Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/090312

Solving Fractional Partial Difference Equations Using the Discrete Homotopy Analysis Method

Figen Özpınar

Bolvadin Vocational School, Afyon Kocatepe University, TR-03300 Afyonkarahisar, Turkey

Received: 10 Nov. 2021, Revised: 12 Jan. 2022, Accepted: 22 Feb. 2022
Published online: 1 Jul. 2023

Abstract: In this paper, we propose the discrete homotopy analysis method(DHAM) to solve time-fractional difference equations. The fractional differences are described by Caputo's sense. Several illustrative examples present the capability of DHAM for wide classes of fractional partial difference equations.

Keywords: Discrete homotopy analysis method, fractional order, partial difference equations, Caputo-like delta difference.

1 Introduction

Fractional calculus has been considerable interest in numerous fields of science and engineering, such as electrical networks, chemical physics, optics and signal processing, diffusion, viscoelasticity, and so on [1,2,3,4,5]. The fractional difference is a discretized version of the fractional derivative. Recently, discrete fractional calculus has been increasing in popularity [1,6,7,8,9].
Developing correct methods to solve partial differential equations has become important. Some of the (semi) analytical methods for partial differential equations are the Adomian decomposition method(ADM)[5], variational iteration method(VIM)[10], homotopy perturbation method(HPM)[11], and so on. Another analytical approach is the homotopy analysis method(HAM)[12,13]. HAM is independent of small/large physical parameters, unlike the other methods. Very recently, Zhu et al. [14] introduced the discrete homotopy analysis method(DHAM). This method can be employed for solving complex problems containing discontinuity in fluid characteristics and geometry of the problem.
In this study, we propose the discrete homotopy analysis method for solving fractional partial difference equations. Fractional differences are described by Caputo's sense. We shall demonstrate the applicability of DHAM to standard fractional partial difference equations through several test examples.

2 Preliminaries

In this section, we introduce some definitions and properties of discrete fractional calculus which are used further in this paper.

Definition 1([7,8]). *If $f:\mathbb{N}_a\to\mathbb{R}$ and $\kappa>0$ then the fractional sum of order κ, denoted by ${}_a\Delta_t^{-\kappa}$, is defined by*

$$ {}_a\Delta_t^{-\kappa}f(t)=\frac{1}{\Gamma(\kappa)}\sum_{\tau=a}^{t-\kappa}\big(t-\sigma(\tau)\big)^{(\kappa-1)}f(\tau),\qquad t\in\mathbb{N}_{a+\kappa}, $$

* Corresponding author e-mail: fozpinar@aku.edu.tr

where $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$, $\sigma(\tau) = \tau + 1$.

The trivial sum is

$$_a\Delta_t^{-0} f(t) = f(t), \qquad t \in \mathbb{N}_a.$$

Here

$$t^{(\kappa)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\kappa)}$$

is falling function.
This definition is analogous to Riemann-Liouville fractional integral.
Throughout, we assume that if $(t+1-\kappa) \in \{0, -1, -2, \ldots\}$, then $t^{(\kappa)} = 0$.
The power rule was proved in [7]

$$\Delta_t^{-\kappa} t^{(\upsilon)} = \frac{\Gamma(\upsilon+1)}{\Gamma(\kappa+\upsilon+1)} t^{(\kappa+\upsilon)}, \qquad \upsilon \in \mathbb{R}^+.$$

Definition 2([1,6]). *If* $f : \mathbb{N}_a \to \mathbb{R}$ *and* $\kappa > 0$, *then the Caputo-like delta fractional difference of the* κ *order, denoted by* ${}_a^C\Delta_t^\kappa$, *is defined by*

$$\begin{aligned} {}_a^C\Delta_t^\kappa f(t) &= {}_a^C\Delta_t^{-(l-\kappa)} \Delta^l f(t) \\ &= \frac{1}{\Gamma(l-\kappa)} \sum_{\tau=a}^{t-(l-\kappa)} (t-\sigma(\tau))^{(l-\kappa-1)} \Delta^l f(\tau), \qquad t \in \mathbb{N}_{a+(l-\kappa)}, \end{aligned}$$

where $l \in \mathbb{N}_0$ *is such that* $l-1 < \kappa \le l$.
${}_a^C\Delta_t^\kappa$ *maps functions defined on* $\mathbb{N}_a$ *to functions defined on* $\mathbb{N}_{a+(l-\kappa)}$.

In particular, if $0 < \kappa \le 1$, then

$${}_a^C\Delta_t^\kappa f(t) = \frac{1}{\Gamma(1-\kappa)} \sum_{\tau=a}^{t-(1-\kappa)} (t-\sigma(\tau))^{(-\kappa)} \Delta f(\tau),$$

where $\Delta f(\tau) = f(\tau+1) - f(\tau)$.
If κ is an integer, then this difference takes the ordinary forward difference operator

$${}_a^C\Delta_t^\kappa = \Delta^\kappa, \qquad \kappa = 1, 2, 3, \ldots$$

Proposition 1([?]). *If* $f : \mathbb{N}_a \to \mathbb{R}$ *and* $\kappa > 0$, *then*

$${}_{a+(l-\kappa)}\Delta_t^{-\kappa} {}_a^C\Delta_t^\kappa f(t) = f(t) - \sum_{s=0}^{l-1} \frac{(t-a)^{(s)}}{s!} \Delta^s f(a),$$

where $l \in \mathbb{N}_0$ *is such that* $l-1 < \kappa \le l$.
For special case $0 < \kappa \le 1$,

$${}_{a+(1-\kappa)}\Delta_t^{-\kappa} {}_a^C\Delta_t^\kappa f(t) = f(t) - f(a).$$

3 Discrete homotopy analysis method for the fractional difference equations

To illustrate the basic idea of the discrete homotopy analysis method(DHAM), we consider the following fractional partial difference equation:

$$ {}_a^C\Delta_t^{\kappa} U_{k,t} + L(U_{k,t}) + N(U_{k,t}) = g_{k,t}, \qquad l-1<\kappa\le l, \qquad t\in\mathbb{N}_{a+(l-\kappa)}, \qquad k\in\mathbb{N}_0, \tag{1}$$

subject to the initial conditions

$$ {}_a^C\Delta_t^{s} U_{k,a} = f_{k_s}, \qquad s=0,1,\dots,l-1, \tag{2}$$

where L is a linear difference and N is nonlinear difference operator, $g_{k,t}$ is the source term.
Because of the original Eq(1) contains the linear operator ${}_a^C\Delta_t^{\kappa}$, we can choose the auxiliary linear operator

$$ \pounds[U_{k,t}] = {}_a^C\Delta_t^{\kappa} U_{k,t}. \tag{3}$$

For simplicity, we can define the nonlinear operator

$$ \aleph[U_{k,t}] = {}_a^C\Delta_t^{\kappa} U_{k,t} + L(U_{k,t}) + N(U_{k,t}) - g_{k,t}. \tag{4}$$

Similarly to continuous HAM, we can construct the so-called *zeroth*-order deformation equation by means of the discrete HAM(DHAM)

$$ (1-p)\pounds[\varphi_{k,t}(p) - U_{k_0,t}] = p\hbar H_{k,t}\,\aleph[\varphi_{k,t}(p)], \tag{5}$$

subject to the initial conditions

$$ {}_a^C\Delta_t^{s}\varphi_{k,a}(p) = f_{k_s}, \qquad s=0,1,\dots,l-1, \tag{6}$$

where $p\in[0,1]$ is an embedding parameter, $\hbar\neq 0$ is an auxiliary parameter which is called convergence-control parameter, $H_{k,t}$ is a nonzero auxiliary function, $U_{k_0,t}$ denotes an initial guess of $U_{k,t}$, $\varphi_{k,t}(p)$ is an unknown function on the independent variables *k,t,p*. Obviously, when $p=0$ and $p=1$, it holds

$$ \varphi_{k,t}(0) = U_{k_0,t} \qquad and \qquad \varphi_{k,t}(1) = U_{k,t}, \tag{7}$$

respectively. According to DHAM we expend $\varphi_{k,t}(p)$ in Taylor series, with respect to p as follows:

$$ \varphi_{k,t}(p) = U_{k_0,t} + \sum_{m=1}^{\infty} U_{k_m,t}\,p^m, \tag{8}$$

where

$$ U_{k_m,t} = \frac{1}{m!}\frac{\partial^m\varphi_{k,t}(p)}{\partial p^m}\bigg|_{p=0}. \tag{9}$$

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter and the auxiliary function are properly chosen such that the series (8) is convergent at $p=1$. Thus, according to (7) we have

$$ U_{k,t} = U_{k_0,t} + \sum_{m=1}^{\infty} U_{k_m,t}. \tag{10}$$

When $\hbar=-1$ and $H_{k,t}=1$, Eq(5) becomes

$$ (1-p)\pounds[\varphi_{k,t}(p) - U_{k_0,t}] + p\,\aleph[\varphi_{k,t}(p)] = 0, $$

used in the discrete homotopy perturbation method[15].
To obtain *mth*-order deformation equation let us define the vector:

$$ \overrightarrow{U_m} = \{U_{k_0,t}, U_{k_1,t}, U_{k_2,t}, \dots, U_{k_m,t}\}. $$

Differentiating Eq(5) m-times with respect to the embedding parameter p, then setting $p=0$ and finally dividing by $m!$, we have mth-order deformation equation as follows:

$$\pounds\left[U_{k_m,t}-\chi_m U_{k_{m-1},t}\right]=\hbar H_{k,t}\Re_m\left[\overrightarrow{U_{m-1}}\right], \tag{11}$$

subject to the initial conditions

$${}_a^C\Delta_t^s U_{k_m,a}=0, \qquad s=0,1,\ldots,l-1, \tag{12}$$

where

$$\Re_m\left[\overrightarrow{U_{m-1}}\right]=\frac{1}{(m-1)!}\frac{\partial^{m-1}\aleph\left[\varphi_{k,t}(p)\right]}{\partial p^{m-1}}\bigg|_{p=0}, \tag{13}$$

$$\chi_m=\begin{cases}0,\ m\leq 1\\ 1,\ m>1\end{cases}.$$

Applying the fractional sum ${}_{a+(l-\kappa)}\Delta_t^{-\kappa}$ to both sides of Eq(11), we have

$$U_{k_m,t}=\chi_m U_{k_{m-1},t}+\hbar_{a+(l-\kappa)}\Delta_t^{-\kappa}\left[H_{k,t}\Re_m\left[\overrightarrow{U_{m-1}}\right]\right]. \tag{14}$$

So that mth-order deformation equation Eq(11) is linear, it can be solved easily.

4 Test examples

In this section, to demonstrate the applicability of our approach, we shall apply the DHAM to several fractional partial difference equations.

*Example 1.*Let us consider the following fractional difference equation

$${}_0^C\Delta_t^\kappa U_{k,t}=\eta U_{k,t}, \qquad k\in\mathbb{N}_0, \qquad t\in\mathbb{N}_{1-\kappa}, \qquad 0<\kappa\leq 1, \tag{15}$$

subject to initial condition

$$U_{k,0}=a_0, \qquad a_0\in\mathbb{R}. \tag{16}$$

By means of DHAM, we choose the linear operator

$$\pounds\left[\varphi_{k,t}(p)\right]={}_0^C\Delta_t^\kappa\varphi_{k,t}(p), \tag{17}$$

with property $\pounds\left[c\right]=0$, where c is constant. We define nonlinear operator

$$\aleph\left[\varphi_{k,t}(p)\right]={}_0^C\Delta_t^\kappa\varphi_{k,t}(p)-\eta\varphi_{k,t}(p). \tag{18}$$

We construct the *zeroth*-order deformation equation by Eq(5). It is obvious that $p=0$ and $p=1$, we can write

$$\varphi_{k,t}(0)=U_{k_0,t}=U_{k,0} \qquad , \qquad \varphi_{k,t}(1)=U_{k,t}. \tag{19}$$

Thus we obtain the mth-order deformation equation for $m\geq 1$:

$$\pounds\left[U_{k_m,t}-\chi_m U_{k_{m-1},t}\right]=\hbar H_{k,t}\Re_m\left[\overrightarrow{U_{m-1}}\right]$$

or

$$U_{k_m,t}=\chi_m U_{k_{m-1},t}+\hbar_{(1-\kappa)}\Delta_t^{-\kappa}\left[H_{k,t}\Re_m\left[\overrightarrow{U_{m-1}}\right]\right], \tag{20}$$

where

$$\begin{aligned}\Re_m\left[\overrightarrow{U_{m-1}}\right] =&\frac{1}{(m-1)!}\frac{\partial^{m-1}\aleph\left[\varphi_{k,t}(p)\right]}{\partial p^{m-1}}\Bigg|_{p=0}\\ =&{}_0^C\Delta_t^{\kappa}U_{k_{m-1},t}-\eta U_{k_{m-1},t}.\end{aligned} \tag{21}$$

In order to the approximations of $U_{k,t}$ are only depend on auxiliary parameter $\hbar$, we select $H_{k,t}=1$. Thus we substitute the initial condition (16) in to (20) we have

$$\begin{aligned}U_{k_1,t} =&-\hbar\eta a_0\frac{t^{(\kappa)}}{\Gamma(\kappa+1)},\\ U_{k_2,t} =&-\hbar(1+\hbar)\eta a_0\frac{t^{(\kappa)}}{\Gamma(\kappa+1)}+\hbar^2\eta^2 a_0\frac{\left(t+(\kappa-1)\right)^{(2\kappa)}}{\Gamma(2\kappa+1)},\\ U_{k_3,t} =&-\hbar(1+\hbar)^2\eta a_0\frac{t^{(\kappa)}}{\Gamma(\kappa+1)}+2\hbar^2(1+\hbar)\eta^2 a_0\frac{\left(t+(\kappa-1)\right)^{(2\kappa)}}{\Gamma(2\kappa+1)}\\ &-\hbar^3\eta^3 a_0\frac{\left(t+2(\kappa-1)\right)^{(3\kappa)}}{\Gamma(3\kappa+1)},\\ U_{k_4,t} =&-\hbar(1+\hbar)^3\eta a_0\frac{t^{(\kappa)}}{\Gamma(\kappa+1)}+3\hbar^2(1+\hbar)^2\eta^2 a_0\frac{\left(t+(\kappa-1)\right)^{(2\kappa)}}{\Gamma(2\kappa+1)}\\ &-3\hbar^3(1+\hbar)\eta^3 a_0\frac{\left(t+2(\kappa-1)\right)^{(3\kappa)}}{\Gamma(3\kappa+1)}+\hbar^4\eta^4 a_0\frac{\left(t+3(\kappa-1)\right)^{(4\kappa)}}{\Gamma(4\kappa+1)},\\ &\vdots\end{aligned}$$

and so on. Hence, the DHAM series solution is

$$\begin{aligned}U_{k,t} =&U_{k_0,t}+U_{k_1,t}+U_{k_2,t}+U_{k_3,t}+U_{k_4,t}+\ldots\\ =&a_0-\hbar\left[1+(1+\hbar)+(1+\hbar)^2+(1+\hbar)^3+\ldots\right]\eta a_0\frac{t^{(\kappa)}}{\Gamma(\kappa+1)}\\ &+\hbar^2\left[1+2(1+\hbar)+3(1+\hbar)^2+\ldots\right]\eta^2 a_0\frac{\left(t+(\kappa-1)\right)^{(2\kappa)}}{\Gamma(2\kappa+1)}\\ &-\hbar^3\left[1+3(1+\hbar)+\ldots\right]\eta^3 a_0\frac{\left(t+2(\kappa-1)\right)^{(3\kappa)}}{\Gamma(3\kappa+1)}+\hbar^4\eta^4 a_0\frac{\left(t+(\kappa-1)\right)^{(4\kappa)}}{\Gamma(4\kappa+1)}+\ldots\end{aligned}$$

Setting $\hbar=-1$, we get the solution in the following form:

$$\begin{aligned}U_{k,t} =&a_0+\eta a_0\frac{t^{(\kappa)}}{\Gamma(\kappa+1)}+\eta^2 a_0\frac{\left(t+(\kappa-1)\right)^{(2\kappa)}}{\Gamma(2\kappa+1)}+\eta^3 a_0\frac{\left(t+2(\kappa-1)\right)^{(3\kappa)}}{\Gamma(3\kappa+1)}\\ &+\eta^4 a_0\frac{\left(t+3(\kappa-1)\right)^{(4\kappa)}}{\Gamma(4\kappa+1)}+\ldots\end{aligned}$$

$$U_{k,t}=a_0\sum_{n=0}^{\infty}\eta^n\frac{\left(t+(n-1)(\kappa-1)\right)^{(n\kappa)}}{\Gamma(n\kappa+1)}=a_0E_{(\kappa)}(\eta,t), \tag{22}$$

where $E_{(\kappa)}$ is the discrete(like) Mittag-Leffler function[**?**].

Example 2. In this example, we consider the following fractional order discrete diffusion equation

$$ {}_0^C\Delta_t^\kappa U_{k,t} = \Delta_k^2 U_{k,t} + k\Delta_k U_{k,t} + U_{k,t}, \qquad k \in \mathbb{N}_0, \qquad t \in \mathbb{N}_{1-\kappa}, \qquad 0 < \kappa \leq 1, \tag{23} $$

subject to initial condition

$$ U_{k,0} = k, \tag{24} $$

where Δ_k is the forward partial difference which is defined as usual, i.e., $\Delta_k U_{k,t} = U_{k+1,t} - U_{k,t}$ and $\Delta_k^2 U_{k,t} = \Delta_k(\Delta_k U_{k,t}) = U_{k+2,t} - 2U_{k+1,t} + U_{k,t}$.
By means of DHAM, we consider the following linear operator:

$$ \pounds\left[\varphi_{k,t}(p)\right] = {}_0^C\Delta_t^\kappa \varphi_{k,t}(p), \tag{25} $$

with property $\pounds\left[c\right] = 0$, where c is constant. We define nonlinear operator as

$$ \aleph\left[\varphi_{k,t}(p)\right] = {}_0^C\Delta_t^\kappa \varphi_{k,t}(p) - \Delta_k^2 \varphi_{k,t}(p) - k\Delta_k \varphi_{k,t}(p) - \varphi_{k,t}(p). \tag{26} $$

We construct the *zeroth*-order deformation equation by Eq(5). It is obvious that for $p = 0$ and $p = 1$, we can write

$$ \varphi_{k,t}(0) = U_{k_0,t} = U_{k,0} \qquad , \qquad \varphi_{k,t}(1) = U_{k,t}. \tag{27} $$

Thus we obtain the *mth*-order deformation equation for $m \geq 1$:

$$ U_{k_m,t} = \chi_m U_{k_{m-1},t} + \hbar_{(1-\kappa)}\Delta_t^{-\kappa}\left[H_{k,t}\Re_m\left[\overrightarrow{U_{m-1}}\right]\right], \tag{28} $$

where

$$ \Re_m\left[\overrightarrow{U_{m-1}}\right] = {}_0^C\Delta_t^\kappa U_{k_{m-1},t} - \Delta_k^2 U_{k_{m-1},t} - k\Delta_k U_{k_{m-1},t} - U_{k_{m-1},t}. \tag{29} $$

In order to the approximations of $U_{k,t}$ are only depend on auxiliary parameter $\hbar$, we select $H_{k,t} = 1$.
Thus we substitute the initial condition (24) in to (28) we have

$$
\begin{aligned}
U_{k_1,t} &= -\hbar 2k\frac{t^{(\kappa)}}{\Gamma(\kappa+1)},\\
U_{k_2,t} &= -\hbar(1+\hbar)2k\frac{t^{(\kappa)}}{\Gamma(\kappa+1)} + \hbar^2 2^2 k\frac{(t+(\kappa-1))^{(2\kappa)}}{\Gamma(2\kappa+1)},\\
U_{k_3,t} &= -\hbar(1+\hbar)^2 2k\frac{t^{(\kappa)}}{\Gamma(\kappa+1)} + 2\hbar^2(1+\hbar)2^2 k\frac{(t+(\kappa-1))^{(2\kappa)}}{\Gamma(2\kappa+1)}\\
&\quad - \hbar^3 2^3 k\frac{(t+2(\kappa-1))^{(3\kappa)}}{\Gamma(3\kappa+1)},\\
U_{k_4,t} &= -\hbar(1+\hbar)^3 2k\frac{t^{(\kappa)}}{\Gamma(\kappa+1)} + 3\hbar^2(1+\hbar)^2 2^2 k\frac{(t+(\kappa-1))^{(2\kappa)}}{\Gamma(2\kappa+1)}\\
&\quad - 3\hbar^3(1+\hbar)2^3 k\frac{(t+2(\kappa-1))^{(3\kappa)}}{\Gamma(3\kappa+1)} + \hbar^4 2^4 k\frac{(t+3(\kappa-1))^{(4\kappa)}}{\Gamma(4\kappa+1)},\\
&\quad \vdots
\end{aligned}
$$

and so on. Hence, the DHAM series solution is

$$
\begin{aligned}
U_{k,t} =& k - \hbar\left[1 + (1+\hbar) + (1+\hbar)^2 + (1+\hbar)^3 + \ldots\right]2k\frac{t^{(\kappa)}}{\Gamma(\kappa+1)}\\
&+ \hbar^2\left[1 + 2(1+\hbar) + 3(1+\hbar)^2 + \ldots\right]2^2 k\frac{(t+(\kappa-1))^{(2\kappa)}}{\Gamma(2\kappa+1)}\\
&- \hbar^3\left[1 + 3(1+\hbar) + \ldots\right]2^3 k\frac{(t+2(\kappa-1))^{(3\kappa)}}{\Gamma(3\kappa+1)} + \hbar^4 2^4 k\frac{(t+3(\kappa-1))^{(4\kappa)}}{\Gamma(4\kappa+1)} + \ldots
\end{aligned}
$$

Setting $\hbar = -1$, we obtain the solution in the following form

$$\begin{aligned} U_{k,t} =& k + 2k\frac{t^{(\kappa)}}{\Gamma(\kappa+1)} + 2^2 k\frac{(t+(\kappa-1)^{(2\kappa)})}{\Gamma(2\kappa+1)} + 2^3 k\frac{(t+2(\kappa-1))^{(3\kappa)}}{\Gamma(3\kappa+1)} \\ &+ 2^4 k\frac{(t+3(\kappa-1))^{(4\kappa)}}{\Gamma(4\kappa+1)} + \dots \end{aligned}$$

$$U_{k,t} = k\sum_{n=0}^{\infty} 2^n \frac{\left(t+(n-1)(\kappa-1)\right)^{(n\kappa)}}{\Gamma(n\kappa+1)} = kE_{(\kappa)}(2,t). \tag{30}$$

5 Conclusion

The discrete homotopy analysis method is implemented to fractional partial difference equations. The convergence region of the solution series obtained by DHAM can be controlled by introducing an auxiliary parameter. Obtained results show that DHAM can provide highly accurate solutions for fractional partial difference equations.

References

[1] T. Abdeljawad, On Riemann and Caputo fractional differences, *Comput. Math. Appl.* **62**, 1602–1611 (2011).
[2] R. Herrmann, *Fractional calculus-an introduction for physicists*, World Scientific, Singapore, 2011.
[3] R. E. Mickens, *Nonstandard finite difference models of differential equations*, World Publ. Co., Singapore, 1994.
[4] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.
[5] A. M. Wazwaz, A reliable modification of Adomian's decomposition method, *Appl. Math. Comput.* **102**, 77-86 (1999).
[6] G. A. Anastassiou, About discrete fractional calculus with inequalities, *Intel. Math. Comput. Anal. Intel. Sys. Refer. Lib.* **5**, 575–585 (2011).
[7] F. M. Atıcı and P. W. Eloe, A transform method in discrete fractional calculus, *Int. J. Differ. Equ. 2*, 165–176 (2007).
[8] F. M. Atıcı and P. W. Eloe, Initial value problems in discrete fractional calculus, *Proc. Amer. Math Soc.* **137**, 981–989 (2009).
[9] F. M. Atıcı and S. Sengül, Modelling with fractional difference equations, *J. Math. Anal. Appl.* **369**, 1–9 (2010).
[10] J. H. He, Variational iteration method-a kind of nonlinear analytical technique: some examples, *Int. J. Nonlin. Mech.* **34**, 699–708 (1999).
[11] J. H. He, Homotopy perturbation technique, *Comp. Meth. Appl. Mech. Engn.* **178**, 257–262 (1999).
[12] S. J. Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems*, Ph.D Thesis, Shanghai Jiao Tong University, 1992.
[13] S. J. Liao, *Beyond Perturbation: Introduction to Homotopy Analysis Method*, Chapman and Hall/CRC Press, Boca Raton, 2003.
[14] H. Zhu, H. Shu and M. Ding, Numerical solutions of partial differential equations by discrete homotopy analysis method, *Appl. Math. Comput.* **216**, 3592–3605 (2010).
[15] H. Zhu and M. Ding, The discrete homotopy perturbation method for solving Burgers' and heat equations, *J. Inf. Comput. Sci* **11**, 1647–1657 (2014).