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On a Certain Triple Construction of *GMS*-Algebras

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Abstract: In this paper we introduce a certain subclass $\mathbf{GK_2}$ of the class **GMS** of all generalized *MS*-algebras. A simple triple construction of principal generalized K_2-algebras which works with pairs of elements only is given. We also characterize isomorphisms of these algebras by means of triples. Finally we introduce a notion of congruence pairs for the class of principal generalized K_2-algebras. Then, we will consider the representation of congruences on principal GK_2-algebras in terms of congruence pairs on some underlying simpler structures.

Keywords: *MS*-algebras; Generalized *MS*-algebras; Kleene algebras; Generalized Kleene algebras ; K_2-algebras; Congruence pairs.

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1 Introduction

In 1983 T. S. Blyth and J. C. Varlet [9] introduced MS-algebras which are algebras of type (2,2,1,0,0) abstracting de Morgan algebras and Stone algebras. In [10] they investigated the lattice of subvarieties of *MS*-algebras and characterized its members by identities. In 1996 Ševčovič [19] investigated a larger variety of algebras containing *MS*-algebras, so-called generalized *MS*-algebras (*GMS*-algebras). In such algebras the distributive identity need not be necessarily satisfied. In [11] T. S. Blyth and J. C. Varlet presented a construction of some *MS*-algebras from the subvariety $\mathbf{K_2}$ from Kleene algebras and distributive lattices. This was a construction by means of triples which were successfully used in construction of Stone algebras (see [13,14]), distributive *p*-algebras (see [16]), modular *p*-algebras (see [17]), etc. T. S. Blyth and J. V. Varlet [12] improved their construction from [11] by means of quadruples and they showed that each member of $\mathbf{K_2}$ can be constructed in this way. In [15] M. Haviar presented a simple quadruple construction of locally bounded K_2-algebras which works with pairs of elements only. In 2012 A. Badawy, D. Guffova and M. Haviar [5] introduced a simple triple construction of principal *MS*-algebras and they showed that there exists a one-to-one correspondence between the principal *MS*-algebras and the principal *MS*-triples. They also introduced the class of decomposable*MS*-algebras containing the class of principal *MS*-algebras and they presented a triple construction of decomposable *MS*-algebras generalizing the construction of principal *MS*-algebras. Moreover, they proved that there exists a one-to-one correspondence between the decomposable *MS*-algebras and the decomposable *MS*-triples. Recently, A. Badawy [1] introduced a quadruple construction of the class of all modular *GMS*-algebras. Also, A. Badawy [2, 3] and [4] introduced the notion of d_L-filters of principal *MS*-algebras, the notion of De Morgan filters of decomposable *MS*-algebras and the congruences induced by De Morgan filters of decomposable *MS*-algebras, respectively. A. Badawy and M.S. Rao [6] introduced the notion of closure ideals of *MS*-algebras. R. Beazer [3] introduced the notion of congruence pairs for K_2-algebras.

The aim of this paper is to introduce a subvariety of *GMS*-algebras containing the variety of K_2-algebras, the so-called generalized K_2-algebras. We introduce and construct principal generalized K_2-algebras from generalized Kleene algebras and bounded lattices by means of triples. Also we define isomorphism between two principal GK_2-triples and we show that two principal GK_2-algebras are isomorphic if and only if their associated principal GK_2-triples are isomorphic. In the final part of this paper, we introduce the concept of congruence pairs for the class of principal GK_2-algebras.

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Then, we show that every congruence relation θ on a principal generalized K_2-algebra L can be uniquely determined by a congruence pair (θ_1,θ_2), where $\theta_1 \in Con(L^{\circ\circ})$ and $\theta_2 \in Con(D(L))$.

2 Preliminaries

In this section, we present certain definitions and important results taken mostly from [7,9,10] and [17], those will be required in the paper.

An MS-algebra is an algebra $(L;\vee,\wedge,^{\circ},0,1)$ of type $(2,2,1,0,0)$ where $(L;\vee,\wedge,0,1)$ is a bounded distributive lattice and the unary operation $^{\circ}$ satisfies:

(1) $x \leq x^{\circ\circ}$
(2) $(x\wedge y)^{\circ} = x^{\circ} \vee y^{\circ}$,
(3) $1^{\circ} = 0$.

The class **MS** of all MS-algebras forms a variety. The members of the subvariety **M** of **MS** defined by the identity

(4) $x = x^{\circ\circ}$

are called de Morgan algebras and the members of the subvariety **K** of **M** defined by the identity

(5) $x\wedge x^{\circ} \leq y \vee y^{\circ}$

are called Kleene algebras. The subvariety $\mathbf{K_2}$ of **MS** is defined by the additional two identities:

(6) $x\wedge x^{\circ} = x^{\circ\circ}\wedge x^{\circ}$,
(7) $x\wedge x^{\circ} \leq y\vee y^{\circ}$.

The class **S** of all Stone algebras is a subvariety of **MS** and is characterized by the identity

(8) $x\wedge x^{\circ} = 0$.

The subvariety **B** of **MS** characterized by the identity

(9) $x\vee x^{\circ} = 1$

is called the class of Boolean algebras.

A generalized de Morgan algebra (or GM-algebra) is a universal algebra $(L;\vee,\wedge,^{-},0,1)$ where $(L;\vee,\wedge,0,1)$ is a bounded lattice and $^{-}$ the unary operation of involution satisfies the identities:

(10) $x = x^{--}$,
(11) $(x\wedge y)^{-} = x^{-}\vee y^{-}$,
(12) $1^{-} = 0$.

A generalized Kleene algebra (GK-algebra) L is a GM-algebra satisfying the identity.

(13) $x\wedge x^{-} \leq y\vee y^{-}$.

A modular GM-algebra L is GM-algebra where $(L;\vee,\wedge,0,1)$ is a modular lattice.

A generalized MS-algebra (or GMS-algebra) is a universal algebra $(L;\vee,\wedge,^{\circ},0,1)$ where $(L;\vee,\wedge,0,1)$ is a bounded lattice and the unary operation $^{\circ}$ satisfies the identities:

(14) $x = x^{\circ\circ}$,
(15) $(x\wedge y)^{\circ} = x^{\circ}\vee y^{\circ}$,
(16) $1^{\circ} = 0$.

A modular GMS-algebra is a GMS-algebra $(L;\vee,\wedge,^{\circ},0,1)$ where $L = (L;\vee,\wedge,0,1)$ is a modular lattice.

The class **GMS** of all GMS-algebras forms a variety and containing the class of all modular GMS-algebras and the latter containing the class **MS** of all MS-algebras.

The main immediate consequences of these axioms are summarized in the following result (see [9]).

Lemma 2.1. Let L be a GMS-algebra. Then we have

(1) $0^{\circ} = 1$,
(2) $x\leq y \Rightarrow x^{\circ} \geq y^{\circ}$,
(3) $x^{\circ} = x^{\circ\circ\circ}$,
(4) $(x\vee y)^{\circ} = x^{\circ}\wedge y^{\circ}$,
(5) $(x\wedge y)^{\circ\circ} = x^{\circ\circ}\wedge y^{\circ\circ}$,
(6) $(x\vee y)^{\circ\circ} = x^{\circ\circ}\vee y^{\circ\circ}$.

3 Principal generalized K_2-algebras

In this section we give a simple triple construction of a principal GK_2-algebra from a triple (K,D,φ), where K is a GK-algebra, D is a bounded lattice and $\varphi : K \to D$ is a lattice homomorphism of K into D.

Firstly we introduce certain GMS-algebras, which are called generalized K_2-algebras (briefly GK_2-algebras).

Definition 3.1. A GK_2-algebra is a GMS-algebra L satisfying

(1) $x\wedge x^{\circ} = x^{\circ\circ}\wedge x^{\circ}$,
(2) $x\wedge x^{\circ} \leq y\vee y^{\circ}$.

The class $\mathbf{GK_2}$ of all GK_2-algebras contains the class $\mathbf{K_2}$ of all K_2-algebras.

A modular GK_2-algebra L is a GK_2-algebra, whenever L is a modular lattice. The class $\mathbf{mGK_2}$ of all modular GK_2-algebras contains the class $\mathbf{K_2}$ and the class of all modular S-algebras.

For any GK_2-algebra L, we have two important subsets of L which play basic roles of this paper, namely $L^{\circ\circ} = \{x\in L : x = x^{\circ\circ}\}$, the set of all closed elements of L and $D(L) = \{x\in L : x^{\circ} = 0\}$, the set of all dense elements of L. One can observe the following.

Lemma 3.2. Let $L \in \mathbf{GK_2}$. Then

(1) $L^{\circ\circ}=\{x\in L: x=x^{\circ\circ}\}$ is a GK-algebra,
(2) $D(L)=\{x\in L: x^{\circ}=0\}$ is a filter of L.

Also, we have two other important subsets of a GK_2-algebra L which are given in the following Lemma.

Lemma 3.3. Let $L\in \mathbf{GK_2}$. Then

(1) $L^{\wedge}=\{x\wedge x^{\circ}: x\in L\}=\{x\in L: x\leq x^{\circ}\}$ is an ideal of L,
(2) $L^{\vee}=\{x\vee x^{\circ}: x\in L\}=\{x\in L: x\geq x^{\circ}\}$ is a filter of L. Moreover $D(L)\subseteq L^{\vee}$.

Proof. (1). Clearly, $0\in L^{\wedge}$. Let $x,y\in L^{\wedge}$. Then $x\leq x^{\circ}$ and $y\leq y^{\circ}$. By Definition 3.1(2), we get $x=x\wedge x^{\circ}\leq y\vee y^{\circ}=y^{\circ}$. It follows that $x^{\circ}\geq y^{\circ\circ}\geq y$. Then $x^{\circ}\wedge y^{\circ}\geq x,y$ implies $(x\vee y)^{\circ}=x^{\circ}\wedge y^{\circ}\geq x\vee y$. Then $x\vee y\in L^{\wedge}$. Let $x\in L^{\wedge}$ be such that $z\leq x$ for some $z\in L$. Then $z\leq x\leq x^{\circ}\leq z^{\circ}$. Hence $z\in L^{\wedge}$. Then $L^{\wedge}$ is an ideal of L.
(2). By duality of (1), we get that $L^{\vee}$ is a filter of L. Let $x\in D(L)$. Then $x=x\vee x^{\circ}\in L^{\vee}$, as $x^{\circ}=0$. Therefore $D(L)\subseteq L^{\vee}$

Now we consider certain algebras of the class of GK_2-algebras which the so-called principal GK_2-algebras.

Definition 3.4. A GK_2-algebra L is called a principal GK_2-algebra if it satisfies the following conditions:

(1) The filter $D(L)$ is principal, i.e. there exists an element $d\in L$ such that $D(L)=[d)$,
(2) $(x\wedge y)\vee d=(x\vee d)\wedge(y\vee d)$ for every $x,y\in L$, i.e. d is a distributive element of L,
(3) $x=x^{\circ\circ}\wedge(x\vee d)$ for every $x\in L$.

Clearly, the class of all principal GK_2-algebras contains the class **GK** of all GK-algebras and the class of all principal modular S-algebras.

Definition 3.5. An (abstract) principal GK_2-triple is (K,D,φ), where

(1) $K=(K;\vee,\wedge,^{\circ},0_K,1_K)$ is a GK-algebra,
(2) $D=(D;\vee,\wedge,0_D,1_D)$ is a bounded lattice,
(3) $\varphi: K\to D$ is a $(0,1)$-lattice homomorphism from K into D and $\varphi(a)=0_D$ for any $a\in K^{\wedge}$.

Let L be a principal GK_2-algebra with the smallest dense element d. Define the map $\varphi(L): L^{\circ\circ}\to[d)$ by $\varphi(L)(a)=a\vee d$ for every $a\in L^{\circ\circ}$.

Lemma 3.6. Let L be a principal GK_2-algebra with the smallest dense element d. Then $(L^{\circ\circ},[d),\varphi(L))$ is a principal GK_2-triple.

Proof. By Lemma 3.2(1), $L^{\circ\circ}$ is a GK-algebra and by Lemma 3.2(2), $D(L)=[d)$ is a bounded lattice. It is easy to observe that $\varphi(L)$ is a $(0,1)$-lattice homomorphism. So we prove only that $\varphi(L)(x)=d$ for any $x\in L^{\circ\circ\wedge}$. Let $x\in L^{\circ\circ\wedge}$. Then $x=a\wedge a^{\circ}$ for some $a\in L^{\circ\circ}$.

$$\begin{aligned}\varphi(L)(a\wedge a^{\circ}) &= (a\wedge a^{\circ})\vee d\\ &= (a\wedge a^{\circ})\vee(d\vee d^{\circ}) \text{ as } d^{\circ}=0\\ &= d\vee d^{\circ} \text{ by (2) of definition 3.1 (2)}\\ &= d.\end{aligned}$$

Therefore $(L^{\circ\circ},[d),\varphi(L))$ is a principal GK_2-triple.

We say that $(L^{\circ\circ},[d),\varphi(L))$ the principal GK_2-triple associated with L.

Now we construct principal GK_2-algebras from principal GK_2-triples, which is one of the main results of this paper.

Theorem 3.7. Let (K,D,φ) be a principal GK_2-triple. Then

$$L=\{(a,x): a\in K, x\in D, x\leq\varphi(a)\}$$

is a principal GK_2-algebra if we define

$$\begin{aligned}(a,x)\vee(b,y) &= (a\vee b, x\vee y)\\ (a,x)\wedge(b,y) &= (a\wedge b, x\wedge y)\\ (a,x)^{\circ} &= (a^{\circ},\varphi(a^{\circ}))\\ 1_L &= (1_K,1_D)\\ 0_L &= (0_K,0_D).\end{aligned}$$

Moreover, $L^{00}\cong K$ and $D(L)\cong D$.

Proof. Clearly L is a sublattice of $K\times D$. It is observed that $0_L=(0_K,0_D)$ and $1_L=(1_K,1_D)$ are the smallest and the greatest elements of L respectively. Then L is a bounded lattice. Now for every $(a,x),(b,y)\in L$, we have

$$(a,x)\wedge(a,x)^{\circ\circ}=(a,x)\wedge(a^{\circ\circ},\varphi(a^{\circ\circ}))=(a\wedge a^{\circ\circ},x\wedge\varphi(a^{\circ\circ}))=(a,x).$$

Then $(a,x)\leq(a,x)^{\circ\circ}$. Also, we have

$$\begin{aligned}[(a,x)\wedge(b,y)]^{\circ} &= ((a\wedge b)^{\circ},\varphi((a\wedge b)^{\circ}))\\ &= (a^{\circ}\vee b^{\circ},\varphi(a^{\circ})\vee\varphi(b^{\circ}))\\ &= (a,x)^{\circ}\vee(b,y)^{\circ},\end{aligned}$$

and

$$1_L^{\circ}=0_L.$$

Therefore L is a GMS-algebra. Now we prove that L is a GK_2-algebra. Recall $\varphi(c)=0_D$, $\forall c\in K^{\wedge}$. For every $(a,x)\in L$, we have

$$\begin{aligned}(a,x)\wedge(a,x)^{\circ} &= (a\wedge a^{\circ},x\wedge\varphi(a^{\circ}))\\ &= (a\wedge a^{\circ},x\wedge\varphi(a)\wedge\varphi(a^{\circ})) \text{ as } x\leq\varphi(a)\\ &= (a\wedge a^{\circ},x\wedge\varphi(a\wedge a^{\circ}))\\ &= (a\wedge a^{\circ},x\wedge 0_D) \text{ as } a\wedge a^{\circ}\in K^{\wedge}\\ &= (a\wedge a^{\circ},0_D),\\ (a,x)^{\circ\circ}\wedge(a,x)^{\circ} &= (a\wedge a^{\circ},\varphi(a)\wedge\varphi(a^{\circ}))\\ &= (a\wedge a^{\circ},x\wedge\varphi(a\wedge a^{\circ}))\\ &= (a\wedge a^{\circ},0_D) \text{ as } a\wedge a^{\circ}\in K^{\wedge}.\end{aligned}$$

Then $(a,x)\wedge(a,x)^{\circ}=(a,x)^{\circ\circ}\wedge(a,x)^{\circ}$. Similarly we can deduce that $(a,x)\wedge(a,x)^{\circ}\leq(b,y)\vee(b,y)^{\circ}$. To prove that the GK_2-algebra L is principal, we firstly proceed to prove that $L^{\circ\circ}$ is a GK-algebra.

$$\begin{aligned}L^{\circ\circ} &= \{(a,x)\in L:(a,x)^{\circ\circ}=(a,x)\}\\ &= \{(a,x)\in L:(a^{\circ\circ},\varphi(a^{\circ\circ}))=(a,x)\}\\ &= \{(a,x)\in L: a\in K, x\in D, x=\varphi(a)\}\\ &= \{(a,\varphi(a)): a\in K\}.\end{aligned}$$

Obviously $L^{\circ\circ} \cong K$ under the isomorphism $(a, \varphi(a)) \to a$. It follows that $L^{\circ\circ}$ is a GK-algebra. Now we prove that $D(L)$ is a principal filter of L.

$$\begin{aligned} D(L) &= \{(a,x) \in L : (a,x)^{\circ} = (0_K, 0_D)\} \\ &= \{(a,x) \in L : (a^{\circ}, \varphi(a^{\circ})) = (0_K, 0_D)\} \\ &= \{(1_K, x) : x \in D\} \end{aligned}$$

Clearly $D(L) \cong D$ under the isomorphism $(1_K, x) \to x$. Then $(1_K, 0_D) \to 0_D$ implies that $(1_K, 0_D))$ is the smallest dense element of L. So $D(L) = [(1_K, 0_D))$
Now, we prove that $(1_K, 0_D)$ is a distributive element of L. For any $(a,x), (b,y) \in L$, we have

$$\begin{aligned} ((a,x) \wedge (b,y)) \vee (1_K, 0_D) &= \\ &= ((a \wedge b) \vee 1_K, (x \wedge y) \vee 0_D) \\ &= ((a \vee 1_K) \wedge (b \vee 1_K), (x \vee 0_D) \wedge (y \vee 0_D)) \\ &= (a \vee 1_K, x \vee 0_D) \wedge (b \vee 1_K, y \vee 0_D) \\ &= ((a,x) \vee (1_K, 0_D)) \wedge ((b,y) \vee (1_K, 0_D)). \end{aligned}$$

Also, we get

$$\begin{aligned} (a,x)^{\circ\circ} \wedge ((a,x) \vee (1_K, 0_D)) &= (a, \varphi(a)) \wedge (a \vee 1_K, x \vee 0_D) \\ &= (a \wedge (a \vee 1_K), \varphi(a) \wedge x)) \\ &= (a,x) \text{ as } x \leq \varphi(a). \end{aligned}$$

Therefore L is a principal GK_2-algebra. The proof is complete.

Corollary 3.8. Let L be a principal GK_2-algebra constructed from the principal GK_2-triple (K, D, φ). Then

(1) $L^{\vee} = \{(a,x) \in L : a \in K^{\vee}\}$,
(2) $L^{\wedge} = \{(a, 0_D) \in L : a \in K^{\wedge}\}$.

Corollary 3.9. Let L be a principal GK_2-algebra constructed from the principal GK_2-triple (K, D, φ). Then

(1) L is a modular GK_2-algebra, whenever K is a modular GK-algebra and D is a modular lattice,
(2) L is a K_2-algebra, whenever K is a Kleene algebra and D is a distributive lattice,
(3) L is a modular S-algebra, whenever K is a Boolean algebra and D is a modular lattice,
(4) L is a Stone algebra, whenever K is a Boolean algebra and D is a distributive lattice.

We shall say that the principal GK_2-algebra L from Theorem 3.7 is associated with the principal GK_2-triple (K, D, φ) and the construction of L described in Theorem 3.7 will be called a principal GK_2-construction.

We illustrate the principal GK_2-construction on the following example.

Example 3.10. Let K be the three-element GK-algebra and let D be the Diamond M_5 (see Figure 1).

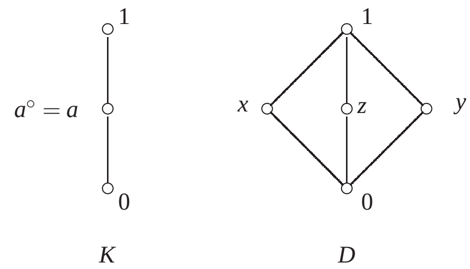

Fig. 1: K is a Kleene algebra and D is a bounded modular lattice

Define a homomorphism $\varphi : K \to D$ by the rule

$$\varphi(0) = \varphi(a) = 0 \text{ and } \varphi(1) = 1.$$

Then (K, D, φ) is a principal GK_2-triple and by the principal GK_2-construction we obtain a principal GK_2-algebra L such that

$$L = \{(0,0), (a,0), (1,0), (1,x), (1,z), (1,y), (1,1)\}$$

and

$$(0,0)^0 = (1,1), (a,0)^0 = (a,0),$$
$$(1,0)^0 = (1,x)^0 = (1,z) = (1,y)^0 = (1,1)^0 = (0,0).$$

The algebra L is represented in Figure 2. The shaded elements form a GK-algebra L^{00} which is obviously isomorphic to K. One can also observe that the filter $D(L)$ is isomorphic to the given lattice D. Moreover, the homomorphism $\varphi(L) : L^{00} \to D(L)$ defined by $\varphi(L)(c, \varphi(c)) = (c, \varphi(c)) \vee (1,0)$ is a $(0,1)$-homomorphism and $\varphi(L)(a,0) = (0,0)$ for all $a \in K^{\wedge}$. Hence the triple $(L^{00}, D(L), \varphi(L))$ is a principal GK_2-triple.

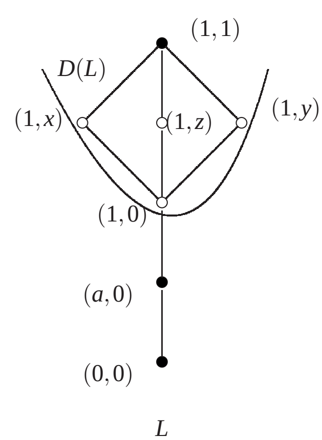

Fig. 2: L is the GK_2-algebra associated with (K, D, φ)

It is observed in the following Theorem that every principal GK_2-algebra can be obtained by the principal GK_2-construction.

Theorem 3.11. Let L be a principal GK_2-algebra with the smallest dense element d. Let $(L^{\circ\circ},[d),\varphi(L))$ be a principal GK_2-triple associated with L. Then the principal GK_2-algebra L_1 associated with $(L^{\circ\circ},[d),\varphi(L))$ is isomorphic to L.

Proof. Define the map $f:L\to L_1$ by $f(x)=(x^{\circ\circ},x\vee d)$ for every $x\in L$. Since $x\vee d\le x^{\circ\circ}\vee d=\varphi(L)(x^{\circ\circ})$, then $f(x)\in L_1$.
Now for any $x,y\in L$, we have

$$\begin{aligned}f(x\wedge y)&=((x\wedge y)^{\circ\circ},(x\wedge y)\vee d)\\&=(x^{\circ\circ}\wedge y^{\circ\circ},(x\vee d)\wedge(y\vee d))\ by\ definition\ 3.4(3)\\&=(x^{\circ\circ},x\vee d)\wedge(y^{\circ\circ},y\vee d)\\&=f(x)\wedge f(y),\end{aligned}$$

and

$$\begin{aligned}f(x\vee y)&=((x\vee y)^{\circ\circ},(x\vee y)\vee d)\\&=(x^{\circ\circ}\vee y^{\circ\circ},(x\vee d)\vee(y\vee d))\\&=(x^{\circ\circ},x\vee d)\vee(y^{\circ\circ},y\vee d)\\&=f(x)\vee f(y),\end{aligned}$$

also

$$\begin{aligned}f(x^{\circ})&=(x^{\circ\circ\circ},x^{\circ}\vee d)\\&=(x^{\circ},\varphi(L)(x^{\circ}))\\&=(f(x))^{\circ}.\end{aligned}$$

Therefore f is a homomorphism of GK_2-algebras. To prove that f is an injective mapping, suppose $f(x)=f(y)$. Then we have $x^{\circ\circ}=y^{\circ\circ}$ and $x\vee d=y\vee d$. Consequently, by Definition 3.4(3), we get

$$x=x^{\circ\circ}\wedge(x\vee d)=y^{\circ\circ}\wedge(y\vee d)=y.$$

Now we prove that f is a surjective map. Let $(a,x)\in L_1$. Set $z=a\wedge x$. Since $x\le\varphi(L)(a),a\in L^{\circ\circ}$ and $x\in D(L)$, then we have

$$\begin{aligned}f(z)&=((a\wedge x)^{\circ\circ},(a\wedge x)\vee d)\\&=(a^{\circ\circ}\wedge x^{\circ\circ},(a\vee d)\wedge(x\vee d))\ by\ definition\ 3.4(3)\\&=(a\wedge 1,(a\vee d)\wedge x)\text{ as }a^{\circ\circ}=a,x^{\circ\circ}=1\text{ and }x\ge d\\&=(a,\varphi(L)(a)\wedge x)\\&=(a,x).\end{aligned}$$

Therefore f is an isomorphism and $L\cong L_1$.

4 Isomorphisms of principal GK_2-algebras

In this section we define an isomorphism between two principal GK_2-triples and we show that two principal GK_2-algebras are isomorphic if and only if their associated principal GK_2-triples are isomorphic.

Definition 4.1. An isomorphism of the principal GK_2-triples (K,D,φ) and (K_1,D_1,φ_1) is a pair (α,β), where α is an isomorphism of K and K_1, β is an isomorphism of D and D_1 such that the diagram

$$\begin{array}{ccc}K&\xrightarrow{\ \varphi\ }&D\\ \alpha\downarrow&&\downarrow\beta\\ K_1&\xrightarrow[\ \varphi_1\ ]{}&D_1\end{array}$$

commutes. The following Theorem shows that the principal GK_2-algebras are represented by the principal GK_2-triples uniquely.

Theorem 4.2. Two principal GK_2-algebras are isomorphic if and only if their associated principal GK_2-triples are isomorphic.

Proof. Let $g:L_1\to L_2$ be an isomorphism of principal GK_2-algebras. It is obvious that $(g|L_1^{\circ\circ},g|D(L_1))$ is an isomorphism between the principal GK_2-triples $(L_1^{\circ\circ},D(L_1),\varphi(L_1))$ and $(L_2^{\circ\circ},D(L_2),\varphi(L_2))$, where $g|L_1^{\circ\circ}$ and $g|D(L_1)$ are restrictions of g to $L_1^{\circ\circ}$ and $D(L_1)$ respectively. Conversely, let (K_1,D_1,φ_1) and (K_2,D_2,φ_2) be the principal GK_2-triples associated to principal GK_2-algebras L_1 and L_2 respectively and let

$$(\alpha,\beta):(K_1,D_1,\varphi_1)\to(K_2,D_2,\varphi_2)$$

be an isomorphism of principal GK_2-triples. Let us denote by $L_1^{'}$ and $L_2^{'}$ the principal GK_2-algebras associated to principal GK_2-triples (K_1,D_1,φ_1) and (K_2,D_2,φ_2), respectively. Consider the mapping $g:L_1^{'}\to L_2^{'}$ defined by the rule $g(a,x)=(\alpha(a),\beta(x))$. It is clear that g is a $(0,1)$-lattice isomorphism.

Moreover, we have

$$\begin{aligned}g((a,x)^{\circ})&=g(a^{\circ},\varphi_1(a^{\circ}))\\&=(\alpha(a^{\circ}),\beta(\varphi_1(a^{\circ})))\\&=(\alpha(a^{\circ}),\varphi_2(\alpha(a^{\circ})))\\&=((\alpha(a))^{\circ},\varphi_2(\alpha(a))^{\circ})\\&=(\alpha(a),\beta(x))^{\circ}\\&=(g(a,x))^{\circ}\end{aligned}$$

Therefore g is an isomorphism of principal GK_2-algebras. The next Theorem 4.3 together with the previous Theorem 4.2 and Theorem 3.11 show that there is a one-to-one correspondence between principal GK_2-algebras and principal GK_2-triples.

Theorem 4.3. Let (K,D,φ) be a principal GK_2-triple and let L be its associated principal GK_2-algebra. Then

$$(L^{00},D(L),\varphi(L))\cong(K,D,\varphi).$$

Proof. From Theorem 3.7, we have the two isomorphisms $\alpha:L^{\circ\circ}\to K$ defined by $\alpha(a,\varphi(a))=a$ and $\beta:D(L)\to D$ defined by $\beta(1_K,x)=x$. It remains to prove that the diagram

$$\begin{array}{ccc} & \varphi(L) & \\ L^{\circ\circ} & \longrightarrow & D(L) \\ \alpha\downarrow & & \downarrow\beta \\ K_1 & \longrightarrow & D \\ & \varphi & \end{array}$$

is commutative. Let $z \in L^{\circ\circ}$. Then $z = (a, \varphi(a))$ for some $a \in K$ and we have

$$\begin{aligned} \beta(\varphi(L)(z)) &= \beta((a,\varphi(a)) \vee (1_K, 0_D)) \\ &= \beta(a \vee 1_K, \varphi(a) \vee 0_D) \\ &= \beta(1_K, \varphi(a)) \\ &= \varphi(a) \\ &= \varphi(\alpha(a, \varphi(a))). \end{aligned}$$

The proof is complete.

5 Congruence pairs of principal GK_2-algebras

In this section we introduce the concept of congruence pairs in principal GK_2-algebras. Also we characterize any congruence relation on a principal GK_2-algebra in terms of congruence pair.

Let L be a GK_2-algebra. For a congruence relation θ on L, let $\theta_{L^{\circ\circ}}$ and $\theta_{D(L)}$ are denote the restrictions of θ to $L^{\circ\circ}$ and $D(L)$ respectively. Obviously, $\theta_{L^{\circ\circ}}$ and $\theta_{D(L)}$ are congruence relations on $L^{\circ\circ}$ and $D(L)$ respectively. We use $Con(L)$ to denote the lattice of all congruences on L. Also we use $\triangle$ and ∇ to denote the identity and universal congruences of L respectively. Thus $(\theta_{L^{\circ\circ}}, \theta_{D(L)}) \in Con(L^{\circ\circ}) \times Con(D(L))$.

Now we introduce the notion of congruence pairs for principal GK_2-algebras.

Definition 5.1. Let L be a principal GK_2-algebra with a smallest dense element d. An arbitrary pair $(\theta_1, \theta_2) \in Con(L^{\circ\circ}) \times Con(D(L))$ is called congruence pair of L if $(a,b) \in \theta_1$ implies $(a \vee d, b \vee d) \in \theta_2$.

From Definition 5.1, we immediately obtain the following results

Lemma 5.2. Let L be a principal GK_2-algebra with a smallest dense element d. Then we have the following

(1) $(\triangle, \Phi)$ is a congruence pairs of L, for every $\Phi \in Con(D(L))$,
(2) (ψ, ∇) is a congruence pairs of L, for every $\psi \in Con(L^{\circ\circ})$.

For the principal GK_2-algebra, we have the following lemma.

Lemma 5.3. Let L be a principal GK_2-algebra with a smallest dense element d. Let (θ_1, θ_2) be a congruence pair. Then $(a,b) \in \theta_1$ and $(x,y) \in \theta_2$ imply $(a \vee x, b \vee y) \in \theta_2$.

Proof. Let $(a,b) \in \theta_1$ and $(x,y) \in \theta_2$. Then by the above Definition 5.1, we get $(a \vee d, b \vee d) \in \theta_2$. It follows that $(a \vee d \vee x, b \vee d \vee y) \in \theta_2$. Since $d \le x, y$, then $(a \vee x, b \vee y) \in \theta_2$. In the following theorem, we give a characterization for congruence pairs of a principal GK_2-algebra. This is a one of the main results of this paper.

Theorem 5.4. Let L be a principal GK_2-algebra with a smallest dense element d. Then every congruence relation θ of L determines a congruence pair $(\theta_{L^{\circ\circ}}, \theta_{D(L)})$. Conversely, every congruence pair (θ_1, θ_2) uniquely determines a congruence relation θ on L satisfying $\theta_{L^{\circ\circ}} = \theta_1$ and $\theta_2 = \theta_{D(L)}$ by the rule

$$(x,y) \in \theta \Leftrightarrow (x^{\circ\circ}, y^{\circ\circ}) \in \theta_1 \text{ and } (x \vee d, y \vee d) \in \theta_2$$

Proof. Let θ be a congruence on L. Then it is clear that $(\theta_{L^{\circ\circ}}, \theta_{D(L)})$ is a congruence pair. Conversely, Let (θ_1, θ_2) be a congruence pair and let θ be the relation define on L by the above rule. Clearly θ is an equivalent relation on L. We proceed to prove that θ is a lattice congruence. Let $(a,b), (a_1,b_1) \in \theta$. Then $(a^{\circ\circ}, b^{\circ\circ}), (a_1^{\circ\circ}, b_1^{\circ\circ}) \in \theta_1$ and $(a \vee d, b \vee d), (a_1 \vee d, b_1 \vee d) \in \theta_2$. Since $\theta_1 \in Con(L^{\circ\circ})$ and $\theta_2 \in Con(D(L))$, then we get

$$((a \wedge a_1)^{\circ\circ}, (b \wedge b_1)^{\circ\circ}) = (a^{\circ\circ} \wedge a_1^{\circ\circ}, b^{\circ\circ} \wedge b_1^{\circ\circ}) \in \theta_1,$$
$$((a \wedge a_1) \vee d, (b \wedge b_1) \vee d) = ((a \vee d) \wedge (a_1 \vee d)\ ,\ (b \vee d) \wedge (b_1 \vee d)) \in \theta_2.$$

and

$$((a \vee a_1)^{\circ\circ}, (b \vee b_1)^{\circ\circ}) = (a^{\circ\circ} \vee a_1^{\circ\circ}, b^{\circ\circ} \vee b_1^{\circ\circ}) \in \theta_1,$$
$$((a \vee a_1) \vee d, (b \vee b_1) \vee d) = ((a \vee d) \vee (a_1 \vee d)\ ,\ (b \vee d) \vee (b_1 \vee d)) \in \theta_2.$$

It follows that $(a \wedge a_1, b \wedge b_1)$, $(a \vee a_1, b \vee b_1) \in \theta$, and therefore θ is preserved by the meet and join operations of L. In order to show that θ is preserved by the unary operation $^\circ$, we let $(a,b) \in \theta$. Then $(a^{\circ\circ}, b^{\circ\circ}) \in \theta_1$. Hence $(a^\circ, b^\circ) \in \theta_1$. By the Definition of congruence pair, $(a^\circ \vee d, b^\circ \vee d) \in \theta_2$. Then $(a^{\circ\circ\circ}, b^{\circ\circ\circ}) \in \theta_1$ and $(a^\circ \vee d, b^\circ \vee d) \in \theta_2$ imply that $(a^\circ, b^\circ) \in \theta$. Therefore θ is a congruence on L.
Now, we show that $\theta_{L^{\circ\circ}} = \theta_1$ and $\theta_{D(L)} = \theta_2$. Let $a, b \in L^{\circ\circ}$ be such that $(a,b) \in \theta_1$. Then $(a^{\circ\circ}, b^{\circ\circ}) \in \theta_1$. By the Definition of congruence pair, we have $(a \vee d, b \vee d) \in \theta_2$. Hence $(a,b) \in \theta$. It follows that $(a,b) \in \theta_{L^{\circ\circ}}$ and $\theta \le \theta_{L^{\circ\circ}}$. Conversely, let $(a,b) \in \theta_{L^{\circ\circ}}$. Then $(a,b) \in \theta$ implies $(a,b) = (a^{\circ\circ}, b^{\circ\circ}) \in \theta_1$. Thus $\theta_{L^{\circ\circ}} \le \theta_1$ Then $\theta_1 = \theta_{L^{\circ\circ}}$. The equality $\theta_2 = \theta_{D(L)}$ follows straight from the definition of θ. For the uniqueness of θ. Let θ and $\acute{\theta}$ be two congruence relation on L with $\theta_{L^{\circ\circ}} = \acute{\theta}_{L^{\circ\circ}} = \theta_1$ and $\theta_{D(L)} = \acute{\theta}_{D(L)} = \theta_2$. Let $(x,y) \in \theta$. Then $(x^{\circ\circ}, y^{\circ\circ}) \in \theta_{L^{\circ\circ}}$ and $(x \vee d, y \vee d) \in \theta_{D(L)}$. Hence $(x^{\circ\circ}, y^{\circ\circ}) \in \acute{\theta}_{L^{\circ\circ}}$ and $(x \vee d, y \vee d) \in \acute{\theta}_{D(L)}$. Therefore $(x^{\circ\circ}, y^{\circ\circ}) \in \acute{\theta}$ and

$(x \vee d, y \vee d) \in \acute{\theta}$. Then we deduce that $(x,y) = (x^{\circ\circ} \wedge (x \vee d), y^{\circ\circ} \wedge (y \vee d)) \in \acute{\theta}$. Hence $\theta \leq \acute{\theta}$. Also, we can get $\acute{\theta} \leq \theta$. Then $\theta = \acute{\theta}$.

A one-to-one correspondence between the congruences lattice of a principal GK_2-algebra L and the set of all congruence pairs of L is obtained immediately by the next corollary.

Corollary 5.5. Let L be a principal GK_2-algebra with a smallest dense element d. Then the set $A(L)$ of congruence pairs of L is a sublattice of $Con(L^{\circ\circ}) \times Con(D(L))$ and $\theta \mapsto (\theta_{L^{\circ\circ}}, \theta_{D(L)})$ is an isomorphism of $Con(L)$ and $A(L)$.

Proof. Let $(\theta_1, \theta_2), (\psi_1, \psi_2) \in A(L)$. Then, it is easy to verify that $(\theta_1 \wedge \psi_1, \theta_2 \wedge \psi_2) \in A(L)$. Now, we proceed to show that $(\theta_1 \vee \psi_1, \theta_2 \vee \psi_2) \in A(L)$. Let $(a,b) \in \theta_1 \vee \psi_1$. Then there is a sequence $a = a_0, a_1, ..., a_n = b$ in $L^{\circ\circ}$ such that $(a_{i-1}, a_i) \in \theta_1 \cup \psi_1$, whenever $1 \leq i \leq n$. Then $(a_{i-1} \vee d, a_i \vee d) \in \theta_2 \cup \psi_2$ by Definition 3.1. Thus we have $a \vee d = a_0 \vee d, a_1 \vee d, \cdots, a_n \vee d = b \vee d \in D(L)$. The above result leads to $(a \vee d, b \vee d) \in \theta_2 \vee \psi_2$ and hence $(\theta_1 \vee \psi_1, \theta_2 \vee \psi_2) \in A(L)$. Thus we conclude that $A(L)$ is a sublattice of $Con(L^{\circ\circ}) \times Con(D(L))$. The last part of the Corollary is obvious and the proof is finished.

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