$$
$$\left.G(u,u,u^*),G(u^*,u^*,u)\right)\precsim 0.$$

By (F_3), we deduce that

$$|G(u^*,u^*,u)| \le q_1\,|G(u,u,u^*)|. \quad (8)$$

Similarly, putting $x=v$ and $y=w^*$ in (H_2), we deduce that

$$|G(u,u,u^*)| \le q_1\,|G(u^*,u^*,u)|. \quad (9)$$

From (8) and (9), we obtain

$$|G(u^*,u^*,u)|\,(1-q_1^2)\le 0,$$

which implies that $|G(u^*,u^*,u)|=0$, i.e, $u^*=u$. Consequently, the pairs (S,Q) and (T,P) have a unique common point of coincidence.

By using (5), (7) and weak compatibility of the pairs (S,Q) and (T,P), we obtain that

$$SQw = QSw \quad \text{and} \quad TPv = PTv. \qquad (10)$$

Then,

$$Su = Qu \quad \text{and} \quad Tu = Pu, \qquad (11)$$

with meaning u is a point of coincidence of the pairs (S,Q) and (T,P).

Now, we prove that u is a common fixed point of S,T,P and Q. Putting $x = u$ and $y = v$ in (H_1), we have

$$H_1\left(G(Tu,Tu,Sv),G(Pu,Pu,Qv),G(Pu,Tu,Tu),G(Qv,Sv,Sv),G(Pu,Sv,Sv),G(Qv,Tu,Tu)\right) \precsim 0.$$

This leads us to

$$H_1\left(G(Tu,Tu,u),G(Tu,Tu,u),G(Tu,Tu,Tu),G(u,u,u),G(Tu,u,u),G(u,Tu,Tu)\right) \precsim 0,$$

that is,

$$H_1\left(G(Tu,Tu,u),G(Tu,Tu,u),0,0,G(u,u,Tu),G(Tu,Tu,u)\right) \precsim 0.$$

By (F_3), we have

$$|G(Tu,Tu,u)| \le q_1 |G(u,u,Tu)|. \qquad (12)$$

Similarly, putting $x = v$ and $y = u$ in (H_2), we deduce that

$$|G(u,u,Tu)| \le q_1 |G(Tu,Tu,u)|. \qquad (13)$$

From (12) and (13), we get

$$|G(Tu,Tu,u)|(1-q_1^2) \le 0,$$

which implies that $G(Tu,Tu,u) = 0$, i.e., $Tu = u$. Thus, $Tu = Pu = u$. Similarly, we can show $Su = Qu = u$. This means that

$$Su = Tu = Pu = Qu = u.$$

i.e., u is a common fixed point of S,T,P and Q.

The uniqueness of the common fixed point of S,T,P and Q is easy consequence of the uniqueness of the common point of coincidence of the pairs (S,Q) and (T,P). Also, the proof is similar in case $u \in Q(X)$. This completes the proof.

Theorem 3.2 The conclusion of Theorem 3.1 remains true if the completeness of $P(X) \cup Q(X)$ is replaced by the completeness of one of the subspaces $S(X), T(X), P(X)$ or $Q(X)$.

The following theorem is a new version of Theorem 3.1 under generalized contractive condition.

Theorem 3.3 Let S,T,P and Q be four self-mappings on a complete complex valued G-metric space (X,G) such that $S(X) \subseteq P(X)$ and $T(X) \subseteq Q(X)$. Assume that there exists $H \in \digamma$ such that for all $x,y \in X$, $x \neq y$,

$$\begin{cases} H_1\big(G(Tx,Tx,Sy),G(Px,Px,Qy),G(Px,Tx,Tx), \\ \qquad G(Qy,Sy,Sy),G(Px,Sy,Sy),\Delta\big) \precsim 0, \\ H_2\big(G(Sx,Sx,Ty),G(Qx,Qx,Py),G(Qx,Sx,Sx), \\ \qquad G(Py,Ty,Ty),G(Qx,Ty,Ty),\nabla\big) \precsim 0. \end{cases} \qquad (14)$$

Where $\Delta = \dfrac{G^2(Qy,Tx,Tx)+G^2(Px,Px,Sy)}{G(Qy,Tx,Tx)+G(Px,Px,Sy)}$ and

$$\nabla = \frac{G^2(Py,Sx,Sx)+G^2(Qx,Qx,Ty)}{G(Py,Sx,Sx)+G(Qx,Qx,Ty)}.$$

If $P(X) \cup Q(X)$ is complete subspace of X, then the pairs (S,Q) and (T,P) have a unique common point of coincidence. Moreover, if the pairs (S,Q) and (T,P) are weakly compatible, then the four mappings have a unique common fixed point.

Proof. Let x_0 be arbitrary points in X. Since $S(X) \subseteq P(X)$ and $T(X) \subseteq Q(X)$, then we can define the sequence $\{x_n\}$ in X as (2).

Since $\{y_n\} \subseteq P(X) \cup Q(X)$. Now, we show that $\{y_n\}$ is a Cauchy sequence. Taking $x = x_{2n+1}$ and $y = x_{2n}$, we get

$$H_1\big(G(Tx_{2n+1},Tx_{2n+1},Sx_{2n}), G(Px_{2n+1},Px_{2n+1},Qx_{2n}),G(Px_{2n+1},Tx_{2n+1}, Tx_{2n+1}),G(Qx_{2n},Sx_{2n},Sx_{2n}),G(Px_{2n+1},Sx_{2n}, Sx_{2n}),\Delta_{x_{2n},x_{2n+1}}\big) \precsim 0.$$

This implies that

$$H_1\big(G(y_{2n+2},y_{2n+2},y_{2n+1}),G(y_{2n+1},y_{2n+1},y_{2n}), G(y_{2n+1},y_{2n+2},y_{2n+2}),G(y_{2n},y_{2n+1},y_{2n+1}), G(y_{2n+1},y_{2n+1},y_{2n+1}), \frac{G^2(y_{2n},y_{2n+2},y_{2n+2})+G^2(y_{2n+1},y_{2n+1},y_{2n+1})}{G(y_{2n},y_{2n+2},y_{2n+2})+G(y_{2n+1},y_{2n+1},y_{2n+1})}\big) \precsim 0,$$

that is,

$$H_1\Big(G(y_{2n+2},y_{2n+2},y_{2n+1}),G(y_{2n+1},y_{2n+1},y_{2n}), G(y_{2n+2},y_{2n+2},y_{2n+1}),G(y_{2n+1},y_{2n+1},y_{2n+1}), G(y_{2n+1},y_{2n+1},y_{2n+1}), \frac{G^2(y_{2n},y_{2n+2},y_{2n+2})+G^2(y_{2n+1},y_{2n+1},y_{2n+1})}{G(y_{2n},y_{2n+2},y_{2n+2})+G(y_{2n+1},y_{2n+1},y_{2n+1})}\Big) \precsim 0,$$

which leads to

$$H_1\left(G(y_{2n+2},y_{2n+2},y_{2n+1}),G(y_{2n+1},y_{2n+1},y_{2n}), G(y_{2n+2},y_{2n+2},y_{2n+1}),G(y_{2n+1},y_{2n+1},y_{2n}) ,0,G(y_{2n},y_{2n+2},y_{2n+2})\right) \precsim 0.$$

By (F_1), (CG_4) and (CG_5) we get

$$H_1\left(G(y_{2n+2},y_{2n+2},y_{2n+1}),G(y_{2n+1},y_{2n+1},y_{2n}), G(y_{2n+2},y_{2n+2},y_{2n+1}),G(y_{2n+1},y_{2n+1},y_{2n}),0, G(y_{2n+2},y_{2n+2},y_{2n+1})+G(y_{2n+1},y_{2n+1},y_{2n})\right) \precsim 0.$$

By (F_2), we obtain

$$|G(y_{2n+2},y_{2n+2},y_{2n+1})| \leq q\,|G(y_{2n+1},y_{2n+1},y_{2n})|.$$

Similarly, by taking $x=x_{2n+2}$ and $y=x_{2n+1}$ in (H_2), we have successively

$$H_2\Big(G(Sx_{2n+2},Sx_{2n+2},Tx_{2n+1}),G(Qx_{2n+2},Qx_{2n+2}, Px_{2n+1}),G(Qx_{2n+2},Sx_{2n+2},Sx_{2n+2}),G(Px_{2n+1}, Tx_{2n+1},Tx_{2n+1}),G(Qx_{2n+2},Tx_{2n+1},Tx_{2n+1}), \nabla_{x_{2n+1},x_{2n+2}}\Big) \precsim 0.$$

This implies that

$$H_2\Big(G(y_{2n+3},y_{2n+3},y_{2n+2}),G(y_{2n+2},y_{2n+2},y_{2n+1}), G(y_{2n+2},y_{2n+3},y_{2n+3}),G(y_{2n+1},y_{2n+2},y_{2n+2}), G(y_{2n+2},y_{2n+2},y_{2n+2}),G(y_{2n+1},y_{2n+3},y_{2n+3})\Big) \precsim 0,$$

that is,

$$H_2\left(G(y_{2n+3},y_{2n+3},y_{2n+2}),G(y_{2n+2},y_{2n+2},y_{2n+1}), G(y_{2n+3},y_{2n+3},y_{2n+2}),G(y_{2n+2},y_{2n+2},y_{2n+1}),0, G(y_{2n+1},y_{2n+2},y_{2n+2})+G(y_{2n+2},y_{2n+3},y_{2n+3})\right) \precsim 0.$$

By (F_1), (CG_4) and (CG_5), one can write

$$H_2\left(G(y_{2n+3},y_{2n+3},y_{2n+2}),G(y_{2n+2},y_{2n+2},y_{2n+1}), G(y_{2n+3},y_{2n+3},y_{2n+2}),G(y_{2n+2},y_{2n+2},y_{2n+1}),0, G(y_{2n+2},y_{2n+2},y_{2n+1})+G(y_{2n+3},y_{2n+3},y_{2n+2})\right) \precsim 0.$$

By (F_2), we have

$$|G(y_{2n+3},y_{2n+3},y_{2n+2})| \leq q\,|G(y_{2n+2},y_{2n+2},y_{2n+1})|.$$

Now, by a similar way (step by step) of the proof of Theorem 3.1, we can finished the proof.

In view of Examples 3.2-3.15 of the paper [10], we get the following results which yielding new contraction conditions in complex valued G-metric spaces.

Corollary 3.1 The end of Theorem 3.1 and 3.3 remains true, if we replace an implicit relation (1) by any of the following:

(i) $G(Tx,Tx,Sy)$
$$\precsim k\max\{G(Px,Px,Qy),G(Px,Tx,Tx), G(Qy,Sy,Sy),G(Px,Sy,Sy),G(Qy,Tx,Tx)\},$$
where $k\in[0,\frac{1}{2})$.

(ii) $G(Sx,Sx,Ty)$
$$\precsim k\max\{G(Qx,Ty,Ty)+G(Py,Sx,Sx), G(Qx,Sx,Sx)\},$$
where $k\in[0,\frac{1}{2})$.

(iii) $G(Tx,Tx,Sy)$
$$\precsim k\max\{G(Px,Tx,Tx)+G(Qy,Tx,Tx), G(Px,Sy,Sy)+G(Qy,Sy,Sy)\},$$
where $k\in[0,\frac{1}{3})$.

(iv) $G(Sx,Sx,Ty)$
$$\precsim k\max\{G(Qx,Qx,Py),G(Qx,Sx,Sx), G(Py,Ty,Ty),\frac{G(Qx,Ty,Ty)+G(Qx,Sx,Sx)}{2}+ \frac{G(Qx,Ty,Ty)+G(Py,Sx,Sx)}{2}\},$$
where $k\in[0,1)$.

(v) $G(Tx,Tx,Sy)$
$$\precsim k\max\{G(Px,Px,Qy),G(Px,Tx,Tx),$$

$G(Qy,Sy,Sy), \frac{G(Qy,Tx,Tx)+G(Px,Px,Qy)}{3}+$
$\frac{G(Px,Tx,Tx)+G(Qy,Tx,Tx)}{3}\},$

where $k \in [0,\frac{1}{2})$.

(vi) $G(Sx,Sx,Ty)$
$\precsim a_1 G(Qx,Qx,Py)+a_2 G(Qx,Sx,Sx)+$
$a_3 G(Py,Ty,Ty)+a_4 \max\{G(Qx,Ty,Ty),$
$G(Py,Sx,Sx),G(Qx,Sx,Sx)\},$

where $a_1,a_2,a_3,a_4 \geq 0$ and
$a_1+a_2+a_3+2a_4<1$.

(vii) $G(Tx,Tx,Sy)$
$\precsim k\max\{G(Px,Tx,Tx),G(Qy,Sy,Sy),$
$G(Px,Sy,Sy),G(Qy,Tx,Tx),G(Px,Tx,Tx)+$
$G(Tx,Tx,Sy)+G(Qy,Tx,Tx),G(Px,Px,Qy)\},$

where $k \in [0,\frac{1}{4})$.

$(viii)$ $G(Sx,Sx,Ty)$
$\precsim a_1 G(Qx,Qx,Py)+a_2 \max\{G(Qx,Ty,Ty)$
$+G(Py,Sx,Sx)+G(Qx,Sx,Sx),G(Py,Ty,Ty)$
$+G(Qx,Sx,Sx)\},$

where $a_1,a_2 \geq 0$ and $a_1+3a_2<1$.

(ix) $G(Tx,Tx,Sy)$
$\precsim \mu_1 G(Px,Px,Qy)+\mu_2 G(Qy,Sy,Sy)$
$+\mu_3 \max\{G(Px,Sy,Sy)+G(Qy,Tx,Tx)$
$+G(Px,Tx,Tx)+G(Tx,Tx,Sy),$
$G(Px,Tx,Tx)+G(Qy,Tx,Tx)\},$

where $\mu_1,\mu_2,\mu_3 \geq 0$ and $\mu_1+\mu_2+4\mu_3<1$.

(x) $G(Sx,Sx,Ty)$
$\precsim \mu_1 G(Qx,Qx,Py)+\mu_2 \max\{G(Qx,Sx,Sx),$
$G(Py,Ty,Ty)\}+\mu_3 \max\{G(Qx,Sx,Sx),$
$G(Qx,Ty,Ty)\}+\mu_4 \max\{G(Qx,Qx,Py),$
$G(Qx,Sx,Sx),G(Py,Ty,Ty),$
$\frac{G(Qx,Ty,Ty)+G(Qx,Sx,Sx)}{2}\}$
$+\mu_5 G(Py,Ty,Ty),$

where $\mu_1,\mu_2,\mu_3,\mu_4,\mu_5 \geq 0$ and
$\mu_1+\mu_2+\mu_3+\mu_4+2\mu_5<1$.

(xi) $G(Tx,Tx,Sy)$
$\precsim \mu_1 G(Px,Px,Qy)+\mu_2 \max\{G(Px,Tx,Tx),$
$G(Qy,Sy,Sy),G(Px,Sy,Sy),G(Qy,Tx,Tx)\},$

where $\mu_1,\mu_2 \geq 0$ and $\mu_1+2\mu_2<1$.

(xii) $G(Sx,Sx,Ty)$
$\precsim \mu_1 G(Qx,Qx,Py)$
$+\mu_2 \left[G(Qx,Sx,Sx)+G(Py,Ty,Ty)\right]$
$+\mu_3 \left[G(Py,Sx,Sx)+G(Qx,Ty,Ty)\right],$

where $\mu_1,\mu_2,\mu_3 \geq 0$ and $\mu_1+2\mu_2+2\mu_3<1$.

$(xiii)$ $G(Sx,Sx,Ty)$
$\precsim \mu_1 G(Qx,Qx,Py)$
$+\mu_2 \left[G(Qx,Sx,Sx)+G(Py,Ty,Ty)\right]$
$+\mu_3 \left[G(Py,Sx,Sx)+G(Qx,Ty,Ty)\right]$
$+\mu_4 \frac{G^2(Qx,Qx,Py)}{1+G(Qx,Qx,Py)},$

where $\mu_1,\mu_2,\mu_3,\mu_4 \geq 0$ and
$\mu_1+2\mu_2+2\mu_3+\mu_4<1$.

(xiv) $G(Tx,Tx,Sy)$
$\precsim \lambda_1 G(Px,Px,Qy)$
$+\lambda_2 \left[G(Px,Tx,Tx)+G(Qy,Sy,Sy)\right]$
$+\lambda_3 \left[G(Qy,Tx,Tx)+G(Px,Sy,Sy)\right],$

where $\lambda_1,\lambda_2,\lambda_3 \geq 0$ and $\lambda_1+2\lambda_2+2\lambda_3<1$.

(xv) $G(Tx,Tx,Sy)$
$\precsim \lambda_1 G(Px,Px,Qy)$
$+\lambda_2 \left[G(Px,Tx,Tx)+G(Qy,Sy,Sy)\right]$
$+\lambda_3 \left[G(Qy,Tx,Tx)+G(Px,Sy,Sy)\right]$

Table 1: Illustrate the existence and uniqueness of fixed points via various constant

ID	Mappings	$(SX \cup TX) \subseteq X$	Constants	Weakly Compatible	Contractions	F. Point
xii	$Sx = Tx = \frac{x+1}{4}$, $Px = Qx = x$.	$SX \cup TX = [\frac{1}{4}, \frac{1}{2}] \subseteq [0,1] = X$	$\mu_1 = \frac{1}{8}$, $\mu_2 = \frac{1}{6}$ and $\mu_3 = \frac{1}{24}$.	$SQ(\frac{1}{3}) = QS(\frac{1}{3})$, $TP(\frac{1}{3}) = PT(\frac{1}{3})$.	$\frac{1}{2}\|x-y\| \leq \frac{29}{48}\|x-y\|$	$\frac{1}{3}$
$xiii$	$Sx = Tx = \frac{x+1}{4}$, $Px = Qx = x$.	$SX \cup TX = [\frac{1}{4}, \frac{1}{2}] \subseteq [0,1] = X$	$\mu_1 = \frac{1}{8}$, $\mu_2 = \frac{1}{6}$, $\mu_3 = \frac{1}{24}$ and $\mu_4 = \frac{1}{32}$.	$SQ(\frac{1}{3}) = QS(\frac{1}{3})$, $TP(\frac{1}{3}) = PT(\frac{1}{3})$.	$\frac{1}{2}\|x-y\| \leq \frac{2}{3}\|x-y\|$	$\frac{1}{3}$
xiv	$Sx = Tx = \frac{2x+1}{6}$, $Px = Qx = x$.	$SX \cup TX = [\frac{1}{6}, \frac{1}{2}] \subseteq [0,1] = X$	$\lambda_1 = \frac{1}{6}$, $\lambda_2 = \frac{1}{8}$ and $\lambda_3 = \frac{1}{10}$.	$SQ(\frac{1}{4}) = QS(\frac{1}{4})$, $TP(\frac{1}{4}) = PT(\frac{1}{4})$.	$\frac{2}{3}\|x-y\| \leq \frac{23}{30}\|x-y\|$	$\frac{1}{4}$
xv	$Sx = Tx = \frac{2x+1}{6}$, $Px = Qx = x$.	$SX \cup TX = [\frac{1}{6}, \frac{1}{2}] \subseteq [0,1] = X$	$\lambda_1 = \frac{1}{8}$, $\lambda_2 = \frac{1}{10}$, $\lambda_3 = \frac{1}{12}$ and $\lambda_4 = \frac{1}{6}$.	$SQ(\frac{1}{4}) = QS(\frac{1}{4})$, $TP(\frac{1}{4}) = PT(\frac{1}{4})$.	$\frac{2}{3}\|x-y\| \leq \frac{169}{180}\|x-y\|$	$\frac{1}{4}$
xvi	$Sx = Tx = \frac{x+1}{4}$, $Px = Qx = x$.	$SX \cup TX = [\frac{1}{6}, \frac{1}{2}] \subseteq [0,1] = X$	$\mu = \frac{1}{3}$	$SQ(\frac{1}{3}) = QS(\frac{1}{3})$, $TP(\frac{1}{3}) = PT(\frac{1}{3})$.	$\frac{1}{2}\|x-y\| \leq \frac{5}{6}\|x-y\|$	$\frac{1}{3}$

$$+\lambda_4 \frac{G^2(Px,Px,Qy)}{1+G(Px,Px,Qy)},$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ and $\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4 < 1$.

(xvi) $G(Sx,Sx,Ty)$

$$\precsim \mu \max\{G(Qx,Ty,Ty) + G(Py,Sx,Sx), G(Py,Sx,Sx)\},$$

where $\mu \in [0, \frac{1}{2})$.

In the following section, we give an example on the contraction conditions $(xii), (xiii), (xiv), (xv)$ and (xvi) of the corollary 3.1.

Example 4.1 Let $X = [0,1]$ and $G : X \times X \times X \longrightarrow \mathbb{C}$ be a complex valued G-metric defined as follows:

$$G(x,y,z) = |x-y| + |y-z| + |z-x|,$$

for all $x,y,z \in X$. Then (X,G) is complex valued $G-$metric space.

By simple calculation some requirements of Corollary 3.1 can be proven in the Table 1:

Acknowledgement

The authors would like to express their thanks to the chief-editor and the referees for their helpful comments and suggestions.

References

[1] A. Azam, B. Fisher and M. Khan, Common fixed point theorems in complex valued metric spaces, Numer. Funct. Anal. optim., **32** (3), 243-253 (2011).

[2] S. Bhatt, S. Chaukiyal and R. C. Dimri, A common fixed point theorem for weakly compatible maps in complex valued metric spaces, Int. J. Math. Sci. Appl., **1** (3), 1385-1389 (2011).

[3] S. Chandok and D. Kumar, Some common fixed point results for rational type contraction mappings in complex valued metric spaces, J. Oper., **2013**, 1-7 (2013).

[4] Hasanen A. Hammad and Manuel De la Sen, Analytical solution of Urysohn integral equations by fixed point technique in complex valued metric spaces, Mathematics 2019, 7, 852; doi:10.3390/math7090852.

[5] M. Imdad and J. Ali, Some common fixed point theorems in fuzzy metric spaces, Mathematical Communication, **11** (2), 153-163 (2007).

[6] S. Kang, B. Singh, V. Gupta and S. Kumar, Contraction Principle in Complex Valued G-Metric Spaces, Int. Journal of Math. Analysis, **7** (52), 2549-2556 (2013).

[7] I. Lin, W. Du and Q. Zheng, Some new fixed point theorems in complex valued G-metric spaces, Abstract and Applied Analysis, **2014**, 1-6 (2014).

[8] V. Popa, Fixed point theorems for implicit contractive mappings, Stud. Cerc. St. Ser. Mat. Univ. Bacau., **7**, 127-133 (1997).

[9] V. Popa, Some fixed point theorems for compatible mappings satisfying an implicit relation, Demo. Math. Wars. Tech. Univ. Inst. Math., **32** (1), 157-163 (1999).

[10] V. Popa, M. Imdad and J. Ali, Using implicit relations to prove unified fixed point theorems in metric and 2-metric spaces, Bull. Malays. Math. Sci. Soc.(2), **33** (1), 105-120 (2010).

[11] V. Popa and A. Patriciu, A general fixed point theorem for pairs of non weakly compatible mappings in G-metric spaces, Novi Sad J. Math., **44** (1), 91-104 (2014).

[12] V. Popa, A general fixed point theorem for implicit cyclic multi-valued contraction mappings, Annales. Math. Sil., **29**, 119-129 (2015).

[13] A. Rafiq, F. Rouzkard, M. Imdad and S. M. Kang, Some common fixed point theorems of weakly compatible mappings in complex valued metric spaces, Mitteilungen Klosterneuburg, **1** (65), 422-432 (2015).

[14] R. A. Rashwan, H. A. Hammad and M. G. Mahmoud, Common fixed point theorems in complex-valued S-metric spaces via implicit relations with applications, Res. Fixed Point Theory Appl., Volume **2019**, Article ID 2018034, 17 pages.

[15] F. Rouzkard and M. Imdad, Some common fixed point theorems on complex valued metric spaces, Computers & Mathematics with Applications, **64** (6), 1866-1874 (2012).

[16] B. Singh, V. Gupta and S. Kumar, Weakly compatible maps using E.A. and (CLR) properties in complex valued G-metric spaces, Journal of Complex System, **2013**, 1-6 (2013).

[17] D. Singh, V. Joshi, M. Imdad and P. Kumam, Unified implicit common fixed point theorems under nonnegative complex valued functions satisfying the identity of indiscernible, Journal of Nonlinear Sciences & Applications (JNSA),**9** (5), 2049-2069 (2016).

[18] N. Singh, D. Singh, A. Badal and V. Joshi, Fixed point theorems in complex valued metric spaces, Journal of the Egyptian Mathematical Society, **24** (3), 402-409 (2016).

[19] W. Sintunavarat, M. B. Zada and M. Sarwar, Common solution of urysohn integral equations with the help of common fixed point results in complex valued metric spaces, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, **111** (2), 531-545 (2017).

R. A. Rashwan having more than 18 years of teaching experience in working as professor in Assiut University, Egypt. He obtained his Ph.D. from Department of Mathematics, Assiut University. He is active researcher in the field of fixed point theory and its applications. He published more than 120 in reputed international Journals of Mathematics.

Hasanen A. Hammad working as Assistant Professor (Lecturer) at Sohag University, Egypt. He received the Ph.D. degree in Mathematics (Functional Analysis) at University of Sohag. His research interests are in the field of functional analysis especially, fixed point theory and its applications, random fixed points and its applications, Optimization and variational inequalities. He has published more than 50 papers in reputed international Journals of Mathematics and he is referee and editor of mathematical journals.

M. G. Mahmoud Assistant Lecturer at South Valley University, Egypt. He received the M.Sc. degree in Mathematics (Functional Analysis) at South Valley University. His research interests are in the field of functional analysis and its applications, beside fixed point theory and its applications. He has published 5 papers in reputed international Journals of Mathematics.