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Construction of Dynamical Field Equation for Circular-Cylindrical Bodies

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Abstract: Newton published his dynamical theory of Gravitation in the year 1686. According to Newton's famous dynamical theory, all interactions in nature are as a result of force. This theory was successful in explaining the gravitational phenomena on earth and the experimental fact of the solar system. It is well known that Newton's dynamical laws of motion and gravitation are founded in terms of invariant rest masses of particles and bodies and cannot be applied to a photon which has no measurable rest mass. There were several attempts at the end of the 19th century to generalize or extend Newton's dynamical theory of gravitation in order to provide better agreement with the experimental data or better consistency to all physical theories and most of these theories are based on the spherical nature of the Earth. In this research work we constructed the laplacian operator for circular cylindrical coordinate using the covariant and the contravariant metric tensors for this field. The constructed laplacian operator was later used in the derivation of dynamical gravitational field equation for circular-cylindrical bodies. The field equation obtained contains $\left(1+\frac{2}{c^2}f\right)$ and a density term contribution ρ which are not found in the existing well-known Newton's dynamical field equations. This field equation differs from the well-known field equation for spherical bodies.

Keywords: Gravitation, Circular-cylindrical, Spacetime, Soliton, General relativity.

1 Introduction

The investigations of cylindrically symmetric space-times can be traced back as far as to 1919 when Levi-Civita (LC) discovered a class of solutions of Einstein's vacuum field equations, corresponding to static cylindrical space-times [1]. The extension of the LC space-times to stationary ones was obtained independently by Lanczos in 1924 and Lewis in 1932. In 1925, Beck studied a class of exact solutions and interpreted them as representing the propagation of cylindrical gravitational waves (GWs) [2].

This class of solutions was later rediscovered by Einstein and Rosen in their seminal work on the studies of the nonlinearity of GWs in 1937 and in the same year van Stockum solved the problem of a rigidly rotating infinitely long cylinder of dust and found explicitly the corresponding metric. In 1957, Bonnor, Weber and Wheeler studied the Einstein-Rosen waves in great details, and since then, cylindrically symmetric space-times have been extensively investigated with various motivations. In particular, to understand some fundamental issues in general relativity (GR), such as the structure of the theory, the nature and formation of space-time singularities, the cosmic censorship and hoop conjectures, one often assumes certain symmetries of space-times, because Einstein's field equations are in general so complicated that it is extremely difficult to study them in their most general form, especially in the times when computers had not been available. In fact, even now it is still very difficult to study space time with only one Killing vector (both analytically and numerically). Therefore, the next step is naturally to consider space-time

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with two Killing vectors, which include cylindrically symmetric space-time [3, 4].Researchers have been able to construct a generalized dynamical gravitational field equation for spherical coordinates. These were successful in explaining Newton's and Einstein's theory of relativity. Diverting from studies for spherical coordinates many scientists and researchers were able to work on cylindrical coordinates. These include Einstein-Maxwell waves, rotating cylindrical GWs and more recently, cylindrical gravitational soliton waves [5, 6].

Despite these studies, they were not being able to derive the field equations for circular cylindrical coordinates. Hence, in this work, we show the way to obtain field equation for circular cylindrical and dynamical gravitational scalar potential exterior and interior to the body by applying the conditions of continuity across all boundaries and normal derivation.

2 Methodologies

Derivation of the Great Metric Tensor for Circular-Cylindrical Bodies

The Cartesian coordinates (x, y, z) are related to the circular cylindrical coordinates (ρ, ϕ, z) as [7, 8]

$$x = \rho cos\phi \tag{2.1}$$

$$y = \rho sin\phi \tag{2.2}$$

$$z = z \tag{2.3}$$

where,

$$0 \leq \rho < \infty, \qquad 0 \leq \emptyset \leq 2\pi$$
$$-\infty < z < \infty$$

The scale factors for cylindrical coordinate is given explicitly as [6,7,8,9,10]

$$h_{11} = 1 \tag{2.4}$$

$$h_{22} = \rho \tag{2.5}$$

$$h_{33} = 1 \tag{2.6}$$

$$h_{00} = -1 \tag{2.7}$$

To express the covariant great metric tensor in circular-cylindrical coordinate, we need to transform the coordinates from Cartesian to circular-cylindrical coordinate as follows$(x, y, z, x^0) \rightarrow (\rho, \phi, z, x^0)$

$$g_{11}(\rho, \phi, z, x^0) = \left(1 + \frac{2}{c^2} f\right)^{-1} \tag{2.8}$$

$$g_{22(\rho,\phi,z,x^0)} = \rho^2 \tag{2.9}$$

$$g_{33(\rho,\phi,z,x^0)} = 1 \tag{2.10}$$

$$g_{00(\rho,\phi,z,x^0)} = -\left(1 + \frac{2}{c^2} f\right) \tag{2.11}$$

$$g_{\mu\nu} = 0, \text{ otherwise.} \tag{2.12}$$

Equations (2.8) – (2.12) are the covariant great metric tensor for circular-cylindrical bodies.

The contarvariant great metric tensors for circular-cylindrical bodies were obtained applying the quotient rule as

$$g^{11} = \left(1 + \frac{2}{c^2} f\right) \tag{2.13}$$

$$g^{22} = \frac{1}{\rho^2} \tag{2.14}$$

$$g^{33} = 1 \tag{2.15}$$

$$g^{00} = -\left(1 + \frac{2}{c^2} f\right)^{-1} \tag{2.16}$$

$$g^{\mu\nu} = 0, \text{ otherwise} \tag{2.17}$$

Derivation of Dynamical Gravitational Field Equation for Circular-Cylindrical Bodies

The well-known Riemannian's Laplacian operator∇_R^2, is given by [5,6,7,8]

$$\nabla_R^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^1}\left(\sqrt{g}. g^{11} \frac{\partial}{\partial x^1}\right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^2}\left(\sqrt{g}. g^{22} \frac{\partial}{\partial x^2}\right)$$
$$+ \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^3}\left(\sqrt{g}. g^{33} \frac{\partial}{\partial x^3}\right) + \frac{1}{\sqrt{g}} \frac{1}{\partial x^0}\left(\sqrt{g}. g^{00} \frac{\partial}{\partial x^0}\right) \tag{2.18}$$

Explicitly in this coordinate system, the laplacian operator is given as

$$= \frac{1}{\rho} \frac{\partial}{\partial \rho}\left[\rho\left(1 + \frac{2f(\rho, \varphi, z)}{c^2}\right) \frac{\partial}{\partial \rho}\right] + \frac{\partial}{\partial \varphi}\left[\frac{1}{\rho^2} \frac{\partial}{\partial \varphi}\right] + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\right]$$
$$+ \frac{1}{c^2} \frac{\partial}{\partial t}\left[-\left(1 + \frac{2f(\rho,\varphi,z)}{c^2}\right)^{-1} \frac{\partial}{\partial t}\right] \tag{2.19}$$

By simplification, equation (2.19) becomes

$$\nabla_R^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left[\rho\left(1 + \frac{2f(\rho, \varphi, z)}{c^2}\right) \frac{\partial}{\partial \rho}\right] + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$
$$+ \frac{1}{c^2} \frac{\partial}{\partial t}\left[-\left(1 + \frac{2f(\rho,\varphi,z)}{c^2}\right)^{-1} \frac{\partial}{\partial t}\right] \tag{2.20}$$

The dynamical field equation for circular cylindrical bodies is given by

$$\nabla_R^2 f(\rho, \varphi, z) = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left[\rho\left(1 + \frac{2f(\rho, \varphi, z)}{c^2}\right) \frac{\partial f(\rho, \varphi, z)}{\partial \rho}\right] + \frac{1}{\rho^2} \frac{\partial^2 f(\rho, \varphi, z)}{\partial \varphi^2} + \frac{\partial^2 f(\rho, \varphi, z)}{\partial z^2}$$
$$+ \frac{1}{c^2} \frac{\partial}{\partial t}\left[-\left(1 + \frac{2f(\rho,\varphi,z)}{c^2}\right)^{-1} \frac{\partial f(\rho,\varphi,z)}{\partial t}\right] = \begin{cases} 0; & \rho > R \\ 4\pi G\rho_0; & \rho < R \end{cases} \tag{2.21}$$

By the symmetry of the distribution of density it follows that, the gravitational field will depend only on the coordinate ρ, then $f(\rho,\phi,z) = f(\rho)$

Equation (2.21) reduces to

$$\nabla_R^2 f(\rho) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\left(1+\frac{2f(\rho)}{c^2}\right)\frac{\partial f(\rho)}{\partial\rho}\right] = \begin{cases} 0; & \rho > R \\ 4\pi G\rho_0; & \rho < R \end{cases} \quad (2.22)$$

Or explicitly as

$$f^{II} + \frac{1}{\rho}f^{I} + \frac{2}{c^2}ff^{II} + \frac{2}{c^2}ff^{I} + \frac{2}{c^2}(f^{I})^2 = \begin{cases} 0; & \rho > R \\ 4\pi G\rho_0; & \rho < R \end{cases} \quad (2.23)$$

3 Conclusions

In this research work we are able to define the covariant as well as the contravariant great metric tensors for circular cylindrical systems. The great metric tensors obtained can be used to determine the affine connections which can further be used to construct geometrical field equations.

Equation (2.20) is the Laplacian operator for circular cylindrical which satisfies the priori of natural laws i.e., independency of the coordinate system used and reduces to second order partial differential equation [11,12,13,14].

The Riemannian Laplacian operator equation (2.20) obtained in this work could be used in the generalization of Maxwell Theory of Electromagnetism in this field based upon Riemannian geometry as well as used in the study of gravito-electric and gravito-magnetic coupling.

The Laplacian operator was applied to obtain the dynamical gravitational field equation for circular-cylindrical bodies. This field equation is open for application to massive bodies of circular-cylindrical coordinates and it contains $\left(1+\frac{2}{c^2}f\right)$ and density term contribution ρ which are not found in the existing well-known Newton's dynamical field equations. This field equation differs from the well-known field equation for spherical bodies. This is as a result of difference in geometry.

It could be observed that our obtained result in equation (2.23) contains unknown Post Newton correction terms of the orders c^{-2} which are for theoretical development, applications and experimental verification.

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