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On the Solution of Linear Partial Differential Equation Related to Diamond and Diamond Bessel Klein Gordon Operator

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Abstract: In this paper,we studied the equation

$$\diamondsuit_B^k\left(\diamondsuit_B+d^2\right)^k u(x)=\sum_{r=0}^{m} c_r\diamondsuit_B^k\left(\diamondsuit_B+d^2\right)^k\delta.$$

We give a sense of Distribution theory considering the properties of the convolution. It was found that the type of above equation depend on the relationship between the value k and m.

Keywords: Dalgaard-Strulik model, energy, economic growth, time delay, limit cycle

1 Introduction

In 2004, Hüseyin Yildirim, M.Zeki Sarikaya and Sermin Öztürk [4,5] first introduced the Bessel diamond operator $\diamondsuit_B^k$ iterated k times, and defined by

$$\diamondsuit_B^k=\left(\left(\sum_{i=1}^{p}B_{x_i}\right)^2-\left(\sum_{j=p+1}^{p+q}B_{x_j}\right)^2\right)^k,\tag{1}$$

where $B_{x_i}=\frac{\partial^2}{\partial x_i^2}+\frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i=2\alpha_i+1,\alpha_i>-\frac{1}{2},x_i>0$. The operator $\diamondsuit_B^k$ can be expressed by $\diamondsuit_B^k=\triangle_B^k\Box_B^k=\Box_B^k\triangle_B^k$, where

$$\triangle_B^k=\left(\sum_{i=1}^{p}B_{x_i}\right)^k \text{ and } \Box_B^k=\left(\sum_{i=1}^{p}B_{x_i}-\sum_{j=p+1}^{p+q}B_{x_j}\right)^k.\tag{2}$$

Hüseyin Yildirim, M.Zeki Sarikaya and Sermin Öztürk [4,5] have shown the convolution form $u(x)=(-1)^kS_{2k}(x)*R_{2k}(x)$ is a unique elementary solution of $\diamondsuit_B^k$ that is

$$\diamondsuit_B^k((-1)^kS_{2k}(x)*R_{2k}(x))=\delta,\tag{3}$$

where $S_{2k}(x)$ and $R_{2k}(x)$ are defined by (9) and (11) with $\alpha=\gamma=2k$ respectively. Next, C. Bunpog and A. Kananthai[2] have first introduced the operator $\left(\diamondsuit_B+m^4\right)^k$ named Diamond Klien-Gordon Bessel operator iterated k times and can be written in the following form

$$\left(\diamondsuit_B+m^4\right)^k=\left(\left(\triangle_B+m^2\right)\left(\Box_B+m^2\right)-m^2\left(\triangle_B+\Box_B\right)\right)^k,\tag{4}$$

where $\Box_B+m^2$ is the Bessel Klien-Gordon operator and $\triangle_B+m^2$ is the Bessel Helmholtz operator defined by

$$\Box_B+m^2=\sum_{i=1}^{p}B_{x_i}-\sum_{j=p+1}^{p+q}B_{x_j}+m^2,\tag{5}$$

and

$$\triangle_B+m^2=\sum_{i=1}^{n}B_{x_i}+m^2.\tag{6}$$

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The purpose of this work, firstly, we study the elementary solution or Green function of the $\left(\diamondsuit_B+d^2\right)^k$, that is

$$\left(\diamondsuit_B+d^2\right)^k G(x)=\delta, \tag{7}$$

where$G(x)$ is the Green function, δ is the Dirac delta distribution, k is a nonnegative integer and $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$. We also consider the convolution of Green function.
Finally, we are finding the solution of the equation

$$\diamondsuit_B^k\left(\diamondsuit_B+d^2\right)^k u(x)=\sum_{r=0}^{m} c_r\diamondsuit_B^k\left(\diamondsuit_B+d^2\right)^k\delta. \tag{8}$$

We use the B-convolution for the generalized function. It was found that the type of the solution (8) that depend on the relationship between the values of k and m are as the following cases:

(1) If $m<k$ and $m=0$, then the solution of (8) is

$$u(x)=c_0((-1)^k S_{2k}(x)*R_{2k}(x))*W_{2k}(x),$$

which is an elementary solution of product of the operator $\diamondsuit_B^k$ and the operator $\left(\diamondsuit_B+d^2\right)^k$ in Theorem 3.1, is the ordinary function for $2k\geq n+2|\upsilon|$, and is a tempered distribution for $2k<n+2|\upsilon|$.

(2) If $0<m<k$ then the solution of (8) is

$$u(x)=\sum_{r=1}^{m} c_r((-1)^{k-r}S_{2(k-r)}(x)*R_{2(k-r)}(x))*W_{2(k-r)}(x),$$

which is an ordinary function for $2k-2r\geq n+2|\upsilon|$ and is tempered distribution for $2k-2r<n+2|\upsilon|$.

(3) If $m\geq k$ and suppose $k\leq m\leq M$, then (8) has the solution

$$u(x)=\sum_{r=k}^{M} c_r\diamondsuit_B^{r-k}\left(\diamondsuit_B+d^2\right)^{r-k}\delta$$

which is only the singular distribution.

Before proceeding that point, the following definitions and some important concepts are needed.

2 Preliminaries

Definition 2.1
Let $x=(x_1,x_2,\dots,x_n),\nu=(\nu_1,\nu_2,\dots,\nu_n)\in\mathbb{R}_n^+$. For any complex number α, we define the distribution family $S_\alpha(x)$ by

$$S_\alpha(x)=\frac{|x|^{\alpha-n-2|\nu|}}{w_n(\alpha)}, \tag{9}$$

where $|x|=x_1^2+x_2^2+\ldots+x_n^2, |\nu|=\nu_1+\nu_2+\ldots+\nu_n$ and,

$$w_n(\alpha)=\frac{\prod_{i=1}^n 2^{\nu_i-\frac{1}{2}}\Gamma(\nu_i+\frac{1}{2})}{2^{n+2|\nu|-2\alpha}\Gamma(\frac{n+2|\nu|-\alpha}{2})} \tag{10}$$

Definition 2.2
Let $x=(x_1,x_2,\dots,x_n),\nu=(\nu_1,\nu_2,\dots,\nu_n)\in\mathbb{R}_n^+$, and denote by

$$V=x_1^2+x_2^2+\cdots+x_p^2-x_{p+1}^2-x_{p+2}^2-\cdots-x_{p+q}^2,$$

the nondegenerated quadratic form. Denote the interior of the forward cone by

$$\Gamma_+=\{x\in\mathbb{R}_n^+ : x_1>0,x_2>0,\dots,x_n>0,V>0\}$$

and $\overline{\Gamma}_+$ denotes its closure. For any complex number γ the distribution family $R_\gamma(x)$ is defined by

$$R_\gamma(x)=\begin{cases}\frac{V^{\frac{\gamma-n-2|\nu|}{2}}}{K_n(\gamma)},, & \text{for } x\in\Gamma_+,\\ 0, & \text{for } x\notin\Gamma_+,\end{cases} \tag{11}$$

where

$$K_n(\gamma)=\frac{\pi^{\frac{n+2|\nu|-1}{2}}\Gamma\left(\frac{2+\gamma-n-2|\nu|}{2}\right)\Gamma\left(\frac{1-\gamma}{2}\right)\Gamma(\gamma)}{\Gamma\left(\frac{2+\gamma-p-2|\nu|}{2}\right)\Gamma\left(\frac{p-\gamma}{2}\right)},$$

where γ is a complex number.

Definition 2.3
Let $x=(x_1,x_2,\dots,x_n)$ be a point of $\mathbb{R}_n^+$, we define the function

$$W_\alpha(x)=\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(\frac{n}{2}+r)}{r!\Gamma(\frac{n}{2})}(m^2)^r(-1)^{\frac{\alpha}{2}+r}S_{\alpha+2r}(x)*R_{\alpha+2r}(x), \tag{12}$$

where the function $S_{\alpha+2r}$ and $R_{\alpha+2r}$ are defined by definition 2.2 and definition 2.3 respectively.
Lemma 2.1

Lemma 1.*Let α and β be complex numbers and $S_\alpha(x)$ be the function defined by (9).Then the following properties are valid*

$$S_0(x)=\delta(x) \tag{13}$$

$$S_{-2k}(x)=(-1)^k\triangle_B^k\delta \tag{14}$$

$$\triangle_B^k\{S_\alpha(x)\}=(-1)^k S_{\alpha-2k}(x) \tag{15}$$

$$S_\alpha(x)*S_\beta(x)=S_{\alpha+\beta}(x), \tag{16}$$

where $\triangle_B^k$ is the Laplace Bessel operator iterated k times and defined by (2).

Proof. [3] . □
Lemma 2.2

Lemma 2. *Let α and β be complex numbers and $R_\gamma(x)$ be the function defined by (10). Then the following properties are valid*

$$R_0(x) = \delta(x) \tag{17}$$

$$R_{-2k}(x) = \square_B^k \delta \tag{18}$$

$$\square_B^k \{S_\gamma(x)\} = S_{\gamma-2k}(x) \tag{19}$$

$$R_\alpha(x) * R_\beta(x) = R_{\alpha+\beta}(x), \tag{20}$$

where $\square_B^k$ is the Ultra-hyperbolic Bessel operator iterated k times and defined by (2).

Proof. [3] . □

Lemma 2.3

Lemma 3. *The functions $S_\alpha(x)$ and $R_\alpha(x)$ defined by (9) and (10) respectively are homogeneous distribution of order $\alpha - n - 2|\upsilon|$ and also tempered distribution.*

Proof. Since $R_\alpha(x)$ and $S_\alpha(x)$ satisfy the Euler equation,that is

$$(\alpha - n - 2|\upsilon|)R_\alpha(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} R_\alpha(x)$$

and

$$(\alpha - n - 2|\upsilon|)S_\alpha(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} S_\alpha(x).$$

We have $R_\alpha(x)$ and $S_\alpha(x)$ as homogeneous distributions of order $\alpha - n - 2|\upsilon|$ and Donoghue [8] has proved that every homogeneous distribution is a tempered distribution. That completes the proof.

Lemma 2.4

Lemma 4. *(The convolution of tempered distribution). The convolution $R_\alpha(x) * S_\alpha(x)$ exists and is a tempered distribution.*

Proof. Choosing supp $R_\alpha(x) = K \subset \Gamma_+$ where K is a compact set, the function $R_\alpha(x)$ is a tempered distribution with compact support and by Donoghue[8] $R_\alpha(x) * S_\alpha(x)$ exists and is a tempered distribution .

Lemma 2.5

Lemma 5. *Given the equation $\diamondsuit_B^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where $\diamondsuit_B^k$ defined by (1). And*

$$u(x) = (-1)^k S_{2k}(x) * R_{2k}(x),$$

where $S_{2k}(x)$ and $R_{2k}(x)$ are defined by (13) and (15) with $\alpha = 2k, \gamma = 2k$ respectively.
We obtain $(-1)^k S_{2k}(x) * R_{2k}(x)$ is an elementary solution of the operator $\diamondsuit_B^k$. That is

$$\diamondsuit_B^k \left((-1)^k S_{2k}(x) * R_{2k}(x) \right) = \delta(x) \tag{21}$$

Proof. [4,5]. □

Lemma 2.6

Lemma 6. *Let α and β be complex numbers. The following formulas are valid*

$$W_0(x) = \delta(x) \tag{22}$$

$$W_\alpha * W_\beta = W_{\alpha+\beta} \tag{23}$$

$$W_\alpha * W_{-2k} = W_{\alpha-2k} \tag{24}$$

Proof. By definition 2.3, we obtain

$$W_0(x) = \delta(x).$$

By definition 2.3 again, we have

$$W_\alpha(x) * W_\beta(x) = \sum_{r=0}^{\infty} \binom{-\frac{\alpha}{2}}{r} (m^2)^r (-1)^{\frac{\alpha}{2}+r} S_{\alpha+2r}(x) * R_{\alpha+2r}(x). * \sum_{s=0}^{\infty} \binom{-\frac{\beta}{2}}{s} (m^2)^s (-1)^{\frac{\beta}{2}+s} S_{\beta+2s}(x) * R_{\beta+2s}(x)$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \binom{-\frac{\alpha}{2}}{r} \binom{-\frac{\beta}{2}}{s} (m^2)^{r+s} (-1)^{\frac{\alpha+\beta}{2}+r+s} \left(S_{\alpha+2r}(x) * R_{\alpha+2r}(x)\right) * \left(S_{\beta+2s}(x) * R_{\beta+2s}(x)\right)$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \binom{-\frac{\alpha}{2}}{r} \binom{-\frac{\beta}{2}}{s} (m^2)^{r+s} (-1)^{\frac{\alpha+\beta}{2}+r+s} \left(S_{\alpha+\beta+2(r+s)}(x) * R_{\alpha+\beta+2(r+s)}(x)\right)$$

$$= \sum_{k=0}^{\infty} (m^2)^k \left[\sum_{r=0}^{k} \binom{-\frac{\alpha}{2}}{r} \binom{-\frac{\beta}{2}}{k-r} \right] (-1)^{\frac{\alpha+\beta}{2}+k} \left(S_{\alpha+\beta+2k}(x) * R_{\alpha+\beta+2k}(x)\right). \tag{25}$$

By properties

$$\sum_{r=0}^{k} \binom{-\frac{\alpha}{2}}{r} \binom{-\frac{\beta}{2}}{k-r} = \binom{-\frac{\alpha+\beta}{2}}{k}.$$

The equation (25) becomes

$$\begin{aligned} W_\alpha(x) * W_\beta(x) &= \sum_{r=0}^{\infty} \binom{-\frac{\alpha+\beta}{2}}{r} (m^2)^r (-1)^{\frac{\alpha+\beta}{2}+k} S_{\alpha+\beta+2k}(x) * R_{\alpha+\beta+2k}(x). \\ &= W_{\alpha+\beta}(x) \end{aligned}$$

Thus,

$$W_\alpha(x) * W_\beta(x) = W_{\alpha+\beta}(x). \tag{26}$$

Putting $\beta = -2k$ in (26), we obtain

$$W_\alpha(x) * W_{-2k}(x) = W_{\alpha-2k}(x). \tag{27}$$

That completes the proof. □

3 Main Results

Theorem 3.1

Theorem 1. *Given the equation*

$$\left(\diamondsuit_B + d^2\right)^k u(x) = \delta(x) \tag{28}$$

for $x \in \mathbb{R}_n^+$ and $\left(\diamondsuit_B + d^2\right)^k$ is the Diamond Klein Gordon operator iterated k times defined by (4), we obtain

$$u(x) = W_{2k}(x) \tag{29}$$

is an elementary solution or Green function of the operator $\left(\diamondsuit_B+d^2\right)^k$ *and* $W_{2k}(x)$ *is defined by (18) with* $\alpha=2k$. *The function* $W_{2k}(x)$ *has the following properties*

$$W_0(x)=\delta(x) \tag{30}$$

and

$$\left(\diamondsuit_B+d^2\right)^k\{W_\alpha(x)\}=W_{\alpha-2k}(x) \tag{31}$$

Proof. In fact,

$$(\diamondsuit_B+d^2)^{-\frac{\alpha}{2}}=\{\diamondsuit_B(1+d^2\diamondsuit_B^{-1})\}^{-\frac{\alpha}{2}}=\diamondsuit_B^{-\frac{\alpha}{2}}(1+d^2\diamondsuit_B^{-1})^{-\frac{\alpha}{2}}$$

and

$$\begin{aligned}(1+d^2\diamondsuit_B^{-1})^{-\frac{\alpha}{2}}\delta&=\sum_{r=0}^{\infty}\binom{-\frac{\alpha}{2}}{r}(d^2\diamondsuit_B^{-1})^r\delta\\&=\sum_{r=0}^{\infty}\binom{-\frac{\alpha}{2}}{r}d^{2r}\diamondsuit_B^{-r}\delta\end{aligned}$$

Thus,

$$\begin{aligned}\diamondsuit_B^{-\frac{\alpha}{2}}(1+d^2\diamondsuit_B^{-1})^{-\frac{\alpha}{2}}&=\diamondsuit^{-\frac{\alpha}{2}}\sum_{r=0}^{\infty}\binom{-\frac{\alpha}{2}}{r}(d^2\diamondsuit_B^{-1})^r\delta\\&=\sum_{r=0}^{\infty}\binom{-\frac{\alpha}{2}}{r}d^{2r}\diamondsuit_B^{-\frac{\alpha}{2}-r}\delta.\end{aligned}$$

From the above equation, we get

$$\begin{aligned}(\diamondsuit_B+d_B^2)^{-\frac{\alpha}{2}}\delta&=\sum_{r=0}^{\infty}\binom{-\frac{\alpha}{2}}{r}d^{2r}\diamondsuit_B^{-\frac{\alpha}{2}-r}\delta.\\&=\sum_{r=0}^{\infty}\binom{-\frac{\alpha}{2}}{r}d^{2r}\triangle_B^{-\frac{\alpha}{2}-r}\square_B^{-\frac{\alpha}{2}-r}\delta.\\&=\sum_{r=0}^{\infty}\binom{-\frac{\alpha}{2}}{r}d^{2r}(-1)^{\frac{\alpha}{2}+r}S_{2(\frac{\alpha}{2}+r)}(x)*R_{2(\frac{\alpha}{2}+r)}(\upsilon)\\&=\sum_{r=0}^{\infty}\binom{-\frac{\alpha}{2}}{r}d^{2r}(-1)^{\frac{\alpha}{2}+r}S_{\alpha+2r}(x)*R_{\alpha+2r}(x)\\&=W_\alpha(x)\end{aligned}$$

If we put $\alpha=-2k$, we obtain

$$\left(\diamondsuit_B+d^2\right)^k\delta=W_{-2k}(x) \tag{32}$$

Putting $k=0$ in (32), we obtain

$$W_0(x)=\delta(x). \tag{33}$$

By lemma 2.6, we have

$$W_\alpha(x)*W_\beta(x)=W_{\alpha+\beta}(x).$$

Putting $\beta=-2k$, we obtain

$$W_\alpha(x)*W_{-2k}(x)=W_{\alpha-2k}(x)$$

$$W_\alpha(x)*\left(\diamondsuit+m^4\right)^k\delta=W_{\alpha-2k}(x)$$

$$\left(\diamondsuit_B+d^2\right)^kW_\alpha(x)*\delta=W_{\alpha-2k}(x). \tag{34}$$

If we put $\alpha=2k$ in (34), we obtain

$$\left(\diamondsuit+d^2\right)^k\delta*W_{2k}(x)=W_0(x).=\delta(x). \tag{35}$$

It follows that $W_{2k}(x)$ is an elementary solution or Green function of the operator $\left(\diamondsuit+d^2\right)^k$. That completes the proof.

Theorem 3.2

Theorem 2.*Let* $\diamondsuit_B^k$ *and* $\left(\diamondsuit+d^2\right)^k$ *be Diamond Bessel operator and Diamond Bessel Klein Gordon operator respectively.For* $0<r<k$

$$\begin{aligned}&\diamondsuit_B^r((-1)^kS_{2k}(x)*R_{2k}(x))*\left(\diamondsuit_B+d^2\right)^rW_{2k}(x)\\&\quad=((-1)^{k-r}S_{2(k-r)}(x)*R_{2(k-r)}(x))*W_{2(k-r)}(x)\end{aligned} \tag{36}$$

and for $k\le m$

$$\diamondsuit_B^r((-1)^kS_{2k}(x)*R_{2k}(x))*\left(\diamondsuit_B+d^2\right)^rW_{2k}(x)=\diamondsuit_B^{m-k}\left(\diamondsuit_B+d^2\right)^{m-k}\delta, \tag{37}$$

where $\left(\diamondsuit_B+d^2\right)^k$ *is the Diamond Bessel Klein Gordon operator iterated* k *times defined by (4),* δ *is the dirac delta distribution and the function* $W_{2k}(x)$ *defined by (18) with* $\alpha=2k$.

Proof. For $0<r<k$, from theorem 3.1, we have

$$\diamondsuit_B^r((-1)^kS_{2k}(x)*R_{2k}(x))*\left(\diamondsuit_B+d^2\right)^kW_{2k}(x))=\delta.$$

We can write the above equation in the following form

$$\diamondsuit_B^{k-r}\diamondsuit_B^r((-1)^kS_{2k}(x)*R_{2k}(x))*\left(\diamondsuit_B+d^2\right)^{k-r}\left(\diamondsuit_B+d^2\right)^rW_{2k}(x)=\delta.$$

or

$$(\diamondsuit_B^{k-r}\delta*\diamondsuit_B^r((-1)^kS_{2k}(x)*R_{2k}(x)))*\left(\diamondsuit_B+d^2\right)^{k-r}\delta*\left(\diamondsuit_B+d^2\right)^rW_{2k}(x)=\delta.$$

We have used the convolution of both sides by $((-1)^{k-r}S_{2(k-r)}(x)*R_{2(k-r))}(x))*W_{2(k-r)}(x)$, we obtain

$$\begin{aligned}&((-1)^{k-r}S_{2(k-r)}(x)*R_{2(k-r)}(x))*W_{2(k-r)}*(\diamondsuit_B^{k-r}\delta*\diamondsuit_B^r((-1)^kS_{2k}(x)*R_{2k}(x)))\\&\quad*\delta*\left(\diamondsuit_B+d^2\right)^rW_{2k}(x)=((-1)^kS_{2k}(x)*R_{2k}(x))*W_{2(k-r)}(x)*\delta.\end{aligned}$$

$$\begin{aligned}&\diamondsuit_B^{k-r}((-1)^{k-r}S_{2(k-r)}(x)*R_{2(k-r)}(x))*\diamondsuit_B^r((-1)^kS_{2k}(x)*R_{2k}(x))\\&\left(\diamondsuit_B+d^2\right)^{k-r}*W_{2(k-r)}*\left(\diamondsuit_B+d^2\right)^rW_{2k}(x)=((-1)^{(k-r)}S_{2(k-r)}(x)*R_{2(k-r)}(x))*W_{2(k-r)}(x).\end{aligned}$$

By property of convolution, we get

$$\begin{aligned}&\delta*\diamondsuit_B^r((-1)^kS_{2k}(x)*R_{2k}(x))*\delta*\left(\diamondsuit_B+d^2\right)^rW_{2k}(x)\\&\quad=((-1)^{(k-r)}S_{2(k-r)}(x)*R_{2(k-r)}(x))*W_{2(k-r)}(x).\end{aligned}$$

$$\begin{aligned}&\diamondsuit_B^r((-1)^kS_{2k}(x)*R_{2k}(x))*\left(\diamondsuit_B+d^2\right)^rW_{2k}(x)\\&\quad=((-1)^{(k-r)}S_{2(k-r)}(x)*R_{2(k-r)}(x))*W_{2(k-r)}(x).\end{aligned}$$

as required. For $k \leq m$

$$\diamondsuit_B^m((-1)^k S_{2k}(x) * R_{2k}(x)) * \left(\diamondsuit_B + d^2\right)^m W_{2k}(x))$$
$$= \diamondsuit_B^{m-k} \diamondsuit_B^k((-1)^k S_{2k}(x) * R_{2k}(x)) * \left(\diamondsuit_B + d^2\right)^{m-k} \left(\diamondsuit_B + d^2\right)^k W_{2k}(x))$$

It follows that

$$\diamondsuit_B^m((-1)^k S_{2k}(x) * R_{2k}(x)) * \left(\diamondsuit_B + d^2\right)^m W_{2k}(x)) = \diamondsuit_B^{m-k} \delta * \left(\diamondsuit_B + d^2\right)^{m-k} \delta.$$

That completes the proof.
Theorem 3.3

Theorem 3._Given the linear differential equation_

$$\diamondsuit_B^k \left(\diamondsuit_B + d^2\right)^k u(x) = \sum_{r=0}^{m} c_r \diamondsuit_B^k \left(\diamondsuit_B + d^2\right)^k \delta, \tag{38}$$

The the type of solution (38) that depend on the relationship between the values of k and m are as the following cases:

(1) If $m < k$ and $m = 0$, then the solution of (38) is

$$u(x) = c_0((-1)^k S_{2k}(x) * R_{2k}(x)) * W_{2k}(x)),$$

which is an elementary solution of the $\left(\diamondsuit_B + d^2\right)^m$ operator in Theorem 3.1.

(2) If $0 < m < k$ then the solution of (38) is

$$u(x) = \sum_{r=1}^{m} c_r((-1)^{k-r} S_{2(k-r))}(x) * R_{2(k-r)}(x)) * W_{2(k-r)}(x)$$

which is an ordinary function for $2k - 2r \geq n + 2|\upsilon|$ and is tempered distribution for $2k - 2r < n + 2|\upsilon|$

(3) If $m \geq k$ and suppose $k \leq m \leq M$, then (38) has the solution

$$u(x) = \sum_{r=k}^{M} c_r \diamondsuit_B^{r-k} \left(\diamondsuit_B + d^2\right)^{r-k} \delta$$

which is only the singular distribution.

Proof. (1) For $m = 0$,
we have $\diamondsuit_B^k \left(\diamondsuit_B + d^2\right)^k u(x) = c_0 \delta$, and by Theorem 3.1 we obtain

$$u(x) = c_0((-1)^k S_{2k}(x) * R_{2k}(x)) * W_{2k}(x)$$

Now, $W_{2k}(x)$ analytic function for $2k \geq n + 2|\upsilon|$ and also $W_{2k}(x)$ exists and is an analytic function by (29).It follows that $W_{2k}(x)$ is an ordinary function for $2k \geq n + 2|\upsilon|$ and is a tempered distribution with $2k < n + 2|\upsilon|$.

(2) For the case $0 < m < k$, we have

$$\diamondsuit_B^k \left(\diamondsuit_B + d^2\right)^k u(x) = \sum_{r=1}^{m} c_r \diamondsuit_B^r \left(\diamondsuit_B + d^2\right)^r \delta,$$
$$= c_1 \diamondsuit_B \left(\diamondsuit_B + d^2\right) \delta + c_2 \diamondsuit_B^2 \left(\diamondsuit_B + d^2\right)^2 \delta + \ldots + c_m \diamondsuit_B^m \left(\diamondsuit_B + d^2\right)^m \delta.$$

Convolving both sides of the above equation by $((-1)^k S_{2k}(x) * R_{2k}(x)) * W_{2k}(x)$, we obtain

$$\begin{aligned}
&((-1)^k S_{2k}(x) * R_{2k}(x)) * W_{2k}(x) * \diamondsuit_B^k \left(\diamondsuit_B + d^2\right)^k u(x) \\
&= c_1((-1)^k S_{2k}(x) * R_{2k}(x)) * W_{2k}(x) \diamondsuit_B \left(\diamondsuit_B + d^2\right) \delta \\
&+ c_2((-1)^k S_{2k}(x) * R_{2k}(x)) * W_{2k}(x) \diamondsuit_B^2 \left(\diamondsuit_B + d^2\right)^2 \delta \\
&\vdots \\
&+ c_m((-1)^k S_{2k}(x) * R_{2k}(x)) * W_{2k}(x) \diamondsuit_B^m \left(\diamondsuit_B + d^2\right)^m \delta
\end{aligned}$$

By properties of convolution, we get

$$\begin{aligned}
&\diamondsuit_B^k((-1)^k S_{2k}(x) * R_{2k}(x)) * \left(\diamondsuit_B + d^2\right)^k W_{2k}(x) * u(x) \\
&= c_1 \diamondsuit_B((-1)^k S_{2k}(x) * R_{2k}(x)) * \left(\diamondsuit_B + d^2\right) W_{2k}(x) \\
&+ c_2 \diamondsuit_B^2((-1)^k S_{2k}(x) * R_{2k}(x)) * \left(\diamondsuit_B + d^2\right)^2 W_{2k}(x) \\
&\vdots \\
&+ c_m \diamondsuit_B^m((-1)^k S_{2k}(x) * R_{2k}(x)) * \left(\diamondsuit_B + d^2\right)^m W_{2k}(x)
\end{aligned}$$

$$\begin{aligned}
u(x) &= c_1 \diamondsuit_B((-1)^k S_{2k}(x) * R_{2k}(x)) * \left(\diamondsuit_B + d^2\right) W_{2k}(x) \\
&+ c_2 \diamondsuit_B^2((-1)^k S_{2k}(x) * R_{2k}(x)) * \left(\diamondsuit_B + d^2\right)^2 W_{2k}(x) + \ldots \\
&+ c_m \diamondsuit_B^m((-1)^k S_{2k}(x) * R_{2k}(x)) * \left(\diamondsuit_B + d^2\right)^m W_{2k}(x)
\end{aligned}$$

By Theorem 3.1 and Theorem 3.2, we obtain

$$\begin{aligned}
u(x) &= c_1((-1)^{k-1} S_{2(k-1)}(x) * R_{2(k-1)}(x)) * W_{2(k-1)}(x) \\
&+ c_2((-1)^{k-2} S_{2(k-2)}(x) * R_{2((k-2)}(x)) * W_{2(k-2)}(x) + \ldots \\
&+ c_m((-1)^{k-m} S_{2(k-m)}(x) * R_{2(k-m)}(x)) * W_{2(k-m)}(x)
\end{aligned}$$

or

$$u(x) = \sum_{r=1}^{m} c_r((-1)^{k-r} S_{2(k-r)}(x) * R_{2(k-r))}(x)) * W_{2(k-r)}(x). \tag{39}$$

Similarly, as in the case(1), $u(x)$ is an ordinary function for $2k - 2r \geq n + 2|\upsilon|$ and is a tempered distribution for $2k - 2r < n + 2|\upsilon|$.

(3) For the case $m \geq k$ and suppose $k \leq m \leq M$, we have

$$\begin{aligned}
\diamondsuit_B^k \left(\diamondsuit_B + d^2\right)^k u(x) &= c_k \diamondsuit_B^k \left(\diamondsuit_B + d^2\right)^k \delta + c_{k+1} \diamondsuit_B^{k+1} \left(\diamondsuit_B + d^2\right)^{k+1} \delta \\
&+ \ldots + c_M \diamondsuit_B^M \left(\diamondsuit_B + d^2\right)_B^M \delta.
\end{aligned} \tag{40}$$

We convolved both sides of the above equation by $((-1)^k S_{2k}(x) * R_{2k}(x)) * W_{2k}(x)$, we obtain

$$u(x) = c_1((-1)^k S_{2k}(x) * R_{2k}(x)) * W_{2k}(x) \Diamond_B^k \left(\Diamond_B + d^2\right)^k \delta$$
$$+ c_2((-1)^k S_{2k}(x) * R_{2k}(x)) * W_{2k}(x) \Diamond_B^{k+1} \left(\Diamond_B + d^2\right)^{k+1} \delta$$
$$+ \ldots + c_m((-1)^k S_{2k}(x) * R_{2k}(x)) * W_{2k}(x) \Diamond_B^M \left(\Diamond_B + d^2\right)^M \delta$$

$$u(x) = c_k \Diamond_B^k((-1)^k S_{2k}(x) * R_{2k}(x)) * \left(\Diamond_B + d^2\right)^k W_{2k}(x)$$
$$+ c_{k+1} \Diamond_B \Diamond_B^k((-1)^k S_{2k}(x) * R_{2k}(x)) * \left(\Diamond_B + d^2\right) \left(\Diamond_B + d^2\right)^k W_{2k}(x)$$
$$+ \ldots + c_M \Diamond_B^{M-k} \Diamond_B^k((-1)^k S_{2k}(x) * R_{2k}(x)) * \Diamond_B^{M-k} \left(\Diamond_B + d^2\right)^k W_{2k}(x)$$

By Theorem 3.1 and Theorem 3.2 again, we obtain

$$u(x) = c_k \delta + c_{k+1} \Diamond_B \left(\Diamond_B + d^2\right) \delta + c_{k+2} \Diamond_B^2 \left(\Diamond_B + d^2\right)^2 \delta$$
$$+ \ldots + c_M \Diamond_B^{M-k} \left(\Diamond_B + d^2\right)^{M-k} \delta$$
$$= \sum_{r=k}^{M} c_r \Diamond_B^{r-k} \left(\Diamond_B + d^2\right)^{r-k} \delta.$$

Since $\left(\Diamond_B + d^2\right)^{r-k} \delta$ is a singular distribution, hence $u(x)$ is only the singular distribution. That completes the proofs.

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